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SOME INEQUALITIES ABOUT CONVOLUTION AND TRIGONOMETRIC APPROXIMATION IN WEIGHTED ORLICZ SPACES

ALI DOĞU, AHMET HAMDI AVŞAR, AND YUNUS EMRE YILDIRIR

Abstract. In this paper, we investigate the estimation problem of a convolution type transform by the best trigonometric approximation numbers in weighted Orlicz spaces, in which the generating Young functions are not necessary to be convex.

1. Introduction

In the approximation theory, some convolution operators are commonly used. There are some applications of this type operators in this theory. Especially, these operators are very convenient for the construction of the approaching polynomials in trigonometric approximation. In different function spaces, the problem of estimation of these operators by means of the sequences of the best approximation numbers is an important problem of the approximation theory. This problem was investigated in classical Orlicz spaces in [7] and weighted Orlicz spaces with Muckenhoupt weights in [9]. On the other hand, a different approach for the Orlicz spaces came up in the paper [3], in which the definition of Orlicz spaces was generalized by Chen saving almost all known properties of these spaces. Later, Akgün developed this approach with Muckenhoupt weights and proved direct and inverse theorems of trigonometric approximation in these spaces [1]. In this work, we investigate the estimation problem mentioned above in these spaces developed by Chen and Akgün. This problem was also investigated in weighted Lorentz spaces [10] and variable exponent Lebesgue spaces [4]. To formulate the main results obtained in this work we need some definitions and notation.

We denote by Φ the class of strictly increasing functions $\phi : [0,\infty) \to [0,\infty)$ such that $\phi(\infty) = \infty$. Let $N[p,q]$ be the class of even functions $\varphi \in \Phi$ such that $\varphi(x)x^{-p}$ is non-decreasing and $\varphi(x)x^{-q}$ is non-increasing as $|x|$ is increasing in $(0,\infty)$. If $p < q$, by $N\langle p,q\rangle$, we denote the class of functions φ in $N[p+\varepsilon, q-\varepsilon]$ for some small number $\varepsilon, \delta > 0$. The class of functions M in $N\langle p,q\rangle$ for some $1 < p \le q < \infty$ will be denoted by Φ_p. The functions in Φ_p, $p > 1$, are continuous and satisfy the condition $M(0) = 0$ and $M \in \Delta_2$, that is, there is a constant $c > 0$ and $u_0 > 0$ such that $M(2u) \le cM(u)$ for $u \ge u_0$. These functions may not be

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convex [3]. If a nonnegative function ω defined on $\mathbf{T} := [0, 2\pi]$ is measurable and positive almost everywhere, then this function is called a weight function.

Let $M \in \Phi_p$, $p > 1$ and ω be a weight function on $\mathbf{T}$. We define $\varphi_M(t) := M(t)/t$. Since $1 < p < q < \infty$, we get $\varphi_M(t) \to \infty$ as $t \to \infty$. By $\psi_M(t)$, we denote the inverse function of positive non-decreasing continuous function $\varphi_M(t)$. We set

$$\Phi_M(x) = \int_0^x \varphi_M(t)dt$$

and

$$\Psi_M(x) = \int_0^x \psi_M(t)dt.$$

Φ_M is a convex function and so Ψ_M is the complementary function of Φ_M, in the sense of Young. The space of Lebesgue measurable functions $f : \mathbf{T} \to \mathbb{R}$ with the condition

$$\int_{\mathbf{T}} \Phi_M (c |f(x)|) \omega (x) dx < \infty,$$

for some constant $c > 0$, will be denoted by $L_{M,\omega}(\mathbf{T})$. In this space, we define the Orlicz norm

$$\|f\|_{M,\omega} := \sup_g \left\{ \int_{\mathbf{T}} |f(x)g(x)| \omega (x) dx : \int_{\mathbf{T}} \Psi_M (|g(x)|) \omega(x)dx \leq 1 \right\}$$

and the Luxemburg norm

$$\|f\|_{(\Phi_M),\omega} := \inf \left\{ k > 0 : \int_{\mathbf{T}} \Phi_M \left(k^{-1} |f(x)|\right) \omega (x) dx \leq 1 \right\}.$$

The equivalence

$$\|f\|_{M,\omega} \sim \|f\|_{(\Phi_M),\omega}$$

is valid. Furthermore, the Orlicz norm can be determined by means of the Luxemburg norm

$$\|f\|_{M,\omega} := \sup_g \left\{ \int_{\mathbf{T}} |f(x)g(x)| \omega (x) dx : \|g\|_{(\Psi_M),\omega} \leq 1 \right\}.$$

It is easily seen that $L_{M,\omega}(\mathbf{T}) \subset L^1(\mathbf{T}, \omega)$, where $\omega \in L^1(T)$ and $L_{M,\omega}(\mathbf{T})$ becomes a Banach space with the above norms. This Banach space is called weighted Orlicz space. If we take $M(x) := x^p$, $1 < p < \infty$, then the space $L_{M,\omega}(\mathbf{T})$ turns into the weighted Lebesgue space $L^p(\mathbf{T}, \omega)$.

In this weighted Orlicz space, some trigonometric approximation problems were investigated in [2, 8].

The sequence of the best approximation numbers of $f \in L_{M,\omega}(\mathbf{T})$ by polynomials in $\mathcal{T}_n$ is defined by

$$E_n(f)_{M,\omega} := \inf_{T_n \in \mathcal{T}_n} \|f - T_n\|_{M,\omega}$$

where $\mathcal{T}_n$ is the set of trigonometric polynomials of degree $\leq n$.

A weight function $\omega : \mathbf{T} \to [0, \infty]$ belongs to the Muckenhoupt class A_p [6], $1 < p < \infty$, if

$$\left(\frac{1}{|I|} \int_I \omega(x) dx \right) \left(\frac{1}{|I|} \int_I \omega^{\frac{1}{1-p}}(x) dx \right)^p \leq C_{A_p},$$

with a finite constant C_{A_p} independent of I, where the supremum is taken with respect to all intervals I with length $\leq 2\pi$ and $|I|$ denotes the length of I. The constant C_{A_p} is called the Muckenhoupt constant of ω.

For $f \in L_{M,\omega}(\mathbf{T})$, let

$$f(x) \sim \sum_{k=-\infty}^{\infty} c_k(f) e^{ikx},$$

be the Fourier series of f, where

$$c_k(f) = \frac{1}{2\pi} \int_{\mathbf{T}} f(x) e^{-ikx} dx.$$

The n-th partial sum of Fourier series of f is

$$S_n(x, f) := S_n(f) := \sum_{k=-n}^{n} c_k(f) e^{ikx}$$

for $n = 0, 1, 2, ...$

In [1], it was proved that the operator $S_n : L_{M,\omega}(\mathbf{T}) \to L_{M,\omega}(\mathbf{T})$ is bounded in $L_{M,\omega}(\mathbf{T})$ if $M \in \Phi_p$, $p > 1$, $\omega \in A_p$ and $f \in L_{M,\omega}(\mathbf{T})$. Hence we have

$$\|S_n(f)\|_{(M),\omega} \leq C \|f\|_{(M),\omega}, \qquad n = 0, 1, 2, ... \tag{1.1}$$

and

$$\|f - S_n(f)\|_{(M),\omega} \leq C E_n(f)_{M,\omega}, \qquad n = 0, 1, 2, ... \tag{1.2}$$

The set of trigonometric polynomials is dense in $L_{M,\omega}(\mathbf{T})$ since the hypothesis of Lemma 3 of [5] are fulfilled for $M \in \Phi_p$, $p > 1$ and $\omega \in A_p$. Hence $E_n(f)_{M,\omega} \to 0$ as $n \to \infty$ and the Fourier Series of f converges to f in norm in $L_{M,\omega}(\mathbf{T})$.

For $f \in L_{M,\omega}(\mathbf{T})$, $1 < p, q < \infty$, $\omega \in A_p$, the operator σ_h is defined as

$$(\sigma_h f)(x, u) := \frac{1}{2h} \int_{-h}^{h} f(x + tu) dt, \quad 0 < h < \pi, \ x \in \mathbb{T}, \ -\infty < u < \infty.$$

In [1], it was proved that the Hardy-Littlewood Maximal operator is bounded in $L_{M,\omega}(\mathbf{T})$, if $M \in \Phi_p$, $p > 1$ and $\omega \in A_p$. Therefore the operator σ_h is bounded in $L_{M,\omega}(\mathbf{T})$ under conditions $M \in \Phi_p$, $p > 1$ and $\omega \in A_p$.

Since the space $L_{M,\omega}(\mathbf{T})$ is noninvariant with respect to the usual shift $f(x-hu)$, we define the convolution type transforms by using the mean value function $(\sigma_h f)(x,u)$.

For $f \in L_{M,\omega}(\mathbf{T})$ we define a convolution type transform

$$\int_{-\infty}^{\infty} (\sigma_h f)(x,u) d\mu(u)$$

where $\mu(u)$ is a real function of bounded variation on the real axis. We denote the norm of this transform by $D(f,\mu,h,M)$, i.e.

$$D(f,\mu,h,M) := \left\| \int_{-\infty}^{\infty} (\sigma_h f)(x,u) d\mu(u) \right\|_{M,\omega}.$$

In this paper, we estimate the norm $D(f,\mu,h,M)$ using the best approximation number $E_n(f)_{M,\omega}$.

Throughout this work, the constant c denotes a constant whose values can change even between different occurrences in a chain of inequalities. The relation $\preceq$ is defined as "$A \preceq B \Leftrightarrow$ there is a constant c such that $A \leq cB$"

The remainder of this paper is organized as follows. In Section 2, we state auxiliary results which will be used in the proof of the main results. Main results are formulated and proved in Section 3. Some concluding remarks are drawn in the last section.

2. Auxiliary results

Lemma 2.1. *Let φ be a measurable function of two variables and $\varphi(\cdot,u) \in L_{M,\omega}(\mathbf{T})$. Then*

$$\left\| \int_{\mathbf{T}} \varphi(\cdot,u) du \right\|_{M,\omega} \leq \int_{\mathbf{T}} \|\varphi(\cdot,u)\|_{M,\omega} du.$$

Proof. By Fubini's theorem and Hölder's inequality, we obtain

$$\begin{aligned} \left\| \int_{\mathbf{T}} \varphi(\cdot,u) du \right\|_{M,\omega} &\leq \sup_g \left\{ \int_{\mathbf{T}} \left(\int_{\mathbf{T}} |\varphi(x,u)| du \right) |g(x)| \omega(x) dx : \|g\|_{(\Psi_M),\omega} \leq 1 \right\} \\ &= \sup_g \left\{ \int_{\mathbf{T}} \left(\int_{\mathbf{T}} |\varphi(x,u)| |g(x)| \omega(x) dx \right) du : \|g\|_{(\Psi_M),\omega} \leq 1 \right\} \\ &\leq \sup_g \left\{ \int_{\mathbf{T}} \|\varphi(\cdot,u)\|_{M,\omega} \|g\|_{(\Psi_M),\omega} du : \|g\|_{(\Psi_M),\omega} \leq 1 \right\} \\ &\leq \int_{\mathbf{T}} \|\varphi(\cdot,u)\|_{M,\omega} du. \end{aligned}$$

□ □

Now, we give multiplier theorem and Littlewood-Paley theorem in $L_{M,\omega}(\mathbf{T})$ which are used in the proofs of the main results.

Theorem 2.1 (see [1]). *Let a sequence ξ_k satisfy the conditions*

$$|\xi_k| \le A, \sum_{k=2^{j-1}}^{2^j-1} |\xi_k - \xi_{k+1}| \le A$$

where $A > 0$ is independent of k and j. If $M \in \Phi_p$, $p > 1$, $\omega \in A_p$ and $f \in L_{M,\omega}(\mathbf{T})$ then there exists a function $F \in L_{M,\omega}(\mathbf{T})$ such that the series

$$\frac{\lambda_0 a_0}{2} + \sum_{k=0}^{\infty} \lambda_k (a_k \cos kx + b_k \sin kx)$$

is a Fourier series for F and the inequality

$$\|F\|_{(M),\omega} \le CA \|f\|_{(M),\omega},$$

holds with some constant $C > 0$ independent of f.

Theorem 2.2 (see [1]). *Let $M \in \Phi_p$, $p > 1$, $\omega \in A_p$ and let $f \in L_{M,\omega}(\mathbf{T})$. Then there exist the constants $C > 0$ and $c > 0$ depending only on M and ω such that*

$$c \|f\|_{(M),\omega} \le \left\| \left(\sum_{j=0}^{\infty} \left| \sum_{k=2^{j-1}}^{2^j-1} A_k(x,f) \right|^2 \right) \right\|_{(M),\omega}^{\frac{1}{2}} \le C \|f\|_{(M),\omega}.$$

3. Main results and discussion

The following theorem estimates the quantity $D(f,\mu,h,M)$ by means of the best trigonometric approximation of the function f in the weighted Orlicz spaces.

Theorem 3.1. *Let $M \in \Phi_p$, $p > 1$, $\omega \in A_p$, $f \in L_{M,\omega}(\mathbf{T})$, $M(\sqrt{x})$ be quasiconvex and $\gamma := \min(2, p+\varepsilon)$, where ε is a small positive number. Then for every natural number m*

$$D(f,\mu,h,M) \preceq \left(\sum_{r=0}^{m} E_{2^r-1}^{\gamma}(f)_{M,\omega} \cdot \delta_{2^r,h}^{\gamma} \right)^{1/\gamma} + E_{2^{m+1}}(f)_{M,\omega} \tag{3.1}$$

where

$$\begin{aligned} \delta_{2^r,h} &:= \sum_{l=2^r}^{2^{r+1}-1} |\hat{\mu}(lh) - \hat{\mu}((l+1)h)| + |\hat{\mu}(2^r h)|, \\ \hat{\mu}(x) &:= \int_{-\infty}^{\infty} \frac{\sin ux}{ux} d\mu(u), \qquad 0 < h \le \pi. \end{aligned}$$

Theorem 3.2. *Let $M \in \Phi_p$, $p > 1$, $\omega \in A_p$ and $f \in L_{M,\omega}(\mathbf{T})$. Suppose that the function $F(x)$ satisfies the conditions*

$$\|F(x)\| \le c_1, \sum_{k=2^{\mu}}^{2^{\mu+1}-1} |F(kh) - F((k+1)h)| \le c_2, \ h \le 2^{-m-1}.$$

with some constants c_1, c_2. *If* μ_1 *and* μ_2 *are the functions satisfying the condition*

$$\hat{\mu}_1(x) = \hat{\mu}_2(x)F(x), \qquad |x| < 1$$

then

$$D(f, \mu_1, h, M) \preceq D(f, \mu_2, h, M) + E_{2^{m+1}}(f)_{M,\omega}.$$

Defining the convolution operator by means of the usual shift $f(x+t)$, the similar theorems were proved in Orlicz spaces in [7]. Using the operator $(\sigma_h f)$, the analogues of these theorems were obtained in weighted Orlicz spaces, in which the generating Young function is convex (see [9]). Also, in [4] and [10], the similar results were obtained in the variable exponent Lebesgue spaces and the weighted Lorentz spaces, respectively.

Note that the estimate (3.1) is sharper version of the estimates obtained in [7, 9, 4, 10]. So, Theorem 3.1 is an improvement of the theorems in [7, 9, 4, 10] in weighted Orlicz spaces, in the sense that the generating Young functions are not necessary to be convex.

Proof of Theorem 3.1. Let $m \in \mathbb{N}$, $h \leq 2^{-m-1}$ and $S_{2^{m+1}}$ be 2^{m+1}-th partial sum of Fourier series of the function $f \in L_{M,\omega}(\mathbf{T})$. By the definition of the quantity $D(f, \mu, h, M)$ and the properties of the norm we have

$$\begin{aligned} D(f, \mu, h, M) &= \left\| \int_{-\infty}^{\infty} (\sigma_h f)(x, u) d\mu(u) \right\|_{M,\omega} \\ &\leq \left\| \int_{-\infty}^{\infty} \left[(\sigma_h f)(x, u) - (\sigma_h S_{2^{m+1}} f)(x, u) \right] d\mu(u) \right\|_{M,\omega} + \\ &\quad + \left\| \int_{-\infty}^{\infty} (\sigma_h S_{2^{m+1}} f)(x, u) d\mu(u) \right\|_{M,\omega}. \end{aligned}$$

Considering (1.2), Lemma 2.1 and boundedness of the operator σ_h, we obtain

$$D(f, \mu, h, M) \leq \left\| \int_{-\infty}^{\infty} (\sigma_h S_{2^{m+1}})(x, u) d\mu(u) \right\|_{M,\omega} + cE_{2^{m+1}}(f)_{M,\omega}.$$

Without losing generality, we assume that the Fourier series of f is

$$\sum_{k=1}^{\infty} c_k(f) e^{ikx}.$$

Then, we get

$$\begin{aligned}
\int_{-\infty}^{\infty}\left(\sigma_h S_{2^{m+1}}\right)(x,u)d\mu(u) &= \int_{-\infty}^{\infty}\left(\frac{1}{2h}\int_{-h}^{h} S_{2^{m+1}}(x+tu)dt\right)d\mu(u) \\
&= \int_{-\infty}^{\infty}\left(\frac{1}{2h}\int_{-h}^{h}\sum_{k=1}^{2^{m+1}-1} c_k(f)e^{ik(x+tu)}dt\right)d\mu(u) \\
&= \int_{-\infty}^{\infty}\left(\frac{1}{2h}\sum_{k=1}^{2^{m+1}-1} c_k(f)e^{ikx}\int_{-h}^{h} e^{iktu}dt\right)d\mu(u) \\
&= \sum_{k=1}^{2^{m+1}-1} c_k(f)e^{ikx}\int_{-\infty}^{\infty}\frac{e^{ikhu}-e^{-ikhu}}{2ikhu}d\mu(u) \\
&= \sum_{k=1}^{2^{m+1}-1} c_k(f)e^{ikx}\hat{\mu}(kh). \qquad (3.2)
\end{aligned}$$

Therefore, we have

$$D\left(f,\mu,h,M\right)\leq\left\|\sum_{k=1}^{2^{m+1}-1} c_k(f)e^{ikx}\hat{\mu}(kh)\right\|_{M,\omega}+cE_{2^{m+1}}(f)_{M,\omega}.$$

From Theorem 2.2, we obtain

$$\begin{aligned}
\left\|\sum_{k=1}^{2^{m+1}-1} c_k(f)e^{ikx}\hat{\mu}(kh)\right\|_{M,\omega} &\leq c\left\|\left(\sum_{k=0}^{m}\left|\sum_{l=2^k}^{2^{k+1}-1} c_l(f)e^{ilx}\hat{\mu}(lh)\right|^2\right)^{1/2}\right\|_{M,\omega} \\
&:= c\left\|\left(\sum_{k=0}^{m}\Delta_{k,\mu}^2\right)^{1/2}\right\|_{M,\omega}.
\end{aligned}$$

Since we have (see [1])

$$\left\|\left(\sum_{k=0}^{m}\Delta_{k,\mu}^2\right)^{1/2}\right\|_{M,\omega}\leq C\left(\sum_{k=0}^{m}\left\|\Delta_{k,\mu}\right\|_{M,\omega}^{\gamma}\right)^{1/\gamma}$$

we get

$$\left\|\sum_{k=1}^{2^{m+1}-1} c_k(f)e^{ikx}\hat{\mu}(kh)\right\|_{M,\omega}\leq C\left(\sum_{k=0}^{m}\left\|\Delta_{k,\mu}\right\|_{M,\omega}^{\gamma}\right)^{1/\gamma}.$$

If we apply the Abel transform to $\Delta_{k,\mu}$, we obtain that

$$\begin{aligned}
\Delta_{k,\mu} &= \sum_{l=2^k}^{2^{k+1}-1}\left[S_l(f,x)-S_{2^{k+1}-1}(f,x)\right]\left[\hat{\mu}(lh)-\hat{\mu}((l+1)h)\right]+ \\
&\quad+\left[S_{2^{k+1}-1}(f,x)-S_{2^k-1}(f,x)\right]\hat{\mu}(2^k h).
\end{aligned}$$

From (1.2) it follows that

$$\begin{aligned}\|\Delta_{k,\mu}\|_{M,\omega} &\le \sum_{l=2^k}^{2^{k+1}-1} \left\|S_l(f,x) - S_{2^{k+1}-1}(f,x)\right\|_{M,\omega} |\hat{\mu}(lh) - \hat{\mu}((l+1)h)| \\ &\quad + \left\|S_{2^{k+1}-1}(f,x) - S_{2^k-1}(f,x)\right\|_{M,\omega} \left|\hat{\mu}(2^k h)\right| \\ &\le cE_{2^k-1}(f)_{M,\omega}\delta_{2^k,h}.\end{aligned}$$

Then

$$\left\|\sum_{k=1}^{2^{m+1}-1} c_k(f)e^{ikx}\hat{\mu}(kh)\right\|_{M,\omega} \le c\left(\sum_{k=0}^{m} E^{\gamma}_{2^k-1}(f)_{M,\omega}\delta^{\gamma}_{2^k,h}\right)^{1/\gamma}.$$

This completes the proof. □

Proof of Theorem 3.2. For $f \in L_{M,\omega}(\mathbf{T})$, from the properties of the norm and (1.2) it follows that

$$D\left(f,\mu_1,h,M\right) \le \left\|\int_{-\infty}^{\infty} \left(\sigma_h S_{2^{m+1}} f\right)(x,u)d\mu_1(u)\right\|_{M,\omega} + cE_{2^{m+1}}(f)_{M,\omega}.$$

Using (3.2), (1.1), Theorem 2.1 and the properties of the function $F(x) = \hat{\mu}_1(x)\left(\hat{\mu}_2(x)\right)^{-1}$, we obtain

$$\begin{aligned}\left\|\int_{-\infty}^{\infty} \left(\sigma_h S_{2^{m+1}}\right)(x)d\mu_1(u)\right\|_{M,\omega} &= \left\|\sum_{r=1}^{2^{m+1}-1} c_r(f)e^{irx}\hat{\mu}_2(rh)F(rh)\right\|_{M,\omega} \preceq \\ &\preceq \left\|\sum_{r=1}^{2^{m+1}-1} c_r(f)e^{irx}\hat{\mu}_2(rh)\right\|_{M,\omega} = \\ &= \left\|\int_{-\infty}^{\infty} \left(\sigma_h S_{2^{m+1}}(f)\right)(x)d\mu_2(u)\right\|_{M,\omega} \le \\ &\le \left\|\int_{-\infty}^{\infty} f(x)d\mu_2(u)\right\|_{M,\omega}.\end{aligned}$$

This completes the proof. □

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Ali Doğu
Balikesir University, Istitute of Science, Department of Mathematics, 10145, Balikesir, Turkey.
E-mail address: `dogualii19831227@gmail.com`

Ahmet Hamdi Avşar
Balikesir University, Faculty of Education, Department of Mathematics, 10145, Balikesir, Turkey.
E-mail address: `ahmet.avsar@balikesir.edu.tr`

Yunus Emre Yildirir
Balikesir University, Faculty of Education, Department of Mathematics, 10145, Balikesir, Turkey.
E-mail address: `yildirir@balikesir.edu.tr`

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