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SOLVABILITY OF SINGULAR MULTI-POINT BOUNDARY VALUE PROBLEMS

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Abstract. The aim of this paper is the study of a multipoint boundary value problem for second order differential equations in both regular and singular cases. The main tools are upper and lower solutions method and Schauder's fixed point theorem.

1. Introduction

The study of nonlinear boundary value problems is an important and difficult area of research in differential equations. To obtain the existence results for nonlinear boundary value problems, there exists a variety of techniques such as Mawhin theory for the resonance case and fixed point theorems for the non resonance case. Another powerful tool for proving existence of solutions is the method of upper and lower solutions. The main idea behind this method is to modify the given problem, prove existence results for the modified problem, and then establish the existence of solutions for the given problem. It is to be pointed here that by using this method we prove not only the existence of solution but also give its location between what is called the lower and the upper solutions. This fact makes the mentioned method a strong tool in nonlinear analysis. This method was introduced for the first time by Picard in 1893 and has been developed in 1931 by Dragoni [3] for a Dirichlet problem. Later a large number of works were devoted to this theory, thus, we will refer here to the study of first and second order differential equations with various forms of function f subject to different type of boundary conditions like Neumann, periodic or Dirichlet conditions.

Multipoint boundary value problems for second order ordinary differential equations with nonlinearity depending on the first derivative have been investigated by many authors. In most existing papers, the nonlinear term took one of the following forms:

$$u''(t) = \mu f\left(t, u(t), u'(t)\right),$$

$$u''(t) = f\left(t, u(t), u'(t)\right) + e(t),$$

$$u''(t) = q(t) f\left(t, u(t), u'(t)\right).$$

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Ntouyas and Tsamatos [10] considered the first form without any singularity of the function f and proved existence results for the boundary value problem

$$u''(t) = \mu f\left(t, u(t), u'(t)\right), \quad 0 < t < 1,$$

$$u(0) = 0, \quad u(1) = \sum_{k=1}^{m} \lambda_k u(\eta_k),$$

where $f : [0,1] \times (\mathbb{R}^n)^2 \to \mathbb{R}^n$ is L^1-Carathéodory, $0 < \eta_k < 1$, $\lambda_k \in \mathbb{R}$, $k = 1,..,m$, $\sum_{k=1}^{m} \lambda_k \eta_k \neq 1$. Besides, $\mu \in (0, \mu_0]$ with a suitable constant μ_0, under the hypothesis

$$|f(t,u,v)| \leq M_1(t)\,\zeta_1(|u|) + M_2(t)\,\zeta_2(|v|),$$

where ζ_1 and ζ_2 are nondecreasing functions and $M_1,\ M_2 \in L^1([0,1], \mathbb{R})$.

Przeradzki and Stańczy [11] studied the first form for $\mu = 1$ subject to boundary conditions considered in [10], where $f : [0,1] \times (\mathbb{R}^n)^2 \to \mathbb{R}^n$ is continuous, $\lambda_k \in (0, \infty)$, $0 < \eta_k < 1$, $k = 1,..,m$, $0 < \eta_k < ..... < \eta_m < 1$, with $\sum_{k=1}^{m} \lambda_k \eta_k = 1$ (in this case the problem is at resonance). They applied Mawhin continuation theorem to prove the existence results for both scalar and multidimensional versions.

The second form has been studied by several authors with different multi-point boundary conditions under different conditions on f, where f can be continuous, Carathéodory or L^1-Carathéodory and the function $e(t)$ belongs to $L^1[0,1]$. For more details see [4, 8]. For the last form we refer the reader to [14].

In this paper, we will use the upper and lower solutions method to prove the existence of at least one solution for the boundary value problem BVP:

$$u''(t) + f\left(t, u(t), u'(t)\right) = 0, \quad 0 < t < 1, \tag{1.1}$$

$$u(0) = 0, \quad u(1) = \sum_{k=1}^{m} \lambda_k u(\eta_k), \tag{1.2}$$

where f is a continuous function that can be regular or singular at the points $t = 0$, $t = 1$, $u = 0$ and $0 < \eta_k < 1$, $\lambda_k > 0$ $(k = 1,..,m)$, $\sum_{k=1}^{m} \lambda_k \eta_k \neq 1$, (in this case the boundary value problem (1.1)-(1.2) is not at resonance)

Differential equation (1.1) associated with periodic, Dirichlet, nonlocal and other boundary conditions has been investigated in many papers. We cite here especially those which used the lower and upper solutions method.

Jiang [7] studied (1.1) with the boundary conditions $u(0) = u(1) = 0$. Assuming that the singularity may appear at $t = 0$, $t = 1$ and $u = 0$, and that the function f may be superlinear at $u = \infty$ and changes sign, he proved the existence results by upper and lower solutions method.

Jia and Liu [6] considered (1.1) with $u'(0) = a$, $u(1) = b$, as boundary conditions, where $f \in C\left([0,1] \times \mathbb{R}^2, \mathbb{R}\right)$ and $a, b \in \mathbb{R}$. They obtained the existence and the uniqueness of solution via upper and lower solutions method and by considering a special cone.

Other authors combined the method discussed above by means of other techniques. Minghe and Chang [9] used the upper and lower solutions method together with Leray-Schauder degree for (1.1), for the three point boundary conditions mentioned below:

$$\psi\left(u(0), u'(0)\right) = 0,\ \psi^*\left(u(1), u'(1)\right) = g(u(\eta)),$$

where ψ and ψ^* are two continuous functions. Then they applied the quasi-linearization method for the same differential equation (1.1) with the following boundary conditions

$$au(0) - bu'(0) = c,\ u(1) = g(u(\eta)),$$

where $a \geq 0$, $b \geq 0$, $a + b > 0$, $c \in \mathbb{R}$.

Let us remark that in the literature most papers have focused on regular problems and only few of them have treated singular ones, when f depends on the first derivative. Motivated by this fact, we will focus on a singular one in the present work.

It is to be mentioned here that the greatest step in the study of boundary value problem with singularity was taken by Habets and Zanolin [5] for a generalized Emden-Fowler equation:

$$u''(t) + f(t, u(t)) = 0, \quad 0 < t < 1,$$
$$u(0) = u(1) = 0,$$

where the singularity of the function f are at $t = 0$, $t = 1$ and $u = 0$.

Subsequently, De Coster and Habets [2] have pointed out more conditions that have been assumed by Habets and Zanolin [5]. For papers dealing with singularity via upper and lower solutions method, one may consult [7, 12, 13, 14].

This paper is organized as follows. In section 2, we define the upper and lower solutions and give a lemma that is needed later. In section 3, we present an existence theorem for the regular case. Finally, in section 4, we establish two existence theorems in the singular cases, the first one is for the singularity at $t = 0$, $t = 1$ while the second one for the case $t = 0$, $t = 1$ and $u = 0$. Finally, we give an example to illustrate the obtained results.

2. Preliminaries

We first present some useful definitions and a useful lemma.

Definition 2.1. (Lower solution) [1]. A function $\alpha \in C^1[0,1] \cap C^2(0,1)$ is a lower solution of the BVP (1.1)-*(1.2)* if:

(a) $\alpha(0) \leq 0$, $\alpha(1) \leq \sum\limits_{k=1}^{m} \lambda_k \alpha(\eta_k)$,

(b)$\alpha''(t) + f(t, \alpha(t), \alpha'(t)) \geq 0$, for all $t \in (0,1)$

Definition 2.2. (Upper solution) [1]. A function $\beta \in C^1[0,1] \cap C^2(0,1)$ is an upper solution of the BVP (1.1)-*(1.2)* if:

(a) $\beta(0) \geq 0$, $\beta(1) \geq \sum\limits_{k=1}^{m} \lambda_k \beta(\eta_k)$,

(b) $\beta''(t) + f(t, \beta(t), \beta'(t)) \leq 0$, for all $t \in (0,1)$.

Now, we look for the solution of the given problem.

Lemma 2.1. *Let $y \in C([0,1],\mathbb{R})$, then the solution of the following linear boundary value problem*

$$\begin{cases} u''(t) = -y(t), & 0 < t < 1, \\ u(0) = 0, \quad u(1) = \sum\limits_{k=1}^{m} \lambda_k u(\eta_k), \end{cases}$$

is given by

$$\begin{aligned} u(t) &= \omega t \int_0^1 (b-s)\, y(s)\, ds - \int_0^t (t-s)\, y(s)\, ds \\ &\quad - \omega t \sum_{k=1}^{m} \lambda_k \int_0^{\eta_k} (\eta_k - s)\, y(s)\, ds, \end{aligned}$$

where $\omega = \frac{1}{1-\sum\limits_{k=1}^{m} \lambda_k \eta_k}$.

Proof. It's easy to verify the solution by remarking that $\sum\limits_{k=1}^{m} \lambda_k \eta_k \neq 1$. □

Throughout this paper, we denote the norm in $C^1([0,1],\mathbb{R})$ by $\|u\|_1 = \max\{\|u\|, \|u'\|\}$, where $\|u\| = \max\limits_{t\in[0,1]} |u(t)|$.

3. Regular case

In this section, we give an existence result for the regular case for the boundary value problem (1.1)-(1.2).

Definition 3.1. (Nagumo condition) Let α, β be the lower and upper solutions of BVP (1.1)-(1.2) such that $\alpha(t) \leq \beta(t)$, $\forall t \in [0,1]$. Consider the set

$$D = \left\{(t,x,y) \in [0,1] \times \mathbb{R}^2 / \alpha(t) \leq x \leq \beta(t)\right\}.$$

A function $f \in C\left([0,1] \times \mathbb{R}^2, \mathbb{R}\right)$ is said to satisfy Nagumo condition on D if there exists a function $H \in C\left(\mathbb{R}^+, \mathbb{R}_0^+\right)$ such that

$$|f(t,x,y)| \leq H(|y|), \ \forall (t,x,y) \in D \text{ and } \ |y| \geq a,$$

and

$$\int_a^{+\infty} \frac{s\,ds}{H(s)} > \max_t \beta(t) - \min_t \alpha(t),$$

where

$$a = \max\{|\alpha(1)|, |\beta(1)|\}.$$

To guarantee the existence of solution of (1.1)-(1.2) we have to find a priori bounds for the derivative of solution. Hence, we need the following lemma.

Lemma 3.1. *[1] Let α, β be a lower and upper solutions for boundary value problem (1.1)-(1.2) such that $\alpha \leq \beta$ and let $f : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ be a continuous function satisfying Nagumo condition on the set D. Then there exists $b > 0$, such that for every solution u of (1.1)-(1.2) with $\alpha(t) \leq u(t) \leq \beta(t)$, $\forall t \in [0,1]$, we have*

$$\|u'\| \leq b.$$

Theorem 3.1. *Let α, β be lower and upper solutions for the boundary value problem (1.1)-(1.2) such that $\alpha(t) \leq \beta(t)$ for every $t \in [0,1]$. Let $f : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ be a continuous function satisfying Nagumo condition on the set D and $\sum_{k=1}^{m} \lambda_k \leq 1$. Then the boundary value problem (1.1)-(1.2) has at least one solution $u \in C^2[0,1]$ such that*

$$\alpha(t) \leq u(t) \leq \beta(t), \forall t \in [0,1].$$

Proof. First, let us introduce the modified problem as follows:

$$u''(t) + F\big(t, u(t), u'(t)\big) = 0, \quad 0 < t < 1, \tag{3.1}$$

subject to the boundary conditions (1.2), where

$$f^*(t,x,y) = \begin{cases} f(t,x,-b), & \text{if } y < -b, \\ f(t,x,y), & \text{if } -b \leq y \leq b, \\ f(t,x,b), & \text{if } y > b, \end{cases}$$

and

$$F(t,x,y) = \begin{cases} f^*(t,\alpha(t),y) + \frac{\alpha(t)-x}{\alpha(t)-x+1}, & \text{if } x < \alpha(t), \\ f^*(t,x,y), & \text{if } \alpha(t) \leq x \leq \beta(t), \\ f^*(t,\beta(t),y) + \frac{\beta(t)-x}{x-\beta(t)+1}, & \text{if } x > \beta(t), \end{cases}$$

where b is chosen equal to the one in Lemma 3.1 or large enough.

We will show that the solutions of the modified problem (3.1)-(1.2) lie in a region where f is unmodified i.e. $\alpha(t) \leq u(t) \leq \beta(t)$, and $-b \leq u'(t) \leq b, \forall t \in [0,1]$ and hence they are solutions of the problem (1.1)-(1.2). The proof will be done in two steps.

Step 1: Existence of solution.

Define the operator $T : C^1[0,1] \to C^1[0,1]$ by

$$\begin{aligned} Tu(t) &= \omega t \int_0^1 (1-s)\ F(s,u(s),u'(s))ds \\ &\quad - \int_0^t (t-s)\, F(s,u(s),u'(s))ds \end{aligned}$$

$$-\omega t \sum_{k=1}^{m} \lambda_k \int_0^{\eta_k} (\eta_k - s)\ F(s,u(s),u'(s))ds, \quad t \in [0,1].$$

It follows from the definition of F that $F(t,x,y)$ is continuous and $|F(t,x,y)| \leq M$ on $[0,1] \times \mathbb{R}^2$, $M = M_0 + 1$ where

$$M_0 = \max\{|f(t,x,y)| : t \in [0,1], \alpha(t) \leq x \leq \beta(t), |y| \leq b\}.$$

We have

$$|Tu(t)| \leq (2\omega + 1) M,$$

and

$$\left|(Tu)'(t)\right| \leq (2\omega + 1) M.$$

Consequently, T maps the closed, bounded and convex set

$$B = \left\{u \in C^1[0,1] : ||u||_1 \leq (2\omega + 1) M\right\}$$

into itself. Furthermore, for every $u \in C^1[0,1]$ and $t_1, t_2 \in [0,1]$, $t_1 < t_2$, we have

$$|(Tu)(t_1) - (Tu)(t_2)| \leq (t_2 - t_1)(2\omega + 1) M,$$

and

$$\left|(Tu)'(t_1) - (Tu)'(t_2)\right| \leq (t_2 - t_1)\, M,$$

therefore $|(Tu)(t_1) - (Tu)(t_2)| \longrightarrow 0$ and $|(Tu)'(t_1) - (Tu)'(t_2)| \longrightarrow 0$ when $t_1 \longrightarrow t_2$. This shows that T is equicontinuous. Consequently, T maps bounded sets into relatively compact sets. In addition T is continuous via dominated convergence theorem. Therefore, the map T is completely continuous. Using Schauder's Theorem, we conclude that T has a fixed point u^* in $C^1[0,1]$, that is a solution for the BVP (3.1)-(1.2), since $u^* = Tu^*$, then u^* belongs to $C^2[0,1]$.

Step 2: Localization of solution. Let us prove that if u is a solution of problem (3.1)-(1.2), then it satisfies

$$\alpha(t) \leq u(t) \leq \beta(t), \forall t \in [0,1].$$

Assume the contrary. Assume there exists $t_0 \in [0,1]$ such that

$$\min_{t\in[0,1]} (u(t) - \alpha(t)) = u(t_0) - \alpha(t_0) < 0,$$

We have the following cases:

Case 1: If $t_0 \in (0,1)$, we get

$$\begin{aligned} 0 &\leq u''(t_0) - \alpha''(t_0) \\ &= -f\left(t_0, \alpha(t_0), \alpha'(t_0)\right) - \frac{\alpha(t_0) - u(t_0)}{\alpha(t_0) - u(t_0) + 1} + f\left(t_0, \alpha(t_0), \alpha'(t_0)\right) < 0, \end{aligned}$$

which is a contradiction. Thus, the minimum of $u - \alpha$ is not achieved at the point t_0.

Case 2: If $t_0 = 0$, we obtain

$$u(0) - \alpha(0) < 0.$$

On the other hand, since u is a solution, $u(0) = 0$ and consequently, $\alpha(0) > 0$. This contradicts the fact that α is a lower solution.

Case 3: If $t_0 = 1$, and since $0 < \eta_k < 1$, for all $k \in \{1, ..., m\}$, then taking the case 1 into account we get

$$u(\eta_k) - \alpha(\eta_k) > \min_{t\in[0,1]} (u(t) - \alpha(t)) = u(1) - \alpha(1),$$

hence

$$u(1) - \alpha(1) \geq \sum_{k=1}^{m} \lambda_k (u(\eta_k) - \alpha(\eta_k)) > \sum_{k=1}^{m} \lambda_k (u(1) - \alpha(1)) \geq u(1) - \alpha(1),$$

which leads to a contradiction. Similarly we prove that $u(t) \leq \beta(t), \forall t \in [0,1]$.

To complete the proof we apply Lemma 3.1 to F, then it yields $\|u'\| \leq b$. □

4. Singular case

Now we give an existence theorem for a general singular problem BVPS in the case where f has singularity at $t = 0$ and $t = 1$:

$$u''(t) + f\left(t, u(t), u'(t)\right) = 0, \quad 0 < t < 1,$$

$$u(0) = c, \quad u(1) = \sum_{k=1}^{m} \lambda_k u(\eta_k),$$

where $c \in \mathbb{R}$. Let us remark that BVP (1.1)-(1.2) is a particular case of BVPS for $c = 0$. Now, by the same way we define the upper and lower solution for the problem BVPS.

A function $\alpha \in C^1[0,1] \cap C^2(0,1)$ is a lower solution (respectively β is upper solution) of the BVPS if:

(a) $\alpha(0) \le c, \alpha(1) \le \sum\limits_{k=1}^{m} \lambda_k \alpha(\eta_k)$, (respectively $\beta(0) \ge c, \beta(1) \ge \sum\limits_{k=1}^{m} \lambda_k \beta(\eta_k)$).

(b) $\alpha''(t) + f(t, \alpha(t), \alpha'(t)) \ge 0$, (respectively $\beta''(t) + f(t, \beta(t), \beta'(t)) \le 0$), for all $t \in (0,1)$.

Theorem 4.1. *Let $f : (0,1) \times \mathbb{R}^2 \to \mathbb{R}$ be a continuous function satisfying Nagumo condition on $D^* = \{(t,u,v) \in (0,1) \times \mathbb{R}^2 / \alpha(t) \le u \le \beta(t)\}$, where α and β are respectively lower and upper solutions of the problem BVPS, such that $\alpha(t) \le \beta(t)$ for any $t \in [0,1]$ and $\sum_{k=1}^{m} \lambda_k \le 1$.*

Then the boundary value problem BVPS has at least one solution $u \in C[0,1] \cap C^2(0,1)$ and

$$\alpha(t) \le u(t) \le \beta(t), \ \forall t \in [0,1].$$

Proof. Let $(a_n)_n, (b_n)_n \subset (0,1)$, be decreasing and increasing sequences respectively, such that $a_n \to 0$, $b_n \to 1$, $a_1 < \eta_k < b_1, k = 1, ..., m$. Consider the following sequence of modified problems

$$u''(t) + f\big(t, u(t), u'(t)\big) = 0, \quad a_n < t < b_n, \tag{4.1}$$

$$u(a_n) = A_n, \quad u(b_n) = \sum_{k=1}^{m} \lambda_k u(\eta_k), \tag{4.2}$$

where (A_n) is a real sequence such that $\alpha(a_n) \le A_n \le \beta(a_n)$ and $A_n \to c$, here α and β are the lower and upper solutions of the problem (BVPS). By following the same ideas of the proof of Theorem 3.1, we show that the regular problem (4.1)-(4.2) has a solution u_n satisfying on $[a_n, b_n]$, $\alpha(t) \le u_n(t) \le \beta(t)$ and $|u_n'(t)| \le b$. Using Arzela-Ascoli Theorem and the fact that the operator derivative is closed in $C[a_1, b_1]$, we can find a subsequence of $(u_n)_n$, that we denote also by $(u_n)_n$, converging in $C^1[a_1, b_1]$. By induction we can find a subsequence $\left(u_n^k\right)_n$ of $\left(u_n^{k-1}\right)_n$ that converges in $C^1[a_k, b_k]$. It follows that the diagonal sequence $(u_n^n)_n$ converges to some function u in $C^1(J)$ for any compact J of $(0,1)$, and we have $\alpha(t) \le u(t) \le \beta(t)$ and $|u'(t)| \le b$. Now, from the fact that the operator derivative is closed, we deduce that $u \in C^2(0,1)$ and

$$u''(t) + f\big(t, u(t), u'(t)\big) = 0, \quad 0 < t < 1.$$

Finally from the continuity of u_n and the fact that $a_n \to 0$ and $b_n \to 1$, we have $\lim_{t \to 0^+} u(t) = c$ and $\lim_{t \to 1^-} u(t) = u(1) = \sum\limits_{k=1}^{m} \lambda_k u(\eta_k)$. □

Now we give an existence theorem in the case where the singularity of f can appear at $t = 0$, $t = 1$ and $u = 0$.

Theorem 4.2. *Let $\sum_{k=1}^{m} \lambda_k = 1$. Assume that the following hypotheses hold:*

(H_1) *The function $f \in C((0,1) \times \mathbb{R}_0 \times \mathbb{R})$ and for any compact set $J \subset (0,1)$, there is $\varepsilon_J > 0$ such that*

$$f(t, x, 0) \ge 0, \quad \forall (t, x) \in J \times (0, \varepsilon_J).$$

(H$_2$) *There exists* $h \in C\left((0,1), \mathbb{R}_0^+\right)$, *such that*

$$\int_0^1 (1-s)\, h(s)\, ds < \infty$$

and

$$|f(t,x,y)| \leq h(t), \ \forall (t,x,y) \in (0,1) \times \mathbb{R}_0 \times \mathbb{R}.$$

Then, the boundary value problem (1.1)-(1.2) has at least one positive solution

$$u \in C\left([0,1], \mathbb{R}^+\right) \cap C^2\left((0,1), \mathbb{R}_0^+\right).$$

Proof. The proof will be carried out in several steps.

Step 1: Definition of approximation problems sequence.

For any $n \in \mathbb{N}$, $n \geq 1$, we consider the compact set

$$J_n = \left[\frac{1}{2^{n+1}}, 1 - \frac{1}{2^{n+1}}\right].$$

Set

$$\sigma_n(t) = \max\left\{\frac{1}{2^{n+1}}, \min\left(t, 1 - \frac{1}{2^{n+1}}\right)\right\}, \ t \in (0,1),$$

and

$$g_n(t,x,y) = \max\left\{f\left(\sigma_n(t), x, y\right), f(t,x,y)\right\}, (t,x,y) \in (0,1) \times \mathbb{R}_0 \times \mathbb{R}.$$

Then we have

$$g_n(t,x,y) = f(t,x,y), \ v \in J_n \times \mathbb{R}_0 \times \mathbb{R}.$$

We also define

$$f_n(t,x,y) = \min\left\{g_1(t,x,y), ..., g_n(t,x,y)\right\}.$$

It's clear that (f_n) is a continuous and decreasing sequence on $(0,1) \times \mathbb{R}_0 \times \mathbb{R}$, and

$$f_n(t,x,y) = f(t,x,y), \ (t,x,y) \in J_n \times \mathbb{R}_0 \times \mathbb{R},$$

which implies that the sequence of functions (f_n) converges uniformly to f on $J_n \times \mathbb{R}_0 \times \mathbb{R}$.

Let us introduce a decreasing sequence $(\varepsilon_n) \subset \mathbb{R}_0^+$ such that $\lim\limits_{n \to +\infty} \varepsilon_n = 0$. Now, we can define the sequence of approximation problems $(BVP)_n$:

$$u''(t) + f_n\left(t, u(t), u'(t)\right) = 0, \quad 0 < t < 1, \tag{4.3}$$

$$u(0) = \varepsilon_n, \quad u(1) = \sum_{k=1}^{m} \lambda_k u(\eta_k). \tag{4.4}$$

Step 2: Construction of lower solution of $(BVP)_n$ (4.3)-(4.4): we have $\alpha(t) = \varepsilon_n$ is a lower solution of $(BVP)_n$, since

$$\alpha''(t) + f_n\left(t, \alpha(t), \alpha'(t)\right) = f_n\left(t, \alpha(t), \alpha'(t)\right) = \min_{1 \leq i \leq n} g_i\left(t, \alpha, \alpha'\right).$$

Moreover, from condition $(\mathbf{H}_1)$ it yields

$$g_i\left(t, \alpha, \alpha'\right) \geq f\left(t, \alpha, \alpha'\right) \geq 0,$$

which implies

$$\alpha''(t) + f_n\left(t, \alpha(t), \alpha'(t)\right) \geq 0.$$

Furthermore, we have

$$\alpha(0) = \varepsilon_n, \ \alpha(1) = \varepsilon_n \leq \sum_{k=1}^{m} \lambda_k \alpha(\eta_k).$$

Step 3: Construction of upper solution of $(BVP)_1$.
From step 1 we have for $(t, x, y) \in (0, 1) \times \mathbb{R}_0 \times \mathbb{R}$:

$$f_1(t, x, y) = \max\{f(\sigma_1(t), x, y), f(t, x, y)\}.$$

Note that

$$f(\sigma_1(t), x, y) \leq h(\sigma_1(t)) \leq \sup_{t \in J_1} h(t) = \tau_1,$$

and $f(t, x, y) \leq h(t)$. Hence $f_1(t, x, y) \leq h(t) + \tau_1$.

Now, let β be the solution of the following boundary value problem

$$z''(t) + h(t) + \tau_1 = 0,$$

$$z(0) = \rho, \quad z(1) = \sum_{k=1}^{m} \lambda_k z(\eta_k),$$

that is given by

$$\beta(t) = \rho - \int_0^t (t - s)(h(s) + \tau_1)\, ds$$

$$+\omega t \int_0^1 (1 - s)(h(s) + \tau_1)\, ds - \omega t \sum_{k=1}^{m} \lambda_k \int_0^{\eta_k} (\eta_k - s)(h(s) + \tau_1)\, ds,$$

where

$$\rho \geq \varepsilon_1 + \tau_1 + \int_0^1 (1 - s) h(s)\, ds, \tag{4.5}$$

and $\varepsilon_1 = \alpha(t)$ is the lower solution of $(BVP)_1$.

Moreover β is an upper solution of $(BVP)_1$ satisfying

$$\beta''(t) + f_1(t, \beta, \beta') \leq \beta''(t) + h(t) + \tau_1 = 0,$$

and

$$\beta(0) \geq \rho > 0, \ \beta(1) \geq \sum_{k=1}^{m} \lambda_k \beta(\eta_k).$$

We claim that $\alpha(t) \leq \beta(t)$. Indeed

$$\beta(t) \geq \rho - \int_0^1 (1 - s)(h(s) + \tau_1)\, ds + \omega t \left(1 - \sum_{k=1}^{m} \lambda_k\right) \int_0^1 (1 - s)(h(s) + \tau_1)\, ds.$$

Since $\sum_{k=1}^{m} \lambda_k = 1$, we have

$$\beta(t) \geq \rho - \int_0^1 (1 - s)(h(s))\, ds - \tau_1.$$

Taking (4.5) into account and the fact that $\varepsilon_1 = \alpha(t)$ is the lower solution of $(BVP)_1$, we get

$$\alpha(t) \leq \beta(t).$$

Using Theorem 4.1, for $c = \varepsilon_1$, we conclude that $(BVP)_1$ has at least one solution $u_1 \in C^1[0,1] \cap C^2(0,1)$ such that

$$\alpha(t) \leq u_1(t) \leq \beta(t).$$

Step 4: Existence of at least one solution of $(BVP)_n$ (4.4)-(4.5).

It's clear that, $\forall n \geq 1$

$$\alpha(t) \leq u_n(t) \leq u_{n-1}(t) \leq ... \leq u_1(t).$$

After taking into account that $\alpha(t)$ and $u_{n-1}(t)$ are lower and upper solutions of $(BVP)_n$(4.4)-(4.5) and applying Theorem 4.1 again, we find that $(BVP)_n$ has at least one solution $u_n \in C[0,1] \cap C^2(0,1)$.

Step 5: Existence of a solution for BVP (1.1)-(1.2).

From the previous steps, we have (u_n) is a bounded sequence on the compact sets J_n. Arzela-Ascoli theorem guarantees the existence of uniformly convergent subsequence of (u_n) in C^1 on each compact subset of $(0,1)$, which we also denote by (u_n). Denote the pointwise limit of the sequence (u_n) by

$$\widetilde{u}(t) = \lim_{n \to +\infty} u_n(t).$$

From the fact that the operator derivative is closed, we deduce that $\widetilde{u} \in C^2(0,1)$ and

$$\widetilde{u}''(t) + f\left(t, \widetilde{u}(t), \widetilde{u}'(t)\right) = 0, \quad 0 < t < 1.$$

Moreover,

$$\widetilde{u}(0) = \lim_{n \to +\infty} \varepsilon_n = 0, \quad \widetilde{u}(1) = \sum_{k=0}^{m} \lambda_k \left(\lim_{n \to +\infty} u_n(\eta_k) \right) = \sum_{k=0}^{m} \lambda_k \widetilde{u}(\eta_k).$$

Finally from the continuity of u_n and the fact that $\lim\limits_{n \longrightarrow +\infty} \varepsilon_n = 0$ and $\lim\limits_{n \longrightarrow +\infty} \sum_{k=0}^{m} \lambda_k u_n(\eta_k) = \sum_{k=0}^{m} \lambda_k \widetilde{u}(\eta_k)$, we can check the continuity of $\widetilde{u}$ at the points $t = 0$ and $t = 1$. This completes the proof. $\square$

Now we give an example of a function f satisfying the assumptions of Theorem 4.2:

Example. Let f be the following function:

$$f(t,x,y) = \frac{e^{-\left|\frac{y}{tx}\right|}}{(1-t)}, (t,x,y) \in (0,1) \times \mathbb{R}_0 \times \mathbb{R}.$$

Note that the assumptions of Theorem 4.2 hold. Indeed, we have

$$f(t,x,0) = \frac{1}{(1-t)} > 0,$$

on any compact set $J \subset (0,1)$. Moreover if we choose $h(t) = \frac{1}{(1-t)} \in C\left((0,1), \mathbb{R}_0^+\right)$, then

$$|f(t,x,y)| \leq h(t), \ \forall (t,x,y) \in (0,1) \times \mathbb{R}_0 \times \mathbb{R}$$

and

$$\int_0^1 (1-s)\, h(s)\, ds = 1 < \infty.$$

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