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SPECTRAL PROPERTIES OF THE PROBLEM OF VIBRATION OF A LOADED STRING IN MORREY TYPE SPACES

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Abstract. In this paper we study the spectral problem for a discontinuous second order differential operator with a spectral parameter in transmission conditions, that arises in solving the problem of vibration of a loaded string with fixed ends. An abstract theorem on the stability of basis properties of multiple systems in a Banach space with respect to certain transformations is proved. This fact is used in the proofs of theorems on the basicity of eigenfunctions of a discontinuous differential operator in Morrey type spaces.

1. Introduction

We consider a model eigenvalue problem for the discontinuous second order differential operator

$$-y''(x) = \lambda y(x)\ , \qquad x \in (0,\frac{1}{3}) \cup (\frac{1}{3},1), \tag{1.1}$$

with the boundary conditions

$$y\,(0) = y\,(1) = 0, \tag{1.2}$$

and with the following discontinuity conditions

$$\left.\begin{array}{l} y(\frac{1}{3}-0) = y(\frac{1}{3}+0), \\ y'(\frac{1}{3}-0) - y'(\frac{1}{3}+0) = \lambda m y(\frac{1}{3}), \end{array}\right\} \tag{1.3}$$

where λ is a spectral parameter, m is a non-zero complex number. Such spectral problems arise when the problem of vibrations of a loaded string with fixed ends is solved by applying the Fourier method [16, 1]. The spectral problems with a discontinuity conditions inside the interval play an important role in mathematics, mechanics, physics and other fields of science. The applications of boundary value problems with discontinuity conditions inside the interval are connected with discontinuous material properties. Nowadays there is a number of papers dedicated to spectral problems for the ordinary differential operator with discontinuity conditions. One can find the similar works in [13, 12, 14, 10, 15, 11].

One of the commonly used methods for solving partial differential equations is the method of separation of variables. This method yields the appropriate spectral problem and in order to justify the method, it is very important the

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question of the expansion of functions of certain class on eigen and association functions of the discrete differential operators. The study of spectral properties of some discrete differential operators motivates the development of new methods for constructing bases. In this context, much attention has been given to the study of basis properties (completeness, minimality and basicity) of systems of special functions, which are frequently eigen and associated functions of differential operators. In the case of discontinuous differential operators, there appear systems of eigenfunctions whose basicity cannot be investigated by previously known methods. In the works [4, 5], an abstract approach to the problem described above is considered and a new method is proposed for constructing bases, which has wide applications in the spectral theory of discontinuous differential operators.

In the paper [8] the problem of oscillation of a loaded string is investigated in the case when the load is placed in the middle of the string and it is shown that an abstract method proposed in [4, 5] can be used in non-standard spaces such as a Morrey type space. The concept of Morrey space was introduced by C. Morrey in 1938. Since then, various problems related to this space have been intensively studied. More details about Morrey spaces can be found in [17]. In [6] the basicity of the exponential system, and in [7]-the perturbed exponential system in Morrey type spaces are proved. The present paper is an extension of the method of [4, 5, 8].

2. Necessary information

Let us give some results from [11], which we will need throughout the paper.

Lemma 2.1. [11] *The spectral problem* (1.1)-(1.3) *has two series of eigenvalues:* $\lambda_{1,n} = (\rho_{1,n})^2$, $n = 1, 2, ...,$ $\lambda_{2,n} = (\rho_{2,n})^2$ $n = 0, 1, 2, ...,$ *where*

$$\left.\begin{array}{l} \rho_{1,n} = 3\pi n, \\ \rho_{2,n} = \frac{3\pi n}{2} + \frac{2+(-1)^n}{\pi m n} + O\left(\frac{1}{n^2}\right). \end{array}\right\} \tag{2.1}$$

The corresponding eigenfunctions are given by the following expressions

$$y_{1,n}(x) = \sin 3\pi n x, x \in [0,1], \qquad n = 1, 2, ..., \tag{2.2}$$

$$y_{2,n}(x) = \left\{ \begin{array}{l} \sin \rho_{2,n}\left(x - \frac{1}{3}\right) + \sin \rho_{2,n}\left(x + \frac{1}{3}\right), \ x \in \left[0, \frac{1}{3}\right], \\ \sin \rho_{2,n}(1 - x), \ x \in \left[\frac{1}{3}, 1\right], \qquad n = 0, 1, 2, ... \end{array} \right. . \tag{2.3}$$

Let us construct an operator L, linearizing the problem (1.1) - (1.3) in the direct sum $L_p(0,\ 1) \oplus C$, where C is the complex plane. Denote by $W_p^2\left(0, \frac{1}{3}\right) \oplus W_p^2\left(\frac{1}{3}, 1\right)$ the space of functions whose restrictions to intervals $\left(0,\ \frac{1}{3}\right)$ and $\left(\frac{1}{3},\ 1\right)$ belong to Sobolev spaces $W_p^2\left(0, \frac{1}{3}\right)$ and $W_p^2\left(\frac{1}{3}, 1\right)$, respectively, where $1 < p < \infty$. Let us define the operator L in the following way. As the domain $D(L)$ we take the manifold

$$\begin{array}{r} D(L) = \left\{\hat{y} = \left(y(x), my\left(\frac{1}{3}\right)\right) : y(x) \in W_p^2\left(0, \frac{1}{3}\right) \oplus W_p^2\left(\frac{1}{3}, 1\right), \right. \\ \left. y(0) = y(1) = 0, \quad y\left(\frac{1}{3} - 0\right) = y\left(\frac{1}{3} + 0\right) \right\}, \end{array} \tag{2.4}$$

and for $\hat{y} \in D(L)$ the operator L is defined by the relation

$$L\hat{y} = \left(-y'';\ y'\left(\frac{1}{3} - 0\right) - y'\left(\frac{1}{3} + 0\right)\right). \tag{2.5}$$

The following lemma is true.

Lemma 2.2. [11] *The operator L defined by expressions* (2.4), (2.5) *is a densely defined closed operator with completely continuous resolvent. The eigenvalues of the operator L and the problem* (1.1) - (1.3) *coincide. If $y(x)$ is the eigenfunction (associated function) of problem* (1.1) - (1.3), *then $\hat{y} = \left(y(x);\ my\left(\frac{1}{3}\right)\right)$ is the eigenvector (associated vector) of the operator L.*

When obtaining the main results, we need some concepts and facts from the theory of bases in a Banach space.

Definition 2.1. Let X be a Banach space. If there exists a sequence of positive integers $\{n_k\}$, such that $n_k < n_{k+1}, n_0 = 0$ and any vector $x \in X$ is uniquely represented in the form

$$x = \sum_{k=0}^{\infty} \sum_{i=n_k+1}^{n_{k+1}} c_i u_i$$

then the system $\{u_n\}_{n\in N} \in X$ is called a basis with parentheses in X.

For $n_k = k$ the system $\{u_n\}_{n\in N}$ forms a usual basis for X.

We need the following easily proved statement.

Statement 2.1. Let the system $\{u_n\}_{n\in N}$ form a basis with parentheses for a Banach space X. If the sequence $\{n_{k+1} - n_k\}_{k\in N}$ is bounded and the condition

$$\sup_n \|u_n\| \, \|\vartheta_n\| < \infty,$$

holds, where $\{\vartheta_n\}_{n\in N}$ is a biorthogonal system, then the system $\{u_n\}_{n\in N}$ forms a usual basis for X.

Recall the following definition.

Definition 2.2. The bases $\{u_n\}_{n\in N}$ of Banach space X is called a p-basis, if for any $x \in X$

$$\left(\sum_{n=1}^{\infty} |\langle x, \vartheta_n\rangle|^p\right)^{\frac{1}{p}} \le M \|x\|,$$

where $\{\vartheta_n\}_{n\in N}$ is a biorthogonal system to $\{u_n\}_{n\in N}$.

Definition 2.3. The sequences $\{u_n\}_{n\in N}$ and $\{\varphi_n\}_{n\in N}$ of Banach space X is called a p-close, if

$$\sum_{n=1}^{\infty} \|u_n - \varphi_n\|^p < \infty.$$

We will also use the following results from [3, 9].

Theorem 2.1. [3] *Let $\{x_n\}_{n\in N}$ form a q-basis for the space X, and the system $\{y_n\}_{n\in N}$ is p- close to $\{x_n\}_{n\in N}$, where $\frac{1}{p} + \frac{1}{q} = 1$. Then the following properties are equivalent:*

i) $\{y_n\}_{n\in N}$ is complete in X;

ii) $\{y_n\}_{n\in N}$ is minimal in X;

iii) $\{y_n\}_{n\in N}$ forms an isomorphic to $\{x_n\}_{n\in N}$ for X.

Let $X_1 = X \oplus C^m$ and $\{\hat{u}_n\}_{n\in N} \subset X_1$ be some minimal system, and $\left\{\hat{\vartheta}_n\right\}_{n\in N} \subset X_1^* = X^* \oplus C^m$ be its biorthogonal system:

$$\hat{u}_n = (u_n; \alpha_{n1}, ..., \alpha_{nm}); \quad \hat{\vartheta}_n = (\vartheta_n; \beta_{n1}, ..., \beta_{nm}).$$

Let $J = \{n_1, ..., n_m\}$ be some set of m natural numbers. Suppose

$$\delta = \det \|\beta_{n_i j}\|_{i,j=\overline{1,m}}.$$

The following theorem is true.

Theorem 2.2. [9] *Let the system $\{\hat{u}_n\}_{n\in N}$ form a basis for X_1. In order for the system $\{u_n\}_{n\in N_J}$, where $N_J = N\backslash J$, form a basis for X it is necessary and sufficient that the condition $\delta \neq 0$ be satisfied. In this case the biorthogonal system to $\{u_n\}_{n\in N_J}$ is defined by*

$$\vartheta_n^* = \frac{1}{\delta}\begin{vmatrix} \vartheta_n & \vartheta_{n1} & \dots & \vartheta_{nm} \\ \beta_{n1} & \beta_{n_1 1} & \dots & \beta_{n_m 1} \\ \dots & \dots & \dots & \dots \\ \beta_{nm} & \beta_{n_1 m} & \dots & \beta_{n_m m} \end{vmatrix}.$$

For $\delta = 0$ the system $\{u_n\}_{n\in N_J}$ is not complete and is not minimal in X.

We will also need some facts about the theory of Morrey-type spaces. Let $J = [a, b]$ be some finite segment of real axis. By $|I|$ we denote the linear Lebesgue measure of the set $I \subset J$. By the Morrey- Lebesgue space $L^{p,\alpha}(J)$, $0 \leq \alpha \leq 1$, $p \geq 1$, we mean a normed space of all functions $f(\cdot)$ measurable on J equipped with a finite norm $\|f\|_{L^{p,\alpha}(J)}$:

$$\|f\|_{L^{p,\alpha}(J)} = \sup_{I\subset J}\left(|I|^{\alpha-1}\int_I |f(\xi)|^p |d\xi|\right)^{\frac{1}{p}} < +\infty.$$

$L^{p,\alpha}(J)$ is a Banach space and $L^{p,1}(J) = L_p(J)$, $L^{p,0}(J) = L_\infty(J)$. The embedding $L^{p,\alpha_1}(J) \subset L^{p,\alpha_2}(J)$ is valid for $0 \leq \alpha_1 \leq \alpha_2 \leq 1$. Thus $L^{p,\alpha}(J) \subset L_p(J)$, $\forall\alpha \in [0,1]$, $\forall p \geq 1$.

Denote by $\tilde{L}^{p,\alpha}(J)$ linear subspace of $L^{p,\alpha}(J)$ consisting of functions whose shifts are continuous in $L^{p,\alpha}(J)$, i.e. $\|f(\cdot+\delta) - f(\cdot)\|_{L^{p,\alpha}(J)} \to 0$ as $\delta \to 0$. The closure of $\tilde{L}^{p,\alpha}(J)$ in $L^{p,\alpha}(J)$ will be denoted by $M^{p,\alpha}(J)$ In [6] the following theorem is proved.

Theorem 2.3. *The exponential system $\left\{e^{i\,nt}\right\}_{n\in Z}$ is the bases in $M^{p,\alpha}(-\pi,\pi)$, $1 < p < +\infty$, $0 < \alpha \leq 1$.*

Using this theorem, it is easy to obtain the following

Statement 2.2. Each of the trigonometric systems $\{\sin nx\}_{n=1}^{\infty}$ and $\{\cos nx\}_{n=0}^{\infty}$ forms a bases for $M^{p,\alpha}(0,\pi)$, $1 < p < +\infty$, $0 < \alpha \leq 1$.

3. Main results

In this section we consider the question of the basicity of the system of eigenvectors of the operator$L$ and eigenfunctions of the problem (1.1)-(1.3) in the spaces $M^{p,\alpha}(0,1) \oplus C$ and $M^{p,\alpha}(0,1)$. Before formulating the main result, we prove one abstract theorem, which is essentially used in the proof of the main theorem.

Let X be a Banach space and $\{u_{kn}\}_{k=\overline{1,m};n\in N}$ be some system in X. Let $a_{ik}^{(n)}$, $i,k=\overline{1,m}$, $n\in N$, be some complex number. Let

$$A_n=\left(a_{ik}^{(n)}\right)_{i,k=\overline{1,m}} \quad \text{and} \quad \Delta_n=\det A_n, \quad n\in N.$$

Let us consider the following system in space X

$$\hat{u}_{kn}=\sum_{i=1}^{m} a_{ik}^{(n)}u_{in}, \quad k=\overline{1,m}; n\in N. \tag{3.1}$$

Theorem 3.1. *If the system $\{u_{kn}\}_{k=\overline{1,m};n\in N}$ forms a basis for X and*

$$\Delta_n\neq 0,\ \forall n\in N, \tag{3.2}$$

then the system $\{\hat{u}_{kn}\}_{k=\overline{1,m};n\in N}$ forms a basis with parentheses for X. If in addition the conditions

$$\sup_n\left\{\ \|A_n\|,\left\|A_n^{-1}\right\|\ \right\}<\infty, \qquad \sup_n\left\{\|u_{kn}\|,\|\vartheta_{kn}\|\right\}<\infty, \tag{3.3}$$

holds, where $\{\vartheta_{kn}\}_{k=\overline{1,m};n\in N}\subset X^*$ is a biorthogonal system to $\{u_{kn}\}_{k=\overline{1,m};n\in N}$, then the system $\{\hat{u}_{kn}\}_{k=\overline{1,m};n\in N}$ forms a usual basis for X.

Proof. From the representation (3.1) and from the minimality of the system $\{u_{kn}\}_{k=\overline{1,m};n\in N}$, it follows the minimality of the system $\{\hat{u}_{kn}\}_{k=\overline{1,m};n\in N}$ and the biorthogonal system has the form

$$\hat{\vartheta}_{in}=\sum_{l=1}^{m} b_{li}^{(n)}\vartheta_{\ln}\ ,\ i=\overline{1,m}:\ n\in N, \tag{3.4}$$

where the numbers $b_{li}^{(n)}$ are elements of the inverse matrix $\left(A_n^{-1}\right)^*$. Taking these expressions into account, for $x\in X$ we have

$$\begin{aligned}&\textstyle\sum_{i=1}^m\left\langle x,\hat{\vartheta}_{in}\right\rangle\hat{u}_{in}=\sum_{i=1}^m\sum_{j=1}^m\sum_{l=1}^m a_{ij}^{(n)}b_{li}^{(n)}\left\langle x,\vartheta_{\ln}\right\rangle u_{jn}=\\&=\textstyle\sum_{j=1}^m\sum_{l=1}^m\left(\sum_{i=1}^m b_{li}^{(n)}a_{ij}^{(n)}\right)\left\langle x,\vartheta_{\ln}\right\rangle u_{jn}=\\&=\textstyle\sum_{j=1}^m\sum_{l=1}^m\delta_{lj}\left\langle x,\vartheta_{\ln}\right\rangle u_{jn}=\sum_{j=1}^m\left\langle x,\vartheta_{jn}\right\rangle u_{jn}.\end{aligned}$$

Consequently

$$S_N(x)=\sum_{n=1}^{N}\sum_{i=1}^{m}\left\langle x,\hat{\vartheta}_{in}\right\rangle\hat{u}_{in}=\sum_{n=1}^{N}\sum_{j=1}^{m}\left\langle x,\vartheta_{jn}\right\rangle u_{jn}=$$

$$=\sum_{j=1}^{m}\sum_{n=1}^{N}\left\langle x,\vartheta_{jn}\right\rangle u_{jn}\to x, \quad \text{as} \quad N\to\infty.$$

Thus, the system $\{\hat{u}_{in}\}_{i=\overline{1,m};\,n\in N}$ forms a basis with parentheses for X.

Now let the conditions (3.3) be satisfied. Then from the representations (3.1) and (3.4) we obtain

$$\sup_{i,n}\left\{\|\hat{u}_{in}\|\ ;\ \left\|\hat{\vartheta}_{in}\right\|\right\}<+\infty.$$

Consequently, the system $\{\hat{u}_{in}\}_{i=\overline{1,m};\, n\in N}$ is uniformly minimal, and by Statement 2.1 it forms the usual basis for X. The theorem is proved. $\square$

The main result of the paper is the following

Theorem 3.2. *The system of eigen and associated vectors of the operator L forms a bases for space $M^{p,\alpha}(0,1)\oplus C,\ 1<p<\infty\ ,\ 0<\alpha\leq 1$.*

Now, let us consider the basicity of the system $\{y_0\}\cup\{y_{i,n}\}_{i=1,2;\, n\in N}^{\infty}$ with a remote function in space $M^{p,\alpha}(0,1)$. Using the Theorem 2.2 following theorem can be proved.

Theorem 3.3. *If from the system of eigen and associated functions of problem (1.1) - (1.3) $\{y_0\}\cup\{y_{i,n}\}_{i=1,2;\, n\in N}^{\infty}$ we eliminate any function $y_{2,n_0}(x)$, corresponding to a simple eigenvalue, then the obtaining system forms a basis for $M^{p,\alpha}(0,1)$, $1<p<\infty,\ 0<\alpha\leq 1$. And if we eliminate any function $y_{1,n_0}(x)$ from this system, then the obtaining system does not form a basis in $M^{p,\alpha}(0,1)$; moreover, in this case the obtained system is not complete and is not minimal in this space.*

Remark 3.1. For $m>0$, the linearizing operator L of the problem (1.1) - (1.3) is a self-adjoint operator in $L_2\oplus C$, and in this case all eigenvalues are real and simple, and to each eigenvalue there corresponds only one eigenvector. If $m<0$, then the operator L is a J-self-adjoint operator in $L_2\oplus C$ and in this case, applying the results of [2], we obtain that all eigenvalues are real and simple, with the exception of, may be either one pair of complex conjugate simple eigenvalues or one non-simple real value. In the case of a complex value m the operator L has an infinite number of complex eigenvalues that are asymptotically simple and, consequently, the operator L can have a finite number of associated vectors. If there are associated vectors, they are determined up to a linear combination with the corresponding eigenvector, and in this case there always exists an associated vector for which $z_{2,n}\left(\frac{1}{3}\right)=0$, as well as an associated vector for which $z_{2,n}\left(\frac{1}{3}\right)\neq 0$.

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