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EXISTENCE AND UNIQUENESS OF NONOSCILLATORY SOLUTIONS OF FIRST-ORDER NEUTRAL DIFFERENTIAL EQUATIONS BY USING BANACH'S THEOREM

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Abstract. In this work, we consider the existence and uniqueness of nonoscillatory solutions to first-order differential equations having both delay and advance terms, known as mixed equations. We use the Banach contraction principle to obtain new sufficient conditions, which are weaker than those known, for the existence and uniqueness of nonoscillatory solutions.

1. Introduction

The problem of the existence of nonoscillatory solutions of neutral differential equations has been studied by several authors in the recent years. For related results we refer the reader to [4], [5], [6], [12] and the references cited therein. We refer the reader to the books [1], [2], [7], [8] on the subject of neutral differential equations. Recently, Zhang, Feng, Yan and Song [11] investigated the existence of nonoscillatory solutions of first-order neutral delay differential equation with variable coefficients

$$\begin{aligned}&\frac{d}{dt}[x(t)+P(t)x(t-\tau)]\\&+Q_1(t)x(t-\sigma_1)-Q_2(t)x(t-\sigma_2)\ =\ 0,\ t\ \geq\ t_0,\end{aligned}$$

they obtained sufficient conditions for the existence of nonoscillatory solutions depending on the four different ranges of $P(t)$. Candan [3] by employing Banach's fixed point theorem discussed the existence of nonoscillatory solutions for the following first-order neutral differential equation

$$\begin{aligned}&\frac{d}{dt}[x(t)+P_1(t)x(t-\tau_1)+P_2(t)x(t+\tau_2)]\\&+Q_1(t)x(t-\sigma_1)-Q_2(t)x(t+\sigma_2)\ =\ 0,\end{aligned}$$

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where $P_i \in C([t_0,\infty),\mathbb{R})$, $Q_i \in C([t_0,\infty),[0,\infty))$, $\tau_i > 0$ and $\sigma_i \geq 0$ for $i = 1,2$. In [9] Kong consider the first-order neutral differential equation

$$\begin{aligned}&\frac{d}{dt}\left[x(t) + P_1(t)x(t-\tau_1) + P_2(t)x(t+\tau_2)\right]\\&+ Q_1(t)g_1(x(t-\sigma_1)) - Q_2(t)g_2(x(t+\sigma_2)) = 0,\end{aligned}$$

and by different cases of the coefficients P_1 and P_2 he discussed the existence of nonoscillatory solutions.

Inspired and motivated by the works mentioned above and by using Banach's fixed point theorem, in this work, we study the first-order neutral differential equation

$$\begin{aligned}&\frac{d}{dt}\left[x(t) + P_1(t)h_1(x(t-\tau_1(t))) + P_2(t)h_2(x(t+\tau_2(t)))\right]\\&+ g_1(t,x(t-\sigma_1(t))) - g_2(t,x(t+\sigma_2(t))) = 0. \qquad (1.1)\end{aligned}$$

We give some new criteria for the existence and uniqueness of nonoscillatory solutions of (1.1). Throughout this paper, the following conditions are assumed to hold.

(1) $P_i \in C([t_0,\infty),\mathbb{R})$, $i = 1,2$.

(2) $h_1, h_2 \in C([0,\infty),[0,\infty))$ satisfy the conditions

$$0 \leq h_1(x) \leq K_1 x,\ 0 \leq h_2(x) \leq K_2 x,$$

and suppose that H_1 the inverse function of h_1 exists and we assume that there exist positive constants L_1 and L_2 such that

$$L_1 x \leq H_1(x) \leq L_2 x.$$

(3) $g_i \in C([t_0,\infty)\times[0,\infty),[0,\infty))$, and $g_i(t,x)$ satisfy the conditions

$$0 \leq g_1(t,x) \leq q_1(t)x + f_1(t),\ 0 \leq g_2(t,x) \leq q_2(t)x + f_2(t).$$

where $q_i, f_i \in C([t_0,\infty),[0,\infty))$, $i = 1,2$.

(4) τ_1 is differentiable and the inverse function φ of $t-\tau_1(t)$ exists, with $\varphi(t) \geq t$, and $t-\tau_1(t)$, $t-\sigma_1(t)$ are increasing functions.

(5) $\tau_i(t) > 0$ and $\sigma_i(t) \geq 0$ for $i = 1,2$.

The following theorem will be used to prove the main results in the next section.

Theorem 1.1 (Banach's Contraction Mapping Principle [10]). *A contraction mapping on a complete metric space has exactly one fixed point.*

2. Main Results

To show that an operator S satisfies the conditions for the contraction mapping principle, we consider different cases for the ranges of the coefficients P_1 and P_2.

Theorem 2.1. *Assume that* $0 \leq P_1(t) \leq p_1 < 1$, $0 \leq P_2(t) \leq p_2 < 1 - p_1$ *and there exist positive constant* M_2 *such that*

$$\int_{t_0}^{\infty}\left[q_1(s)M_2 + f_1(s)\right]ds < \infty,\ \int_{t_0}^{\infty}\left[q_2(s)M_2 + f_2(s)\right]ds < \infty, \qquad (2.1)$$

then (1.1) has a bounded nonoscillatory solution.

Proof. Let Λ be the set of all continuous and bounded functions on $[t_0, \infty)$ with the supremum norm. Set

$$\Omega = \{x \in \Lambda : M_1 \le x(t) \le M_2,\ t \ge t_0\}.$$

It is clear that Ω is a bounded, closed and convex subset of Λ. Because of (2.1), we can choose a $t_1 > t_0$,

$$t_1 \ge t_0 + \max\left\{\sup_{t\ge t_0} \tau_1(t),\ \sup_{t\ge t_0} \sigma_1(t)\right\}, \tag{2.2}$$

sufficiently large such that

$$\int_t^\infty [q_1(s)M_2 + f_1(s)]\, ds \le M_2 - \alpha,\ t \ge t_1, \tag{2.3}$$

$$\int_t^\infty [q_2(s)M_2 + f_2(s)]\, ds \le \alpha - (p_1K_1 + p_2K_2)\, M_2 - M_1,\ t \ge t_1, \tag{2.4}$$

and

$$\int_t^\infty [q_1(s) + q_2(s)]\, ds \le 1 - p_1K_1 - p_2K_2 - \frac{M_1}{M_2},\ t \ge t_1, \tag{2.5}$$

where M_1 and M_2 are positive constants such that

$$(p_1K_1 + p_2K_2)M_2 + M_1 < M_2 \text{ and } \alpha \in ((p_1K_1 + p_2K_2)\, M_2 + M_1, M_2)\,.$$

Consider the operator $S : \Omega \to \Lambda$ define by

$$(Sx)(t) = \begin{cases} \alpha - P_1(t)h_1\left(x(t-\tau_1(t))\right) - P_2(t)h_2\left(x(t+\tau_2(t))\right) \\ +\displaystyle\int_t^\infty [g_1\left(s, x(s-\sigma_1(s))\right) - g_2\left(s, x(s+\sigma_2(s))\right)]ds,\ t \ge t_1, \\ (Sx)(t_1),\ t_0 \le t \le t_1. \end{cases}$$

Clearly, Sx is continuous. For $t \ge t_1$ and $x \in \Omega$, from (2.3) and (2.4) it follows that

$$\begin{aligned} (Sx)(t) &\le \alpha + \int_t^\infty g_1\left(s, x(s-\sigma_1(s))\right) ds \\ &\le \alpha + \int_t^\infty [q_1(s)x(s-\sigma_1(s)) + f_1(s)]\, ds \\ &\le \alpha + \int_t^\infty [q_1(s)M_2 + f_1(s)]\, ds \le M_2, \end{aligned}$$

and

$$\begin{aligned} (Sx)(t) &\ge \alpha - P_1(t)h_1\left(x(t-\tau_1(t))\right) - P_2(t)h_2\left(x(t+\tau_2(t))\right) \\ &\quad - \int_t^\infty g_2\left(s, x(s+\sigma_2(s))\right) ds \\ &\ge \alpha - p_1K_1x(t-\tau_1(t)) - p_2K_2x(t+\tau_2(t)) \\ &\quad - \int_t^\infty [q_2(s)x(s+\sigma_2(s)) + f_2(s)]\, ds \\ &\ge \alpha - p_1K_1M_2 - p_2K_2M_2 - \int_t^\infty [q_2(s)M_2 + f_2(s)]\, ds \ge M_1. \end{aligned}$$

This means that $S\Omega \subset \Omega$. To apply the contraction mapping principle, the remaining is to show that S is a contraction mapping on Ω. Thus, if $x_1, x_2 \in \Omega$ and $t \geq t_1$,

$$\begin{aligned}
|(Sx_1)(t) - (Sx_2)(t)| &\leq P_1(t)\left|h_1\left(x_1(t-\tau_1(t))\right) - h_1\left(x_2(t-\tau_1(t))\right)\right| \\
&+ P_2(t)|h_2\left(x_1(t+\tau_2(t))\right) - h_2\left(x_2(t+\tau_2(t))\right)| \\
&+ \int_t^\infty \left(\left|g_1\left(s, x_1\left(s-\sigma_1(s)\right)\right) - g_1\left(s, x_2\left(s-\sigma_1(s)\right)\right)\right|\right. \\
&+ \left.\left|g_2\left(s, x_1\left(s+\sigma_2(s)\right)\right) - g_2\left(s, x_2\left(s+\sigma_2(s)\right)\right)\right|\right) ds \\
&\leq P_1(t)K_1\left|x_1(t-\tau_1(t)) - x_2(t-\tau_1(t))\right| \\
&+ P_2(t)K_2\left|x_1(t+\tau_2(t)) - x_2(t+\tau_2(t))\right| \\
&+ \int_t^\infty \left(q_1(s)\left|x_1\left(s-\sigma_1(s)\right) - x_2\left(s-\sigma_1(s)\right)\right|\right. \\
&\left.+ q_2(s)\left|x_1\left(s+\sigma_2(s)\right) - x_2\left(s+\sigma_2(s)\right)\right|\right) ds,
\end{aligned}$$

or by (2.5)

$$\begin{aligned}
|(Sx_1)(t) - (Sx_2)(t)| &\leq \left(p_1K_1 + p_2K_2 + \int_t^\infty \left[q_1(s) + q_2(s)\right] ds\right) \|x_1 - x_2\| \\
&\leq \lambda_1 \|x_1 - x_2\|,
\end{aligned}$$

where $\lambda_1 = \left(1 - \frac{M_1}{M_2}\right)$. This implies that

$$\|Sx_1 - Sx_2\| \leq \lambda_1 \|x_1 - x_2\|.$$

Since $\lambda_1 < 1$, S is a contraction mapping on Ω. Thus S has a unique fixed point which is a positive and bounded solution of (1.1). □

Theorem 2.2. *Assume that* $0 \leq P_1(t) \leq p_1 < 1$, $p_1 - 1 < p_2 \leq P_2(t) \leq 0$ *and there exist positive constant* N_2 *such that*

$$\int_{t_0}^\infty \left[q_1(s)N_2 + f_1(s)\right] ds < \infty, \quad \int_{t_0}^\infty \left[q_2(s)N_2 + f_2(s)\right] ds < \infty, \tag{2.6}$$

then (1.1) has a bounded nonoscillatory solution.

Proof. Let Λ be the set of all continuous and bounded functions on $[t_0, \infty)$ with the supremum norm. Set

$$\Omega = \left\{x \in \Lambda : N_1 \leq x(t) \leq N_2,\ t \geq t_0\right\}.$$

It is clear that Ω is a bounded, closed and convex subset of Λ. Because of (2.6), we can choose a $t_1 > t_0$ sufficiently large satisfying (2.2) such that

$$\int_t^\infty \left[q_1(s)N_2 + f_1(s)\right] ds \leq \left(1 + p_2K_2\right) N_2 - \alpha,\ t \geq t_1, \tag{2.7}$$

$$\int_t^\infty \left[q_2(s)N_2 + f_2(s)\right] ds \leq \alpha - p_1K_1N_2 - N_1,\ t \geq t_1, \tag{2.8}$$

and

$$\int_t^\infty \left[q_1(s) + q_2(s)\right] ds \leq 1 - p_1K_1 + p_2K_2 - \frac{N_1}{N_2},\ t \geq t_1, \tag{2.9}$$

where N_1 and N_2 are positive constants such that

$$p_1K_1N_2 + N_1 < (1 + p_2K_2)\, N_2 \text{ and } \alpha \in (p_1K_1N_2 + N_1, (1 + p_2K_2)\, N_2)\,.$$

Consider the operator $S : \Omega \to \Lambda$ define by

$$(Sx)(t) = \begin{cases} \alpha - P_1(t)h_1\left(x(t-\tau_1(t))\right) - P_2(t)h_2\left(x(t+\tau_2(t))\right) \\ +\displaystyle\int_t^\infty [g_1\left(s, x(s-\sigma_1(s))\right) - g_2\left(s, x(s+\sigma_2(s))\right)]ds,\ t \ge t_1, \\ (Sx)(t_1),\ t_0 \le t \le t_1. \end{cases}$$

Clearly, Sx is continuous. For $t \ge t_1$ and $x \in \Omega$, from (2.7) and (2.8) it follows that

$$\begin{aligned} (Sx)(t) &\le \alpha - P_2(t)h_2\left(x(t+\tau_2(t))\right) + \int_t^\infty g_1\left(s, x(s-\sigma_1(s))\right) ds \\ &\le \alpha - P_2(t)K_2x(t+\tau_2(t)) + \int_t^\infty \left[q_1(s)x(s-\sigma_1(s)) + f_1(s)\right] ds \\ &\le \alpha - P_2K_2N_2 + \int_t^\infty \left[q_1(s)N_2 + f_1(s)\right] ds \le N_2, \end{aligned}$$

and

$$\begin{aligned} (Sx)(t) &\ge \alpha - P_1(t)h_1\left(x(t-\tau_1(t))\right) - \int_t^\infty g_2\left(s, x(s+\sigma_2(s))\right) ds \\ &\ge \alpha - p_1K_1x(t-\tau_1(t)) - \int_t^\infty \left[q_2(s)x(s+\sigma_2(s)) + f_2(s)\right] ds \\ &\ge \alpha - p_1K_1N_2 - \int_t^\infty \left[q_2(s)N_2 + f_2(s)\right] ds \ge N_1. \end{aligned}$$

This means that $S\Omega \subset \Omega$. To apply the contraction mapping principle, the remaining is to show that S is a contraction mapping on Ω. Thus, if $x_1, x_2 \in \Omega$ and $t \ge t_1$, by using (2.9), we can obtain

$$\begin{aligned} |(Sx_1)(t) - (Sx_2)(t)| &\le \left(p_1K_1 - p_2K_2 + \int_t^\infty \left[q_1(s) + q_2(s)\right] ds\right) \|x_1 - x_2\| \\ &\le \lambda_2 \|x_1 - x_2\|\,, \end{aligned}$$

where $\lambda_2 = \left(1 - \frac{N_1}{N_2}\right)$. This implies that

$$\|Sx_1 - Sx_2\| \le \lambda_2 \|x_1 - x_2\|\,.$$

Since $\lambda_2 < 1$, S is a contraction mapping on Ω. Thus S has a unique fixed point which is a positive and bounded solution of Eq.(1.1). □

Theorem 2.3. *Assume that* $1 < p_1 \le P_1(t) \le p_{1_0} < \infty$, $0 \le P_2(t) \le p_2 < p_1 - 1$ *and there exist positive constant* M_4 *such that*

$$\int_{t_0}^\infty \left[q_1(s)M_4 + f_1(s)\right] ds < \infty,\ \int_{t_0}^\infty \left[q_2(s)M_4 + f_2(s)\right] ds < \infty, \tag{2.10}$$

then (1.1) has a bounded nonoscillatory solution.

Proof. Let Λ be the set of all continuous and bounded functions on $[t_0,\infty)$ with the supremum norm. Set

$$\Omega=\left\{x\in\Lambda: M_3\leq x(t)\leq M_4,\ t\geq t_0\right\}.$$

It is clear that Ω is a bounded, closed and convex subset of Λ. Because of (2.10), we can choose a $t_1>t_0$,

$$\varphi(t_1)-\sigma_1(\varphi(t_1))\geq t_0 \tag{2.11}$$

sufficiently large such that

$$\int_t^\infty\left[q_1(s)M_4+f_1(s)\right]ds\leq\frac{p_1}{L_2}M_4-\alpha,\ t\geq t_1, \tag{2.12}$$

$$\int_t^\infty\left[q_2(s)M_4+f_2(s)\right]ds\leq\alpha-\left(1+p_2K_2\right)M_4-\frac{p_{1_0}}{L_1}M_3,\ t\geq t_1, \tag{2.13}$$

and

$$\int_t^\infty\left[q_1(s)+q_2(s)\right]ds\leq\frac{p_1}{L_2}-\left(1+p_2K_2\right)-\frac{p_{1_0}M_3}{L_1M_4},\ t\geq t_1, \tag{2.14}$$

where M_3 and M_4 are positive constants such that

$$\left(1+p_2K_2\right)M_4+\frac{p_{1_0}}{L_1}M_3<\frac{p_1}{L_2}M_4 \text{ and } \alpha\in\left(\left(1+p_2K_2\right)M_4+\frac{p_{1_0}}{L_1}M_3,\frac{p_1}{L_2}M_4\right).$$

Consider the operator $S:\Omega\to\Lambda$ define by

$$(Sx)(t)=\begin{cases} H_1\left(\frac{1}{P_1(\varphi(t))}\left[\alpha-x(\varphi(t))-P_2(\varphi(t))h_2\left(x(\varphi(t)+\tau_2(\varphi(t)))\right)\right.\right. \\ \left.\left.+\int_{\varphi(t)}^\infty[g_1\left(s,x(s-\sigma_1(s))\right)-g_2\left(s,x(s+\sigma_2(s))\right)]ds\right]\right),\ t\geq t_1, \\ (Sx)(t_1),\ t_0\leq t\leq t_1. \end{cases}$$

Clearly, Sx is continuous. For $t\geq t_1$ and $x\in\Omega$, from (2.12) and (2.13) it follows that

$$\begin{aligned}(Sx)(t)&\leq L_2\left(\frac{1}{P_1(\varphi(t))}\left[\alpha-x(\varphi(t))-P_2(\varphi(t))h_2\left(x(\varphi(t)+\tau_2(\varphi(t)))\right)\right.\right.\\&\quad\left.\left.+\int_{\varphi(t)}^\infty[g_1\left(s,x(s-\sigma_1(s))\right)-g_2\left(s,x(s+\sigma_2(s))\right)]ds\right]\right)\\&\leq\frac{L_2}{P_1}\left(\alpha+\int_{\varphi(t)}^\infty g_1\left(s,x(s-\sigma_1(s))\right)ds\right)\\&\leq\frac{L_2}{p_1}\left(\alpha+\int_t^\infty\left[q_1(s)x(s-\sigma_1(s))+f_1(s)\right]ds\right)\\&\leq\frac{L_2}{p_1}\left(\alpha+\int_t^\infty\left[q_1(s)M_4+f_1(s)\right]ds\right)\leq M_4,\end{aligned}$$

and

$$\begin{aligned}(Sx)(t) &\geq L_1\left(\frac{1}{P_1(\varphi(t))}\left[\alpha - x(\varphi(t)) - P_2(\varphi(t))h_2\left(x(\varphi(t)+\tau_2(\varphi(t)))\right)\right.\right.\\ &\left.\left.+\int_{\varphi(t)}^{\infty}[g_1\left(s, x(s-\sigma_1(s))\right) - g_2\left(s, x(s+\sigma_2(s))\right)]ds\right]\right)\\ &\geq \frac{L_1}{p_{1_0}}\left(\alpha - x(\varphi(t)) - P_2(\varphi(t))h_2\left(x(\varphi(t)+\tau_2(\varphi(t)))\right)\right.\\ &\left.-\int_{\varphi(t)}^{\infty} g_2\left(s, x(s+\sigma_2(s))\right)ds\right)\\ &\geq \frac{L_1}{p_{1_0}}\left(\alpha - x(\varphi(t)) - p_2K_2x(\varphi(t)+\tau_2(\varphi(t)))\right.\\ &\left.-\int_t^{\infty}[q_2(s)x(s-\sigma_1(s)) + f_2(s)]\,ds\right)\\ &\geq \frac{L_1}{p_{1_0}}\left(\alpha - M_4 - p_2K_2M_4 - \int_t^{\infty}[q_2(s)M_4 + f_2(s)]\,ds\right) \geq M_3.\end{aligned}$$

This means that $S\Omega \subset \Omega$. To apply the contraction mapping principle, the remaining is to show that S is a contraction mapping on Ω. Thus, if $x_1, x_2 \in \Omega$ and $t \geq t_1$,

$$\begin{aligned}|(Sx_1)(t) - (Sx_2)(t)| &\leq \frac{L_2}{P_1(\varphi(t))}\left(|x_1(\varphi(t)) - x_2(\varphi(t))|\right.\\ &+ P_2(t)K_2\left|x_1(\varphi(t)+\tau_2(\varphi(t))) - x_2(\varphi(t)+\tau_2(\varphi(t)))\right|\\ &+\int_{\varphi(t)}^{\infty}\left(q_1(s)|x_1\left(s-\sigma_1\left(s\right)\right) - x_2\left(s-\sigma_1\left(s\right)\right)|\right.\\ &\left.\left.+q_2(s)|x_1\left(s+\sigma_2\left(s\right)\right) - x_2\left(s+\sigma_2\left(s\right)\right)|\right)ds\right),\end{aligned}$$

or by (2.14)

$$\begin{aligned}|(Sx_1)(t) - (Sx_2)(t)| &\leq \frac{L_2}{p_1}\left(1 + p_2K_2 + \int_{\varphi(t)}^{\infty}[q_1(s) + q_2(s)]\,ds\right)\|x_1 - x_2\|\\ &\leq \frac{L_2}{p_1}\left(1 + p_2K_2 + \int_t^{\infty}[q_1(s) + q_2(s)]\,ds\right)\|x_1 - x_2\|\\ &\leq \lambda_3\|x_1 - x_2\|,\end{aligned}$$

where $\lambda_3 = \left(1 - \frac{p_{1_0}L_2M_3}{p_1L_2M_4}\right)$. This implies that

$$\|Sx_1 - Sx_2\| \leq \lambda_3\|x_1 - x_2\|.$$

Since $\lambda_3 < 1$, S is a contraction mapping on Ω. Thus S has a unique fixed point which is a positive and bounded solution of (1.1). □

Theorem 2.4. *Assume that* $1 < p_1 \leq P_1(t) \leq p_{1_0} < \infty$, $1 - p_1 < p_2 \leq P_2(t) \leq 0$ *and there exist positive constant* N_4 *such that*

$$\int_{t_0}^{\infty}[q_1(s)N_4 + f_1(s)]\,ds < \infty,\quad \int_{t_0}^{\infty}[q_2(s)N_4 + f_2(s)]\,ds < \infty, \tag{2.15}$$

then (1.1) has a bounded nonoscillatory solution.

Proof. Let Λ be the set of all continuous and bounded functions on $[t_0, \infty)$ with the supremum norm. Set

$$\Omega = \{x \in \Lambda : N_3 \leq x(t) \leq N_4,\ t \geq t_0\}.$$

It is clear that Ω is a bounded, closed and convex subset of Λ. Because of (2.15), we can choose a $t_1 > t_0$ sufficiently large satisfying (2.11) such that

$$\int_t^\infty [q_1(s)N_4 + f_1(s)]\, ds \leq \left(\frac{p_1}{L_2} + p_2K_2\right) N_4 - \alpha,\ t \geq t_1, \tag{2.16}$$

$$\int_t^\infty [q_2(s)N_4 + f_2(s)]\, ds \leq \alpha - N_4 - \frac{p_{1_0}}{L_1} N_3,\ t \geq t_1, \tag{2.17}$$

and

$$\int_t^\infty [q_1(s) + q_2(s)]\, ds \leq \frac{p_1}{L_2} + p_2K_2 - 1 - \frac{p_{1_0}N_3}{L_1N_4},\ t \geq t_1, \tag{2.18}$$

where N_3 and N_4 are positive constants such that

$$N_4 + \frac{p_{1_0}}{L_1} N_3 < \left(\frac{p_1}{L_2} + p_2K_2\right) N_4 \text{ and } \alpha \in \left(N_4 + \frac{p_{1_0}}{L_1} N_3, \left(\frac{p_1}{L_2} + p_2K_2\right) N_4\right).$$

Consider the operator $S : \Omega \to \Lambda$ define by

$$(Sx)(t) = \begin{cases} H_1\Big(\frac{1}{P_1(\varphi(t))}\big[\alpha - x(\varphi(t)) - P_2(\varphi(t))h_2\left(x(\varphi(t) + \tau_2(\varphi(t)))\right) \\ \quad + \int_{\varphi(t)}^\infty [g_1(s, x(s - \sigma_1(s))) - g_2(s, x(s + \sigma_2(s)))]\, ds\big]\Big),\ t \geq t_1, \\ (Sx)(t_1),\ t_0 \leq t \leq t_1. \end{cases}$$

Clearly, Sx is continuous. For $t \geq t_1$ and $x \in \Omega$, from (2.16) and (2.17) it follows that

$$\begin{aligned} (Sx)(t) &\leq L_2\Bigg(\frac{1}{P_1(\varphi(t))}\Big[\alpha - x(\varphi(t)) - P_2(\varphi(t))h_2\left(x(\varphi(t) + \tau_2(\varphi(t)))\right) \\ &\quad + \int_{\varphi(t)}^\infty [g_1(s, x(s - \sigma_1(s))) - g_2(s, x(s + \sigma_2(s)))]\, ds\Big]\Bigg) \\ &\leq \frac{L_2}{p_1}\left(\alpha - P_2(\varphi(t))h_2\left(x(\varphi(t) + \tau_2(\varphi(t)))\right) + \int_{\varphi(t)}^\infty g_1(s, x(s - \sigma_1(s)))\, ds\right) \\ &\leq \frac{L_2}{p_1}\left(\alpha - p_2K_2 x(\varphi(t) + \tau_2(\varphi(t))) + \int_t^\infty [q_1(s)x(s - \sigma_1(s)) + f_1(s)]\, ds\right) \\ &\leq \frac{L_2}{p_1}\left(\alpha - p_2K_2N_4 + \int_t^\infty [q_1(s)N_4 + f_1(s)]\, ds\right) \leq N_4, \end{aligned}$$

and

$$
\begin{aligned}
(Sx)(t) \geq L_1 \Bigg(& \frac{1}{P_1(\varphi(t))} \Big[\alpha - x(\varphi(t)) - P_2(\varphi(t)) h_2 \left(x(\varphi(t) + \tau_2(\varphi(t)) \right) \\
& + \int_{\varphi(t)}^{\infty} \left[g_1\left(s, x(s - \sigma_1(s))\right) - g_2\left(s, x(s + \sigma_2(s))\right) \right] ds \Big] \Bigg) \\
& \geq \frac{L_1}{p_{1_0}} \left(\alpha - x(\varphi(t)) - \int_{\varphi(t)}^{\infty} g_2\left(s, x(s + \sigma_2(s))\right) ds \right) \\
& \geq \frac{L_1}{p_{1_0}} \left(\alpha - x(\varphi(t)) - \int_{t}^{\infty} \left[q_2(s) x(s - \sigma_1(s)) + f_2(s) \right] ds \right) \\
& \geq \frac{L_1}{p_{1_0}} \left(\alpha - N_4 - \int_{t}^{\infty} \left[q_2(s) N_4 + f_2(s) \right] ds \right) \geq N_3.
\end{aligned}
$$

This means that $S\Omega \subset \Omega$. To apply the contraction mapping principle, the remaining is to show that S is a contraction mapping on Ω. Thus, if $x_1, x_2 \in \Omega$ and $t \geq t_1$, by using (2.18), we can obtain

$$
\begin{aligned}
|(Sx_1)(t) - (Sx_2)(t)| & \leq \frac{L_2}{p_1} \left(1 - p_2 K_2 + \int_{\varphi(t)}^{\infty} \left[q_1(s) + q_2(s) \right] ds \right) \|x_1 - x_2\| \\
& \leq \frac{L_2}{p_1} \left(1 - p_2 K_2 + \int_{t}^{\infty} \left[q_1(s) + q_2(s) \right] ds \right) \|x_1 - x_2\| \\
& \leq \lambda_4 \|x_1 - x_2\|,
\end{aligned}
$$

where $\lambda_4 = \left(1 - \frac{p_{1_0} L_2 N_3}{p_1 L_1 N_4} \right)$. This implies that

$$
\|Sx_1 - Sx_2\| \leq \lambda_4 \|x_1 - x_2\|.
$$

Since $\lambda_4 < 1$, S is a contraction mapping on Ω. Thus S has a unique fixed point which is a positive and bounded solution of (1.1). □

Theorem 2.5. *Assume that $-1 < p_1 \leq P_1(t) \leq 0$, $0 \leq P_2(t) \leq p_2 < 1 + p_1$ and there exist positive constant M_6 such that*

$$
\int_{t_0}^{\infty} \left[q_1(s) M_6 + f_1(s) \right] ds < \infty, \quad \int_{t_0}^{\infty} \left[q_2(s) M_6 + f_2(s) \right] ds < \infty, \tag{2.19}
$$

then (1.1) has a bounded nonoscillatory solution.

Proof. Let Λ be the set of all continuous and bounded functions on $[t_0, \infty)$ with the supremum norm. Set

$$
\Omega = \left\{ x \in \Lambda : M_5 \leq x(t) \leq M_6, \ t \geq t_0 \right\}.
$$

It is clear that Ω is a bounded, closed and convex subset of Λ. Because of (2.19), we can choose a $t_1 > t_0$ sufficiently large satisfying (2.2) such that

$$
\int_{t}^{\infty} \left[q_1(s) M_6 + f_1(s) \right] ds \leq \left(1 + p_1 K_1 \right) M_6 - \alpha, \ t \geq t_1, \tag{2.20}
$$

$$
\int_{t}^{\infty} \left[q_2(s) M_6 + f_2(s) \right] ds \leq \alpha - p_2 K_2 M_6 - M_5, \ t \geq t_1, \tag{2.21}
$$

and

$$\int_t^\infty [q_1(s) + q_2(s)]\, ds \le 1 + p_1K_1 - p_2K_2 - \frac{M_5}{M_6},\ t \ge t_1, \tag{2.22}$$

where M_5 and M_6 are positive constants such that

$$p_2K_2M_6 + M_5 < (1 + p_1K_1)\, M_6 \text{ and } \alpha \in (p_2K_2M_6 + M_5, (1 + p_1K_1)\, M_6)\,.$$

Consider the operator $S : \Omega \to \Lambda$ define by

$$(Sx)(t) = \begin{cases} \alpha - P_1(t)h_1\left(x(t - \tau_1(t))\right) - P_2(t)h_2\left(x(t + \tau_2(t))\right) \\ + \displaystyle\int_t^\infty [g_1\left(s, x(s - \sigma_1(s))\right) - g_2\left(s, x(s + \sigma_2(s))\right)] ds,\ t \ge t_1, \\ (Sx)(t_1),\ t_0 \le t \le t_1. \end{cases}$$

Clearly, Sx is continuous. For $t \ge t_1$ and $x \in \Omega$, from (2.20) and (2.21) it follows that

$$\begin{aligned} (Sx)(t) &\le \alpha - P_1(t)h_1\left(x(t - \tau_1(t))\right) + \int_t^\infty g_1\left(s, x(s - \sigma_1(s))\right) ds \\ &\le \alpha - P_1(t)K_1x(t - \tau_1(t)) + \int_t^\infty [q_1(s)x(s - \sigma_1(s)) + f_1(s)]\, ds \\ &\le \alpha - p_1K_1M_6 + \int_t^\infty [q_1(s)M_6 + f_1(s)]\, ds \le M_6, \end{aligned}$$

and

$$\begin{aligned} (Sx)(t) &\ge \alpha - P_2(t)h_2\left(x(t + \tau_2(t))\right) - \int_t^\infty g_2\left(s, x(s + \sigma_2(s))\right) ds \\ &\ge \alpha - p_2K_2x(t + \tau_2(t)) - \int_t^\infty [q_2(s)x(s + \sigma_2(s)) + f_2(s)]\, ds \\ &\ge \alpha - p_2K_2M_6 - \int_t^\infty [q_2(s)M_6 + f_2(s)]\, ds \ge M_5. \end{aligned}$$

This means that $S\Omega \subset \Omega$. To apply the contraction mapping principle, the remaining is to show that S is a contraction mapping on Ω. Thus, if $x_1, x_2 \in \Omega$ and $t \ge t_1$, by using (2.22), we can obtain

$$\begin{aligned} |(Sx_1)(t) - (Sx_2)(t)| &\le \left(-p_1K_1 + p_2K_2 + \int_t^\infty [q_1(s) + q_2(s)]\, ds\right) \|x_1 - x_2\| \\ &\le \lambda_5 \|x_1 - x_2\|\,, \end{aligned}$$

where $\lambda_5 = \left(1 - \frac{M_5}{M_6}\right)$. This implies that

$$\|Sx_1 - Sx_2\| \le \lambda_5 \|x_1 - x_2\|\,.$$

Since $\lambda_5 < 1$, S is a contraction mapping on Ω. Thus S has a unique fixed point which is a positive and bounded solution of (1.1). □

Theorem 2.6. *Assume that* $-1 < p_1 \le P_1(t) \le 0$, $-1 - p_1 < p_2 \le P_2(t) \le 0$ *and there exist positive constant* N_6 *such that*

$$\int_{t_0}^\infty [q_1(s)N_6 + f_1(s)]\, ds < \infty,\ \int_{t_0}^\infty [q_2(s)N_6 + f_2(s)]\, ds < \infty, \tag{2.23}$$

then (1.1) has a bounded non-oscillatory solution.

Proof. Let Λ be the set of all continuous and bounded functions on $[t_0, \infty)$ with the supremum norm. Set

$$\Omega = \{x \in \Lambda : N_5 \le x(t) \le N_6,\ t \ge t_0\}.$$

It is clear that Ω is a bounded, closed and convex subset of Λ. Because of (2.23), we can choose a $t_1 > t_0$ sufficiently large satisfying (2.2) such that

$$\int_t^\infty [q_1(s)N_6 + f_1(s)]\, ds \le (1 + p_1K_1 + p_2K_2)\, N_6 - \alpha,\ t \ge t_1, \tag{2.24}$$

$$\int_t^\infty [q_2(s)N_6 + f_2(s)]\, ds \le \alpha - N_5,\ t \ge t_1, \tag{2.25}$$

and

$$\int_t^\infty [q_1(s) + q_2(s)]\, ds \le 1 + p_1K_1 + p_2K_2 - \frac{N_5}{N_6},\ t \ge t_1, \tag{2.26}$$

where N_5 and N_6 are positive constants such that

$$N_5 < (1 + p_1K_1 + p_2K_2)\, N_6 \text{ and } \alpha \in (N_5, (1 + p_1K_1 + p_2K_2)\, N_6).$$

Consider the operator $S : \Omega \to \Lambda$ define by

$$(Sx)(t) = \begin{cases} \alpha - P_1(t)h_1(x(t - \tau_1(t))) - P_2(t)h_2(x(t + \tau_2(t))) \\ + \displaystyle\int_t^\infty [g_1(s, x(s - \sigma_1(s))) - g_2(s, x(s + \sigma_2(s)))]ds,\ t \ge t_1, \\ (Sx)(t_1),\ t_0 \le t \le t_1. \end{cases}$$

Clearly, Sx is continuous. For $t \ge t_1$ and $x \in \Omega$, from (2.24) and (2.25) it follows that

$$\begin{aligned} (Sx)(t) &\le \alpha - P_1(t)h_1(x(t - \tau_1(t))) - P_2(t)h_2(x(t + \tau_2(t))) \\ &\quad + \int_t^\infty g_1(s, x(s - \sigma_1(s)))\, ds \\ &\le \alpha - p_1K_1x(t - \tau_1(t)) - p_2K_2x(t + \tau_2(t)) \\ &\quad + \int_t^\infty [q_1(s)x(s - \sigma_1(s)) + f_1(s)]\, ds \\ &\le \alpha - p_1K_1N_6 - p_2K_2N_6 + \int_t^\infty [q_1(s)N_6 + f_1(s)]\, ds \le N_6, \end{aligned}$$

and

$$\begin{aligned} (Sx)(t) &\ge \alpha - \int_t^\infty g_2(s, x(s + \sigma_2(s)))\, ds \\ &\ge \alpha - \int_t^\infty [q_2(s)x(s + \sigma_2(s)) + f_2(s)]\, ds \\ &\ge \alpha - \int_t^\infty [q_2(s)N_6 + f_2(s)]\, ds \ge N_5. \end{aligned}$$

This means that $S\Omega \subset \Omega$. To apply the contraction mapping principle, the remaining is to show that S is a contraction mapping on Ω. Thus, if $x_1, x_2 \in \Omega$

and $t \geq t_1$, by using (2.26), we can obtain

$$\begin{aligned}|(Sx_1)(t) - (Sx_2)(t)| &\leq \left(-p_1 K_1 - p_2 K_2 + \int_t^\infty [q_1(s) + q_2(s)]\, ds\right) \|x_1 - x_2\| \\ &\leq \lambda_6 \|x_1 - x_2\|,\end{aligned}$$

where $\lambda_6 = \left(1 - \frac{N_5}{N_6}\right)$. This implies that

$$\|Sx_1 - Sx_2\| \leq \lambda_6 \|x_1 - x_2\|.$$

Since $\lambda_6 < 1$, S is a contraction mapping on Ω. Thus S has a unique fixed point which is a positive and bounded solution of (1.1). □

Theorem 2.7. *Assume that* $-\infty < p_{1_0} \leq P_1(t) \leq p_1 < -1$, $0 \leq P_2(t) \leq p_2 < -p_1 - 1$ *and there exist positive constant* M_8 *such that*

$$\int_{t_0}^\infty [q_1(s) M_8 + f_1(s)]\, ds < \infty, \quad \int_{t_0}^\infty [q_2(s) M_8 + f_2(s)]\, ds < \infty, \tag{2.27}$$

then (1.1) has a bounded non-oscillatory solution.

Proof. Let Λ be the set of all continuous and bounded functions on $[t_0, \infty)$ with the supremum norm. Set

$$\Omega = \{x \in \Lambda : M_7 \leq x(t) \leq M_8,\ t \geq t_0\}.$$

It is clear that Ω is a bounded, closed and convex subset of Λ. Because of (2.27), we can choose a $t_1 > t_0$ sufficiently large satisfying (2.11) such that

$$\int_t^\infty [q_1(s) M_8 + f_1(s)]\, ds \leq \frac{p_{1_0}}{L_1} M_7 + \alpha,\ t \geq t_1, \tag{2.28}$$

$$\int_t^\infty [q_2(s) M_8 + f_2(s)]\, ds \leq -\left(1 + p_2 K_2 + \frac{p_1}{L_2}\right) M_8 - \alpha,\ t \geq t_1, \tag{2.29}$$

and

$$\int_t^\infty [q_1(s) + q_2(s)]\, ds \leq \frac{p_{1_0} M_7}{L_1 M_8} - \left(1 + p_2 K_2 + \frac{p_1}{L_2}\right),\ t \geq t_1, \tag{2.30}$$

where M_7 and M_8 are positive constants such that

$$-\frac{p_{1_0}}{L_1} M_7 < -\left(1 + p_2 K_2 + \frac{p_1}{L_2}\right) M_8 \text{ and } \alpha \in \left(-\frac{p_{1_0}}{L_1} M_7, -\left(1 + p_2 K_2 + \frac{p_1}{L_2}\right) M_8\right).$$

Consider the operator $S : \Omega \to \Lambda$ define by

$$(Sx)(t) = \begin{cases} H_1\left(\frac{-1}{P_1(\varphi(t))}\left[\alpha + x(\varphi(t)) + P_2(\varphi(t)) h_2\left(x(\varphi(t) + \tau_2(\varphi(t)))\right)\right.\right. \\ \left.\left. -\int_{\varphi(t)}^\infty \left[g_1\left(s, x(s - \sigma_1(s))\right) - g_2\left(s, x(s + \sigma_2(s))\right)\right] ds\right]\right),\ t \geq t_1, \\ (Sx)(t_1),\ t_0 \leq t \leq t_1. \end{cases}$$

Clearly, Sx is continuous. For $t \geq t_1$ and $x \in \Omega$, from (2.28) and (2.29) it follows that

$$\begin{aligned}
(Sx)(t) \leq L_2 \Bigg(& \frac{-1}{P_1(\varphi(t))} \left[\alpha + x(\varphi(t)) + P_2(\varphi(t)) h_2\left(x(\varphi(t) + \tau_2(\varphi(t)))\right)\right. \\
& \left.- \int_{\varphi(t)}^{\infty} \left[g_1\left(s, x(s - \sigma_1(s))\right) - g_2\left(s, x(s + \sigma_2(s))\right)\right] ds \right] \Bigg) \\
\leq & \frac{-L_2}{p_1} \Bigg(\alpha + x(\varphi(t)) + p_2 K_2 x(\varphi(t) + \tau_2(\varphi(t))) \\
& + \int_{\varphi(t)}^{\infty} g_2\left(s, x(s + \sigma_2(s))\right) ds \Bigg) \\
\leq & \frac{-L_2}{p_1} \Bigg(\alpha + x(\varphi(t)) + p_2 K_2 x(\varphi(t) + \tau_2(\varphi(t))) \\
& + \int_{t}^{\infty} \left[q_2(s) x(s + \sigma_2(s)) + f_2(s)\right] ds \Bigg) \\
\leq & \frac{-L_2}{p_1} \left(\alpha + M_8 + p_2 K_2 M_8 + \int_{t}^{\infty} \left[q_2(s) M_8 + f_2(s)\right] ds \right) \leq M_8,
\end{aligned}$$

and

$$\begin{aligned}
(Sx)(t) \geq L_2 \Bigg(& \frac{-1}{P_1(\varphi(t))} \left[\alpha + x(\varphi(t)) + P_2(\varphi(t)) h_2\left(x(\varphi(t) + \tau_2(\varphi(t)))\right)\right. \\
& \left.- \int_{\varphi(t)}^{\infty} \left[g_1\left(s, x(s - \sigma_1(s))\right) - g_2\left(s, x(s + \sigma_2(s))\right)\right] ds \right] \Bigg) \\
\geq & \frac{-L_1}{p_{1_0}} \left(\alpha - \int_{\varphi(t)}^{\infty} g_1\left(s, x(s - \sigma_1(s))\right) ds \right) \\
\geq & \frac{-L_1}{p_{1_0}} \left(\alpha - \int_{t}^{\infty} \left[q_1(s) x(s - \sigma_1(s)) + f_1(s)\right] ds \right) \\
\geq & \frac{-L_1}{p_{1_0}} \left(\alpha - \int_{t}^{\infty} \left[q_1(s) M_8 + f_1(s)\right] ds \right) \geq M_7.
\end{aligned}$$

This means that $S\Omega \subset \Omega$. To apply the contraction mapping principle, the remaining is to show that S is a contraction mapping on Ω. Thus, if $x_1, x_2 \in \Omega$ and $t \geq t_1$, by using (2.30), we can obtain

$$\begin{aligned}
|(Sx_1)(t) - (Sx_2)(t)| & \leq \frac{-L_2}{p_1} \left(1 + p_2 K_2 + \int_{\varphi(t)}^{\infty} \left[q_1(s) + q_2(s)\right] ds \right) \|x_1 - x_2\| \\
& \leq \frac{-L_2}{p_1} \left(1 + p_2 K_2 + \int_{t}^{\infty} \left[q_1(s) + q_2(s)\right] ds \right) \|x_1 - x_2\| \\
& \leq \lambda_7 \|x_1 - x_2\|,
\end{aligned}$$

where $\lambda_7 = \left(1 - \frac{p_{1_0} L_2 M_7}{p_1 L_1 M_8}\right)$. This implies that

$$\|Sx_1 - Sx_2\| \leq \lambda_7 \|x_1 - x_2\|.$$

Since $\lambda_7 < 1$, S is a contraction mapping on Ω. Thus S has a unique fixed point which is a positive and bounded solution of (1.1). $\square$

Theorem 2.8. *Assume that* $-\infty < p_{1_0} \leq P_1(t) \leq p_1 < -1$, $p_1 + 1 < p_2 \leq P_2(t) \leq 0$ *and there exist positive constant* N_8 *such that*

$$\int_{t_0}^{\infty} [q_1(s)N_8 + f_1(s)]\, ds < \infty, \quad \int_{t_0}^{\infty} [q_2(s)N_8 + f_2(s)]\, ds < \infty, \tag{2.31}$$

then (1.1) has a bounded nonoscillatory solution.

Proof. Let Λ be the set of all continuous and bounded functions on $[t_0, \infty)$ with the supremum norm. Set

$$\Omega = \{x \in \Lambda : N_7 \leq x(t) \leq N_8,\ t \geq t_0\}.$$

It is clear that Ω is a bounded, closed and convex subset of Λ. Because of (2.31), we can choose a $t_1 > t_0$ sufficiently large satisfying (2.11) such that

$$\int_{t}^{\infty} [q_1(s)N_8 + f_1(s)]\, ds \leq \alpha + p_2 K_2 N_8 + \frac{p_{1_0}}{L_1} N_7,\ t \geq t_1, \tag{2.32}$$

$$\int_{t}^{\infty} [q_2(s)N_8 + f_2(s)]\, ds \leq -\left(1 + \frac{p_1}{L_2}\right) N_8 - \alpha,\ t \geq t_1, \tag{2.33}$$

and

$$\int_{t}^{\infty} [q_1(s) + q_2(s)]\, ds \leq p_2 K_2 + \frac{p_{1_0}}{L_1}\frac{N_7}{N_8} - 1 - \frac{p_1}{L_2},\ t \geq t_1, \tag{2.34}$$

where N_7 and N_8 are positive constants such that

$$-\left(p_2 K_2 N_8 + \frac{p_{1_0}}{L_1} N_7\right) < -\left(1 + \frac{p_1}{L_2}\right) N_8,$$

and

$$\alpha \in \left(-\left(p_2 K_2 N_8 + \frac{p_{1_0}}{L_1} N_7\right), -\left(1 + \frac{p_1}{L_2}\right) N_8\right).$$

Consider the operator $S : \Omega \to \Lambda$ define by

$$(Sx)(t) = \begin{cases} H_1\left(\frac{-1}{P_1(\varphi(t))}\left[\alpha + x(\varphi(t)) + P_2(\varphi(t)) h_2\left(x(\varphi(t) + \tau_2(\varphi(t)))\right)\right.\right. \\ \left.\left. - \int_{\varphi(t)}^{\infty} [g_1(s, x(s - \sigma_1(s))) - g_2(s, x(s + \sigma_2(s)))] ds\right]\right),\ t \geq t_1, \\ (Sx)(t_1),\ t_0 \leq t \leq t_1. \end{cases}$$

Clearly, Sx is continuous. For $t \geq t_1$ and $x \in \Omega$, from (2.32) and (2.33) it follows that

$$\begin{aligned}(Sx)(t) &\leq L_2\left(\frac{-1}{P_1(\varphi(t))}\left[\alpha + x(\varphi(t)) + P_2(\varphi(t))h_2\left(x(\varphi(t) + \tau_2(\varphi(t)))\right)\right.\right.\\&\left.\left.-\int_{\varphi(t)}^{\infty}[g_1\left(s, x(s-\sigma_1(s))\right) - g_2\left(s, x(s+\sigma_2(s))\right)]ds\right]\right)\\&\leq \frac{-L_2}{p_1}\left(\alpha + x(\varphi(t)) + \int_{\varphi(t)}^{\infty}[g_2\left(s, x(s+\sigma_2(s))\right)]ds\right)\\&\leq \frac{-L_2}{p_1}\left(\alpha + x(\varphi(t)) + \int_t^{\infty}\left[q_2(s)x(s-\sigma_1(s)) + f_2(s)\right]ds\right)\\&\leq \frac{-L_2}{p_1}\left(\alpha + N_8 + \int_t^{\infty}\left[q_2(s)N_8 + f_2(s)\right]ds\right) \leq N_8,\end{aligned}$$

and

$$\begin{aligned}(Sx)(t) &\geq L_1\left(\frac{-1}{P_1(\varphi(t))}\left[\alpha + x(\varphi(t)) + P_2(\varphi(t))h_2\left(x(\varphi(t) + \tau_2(\varphi(t)))\right)\right.\right.\\&\left.\left.-\int_{\varphi(t)}^{\infty}[g_1\left(s, x(s-\sigma_1(s))\right) - g_2\left(s, x(s+\sigma_2(s))\right)]ds\right]\right)\\&\geq \frac{-L_1}{p_{1_0}}\left(\alpha + P_2(\varphi(t))h_2\left(x(\varphi(t) + \tau_2(\varphi(t)))\right) - \int_{\varphi(t)}^{\infty} g_1\left(s, x(s-\sigma_1(s))\right)ds\right)\\&\geq \frac{-L_1}{p_{1_0}}\left(\alpha + p_2K_2x(\varphi(t) + \tau_2(\varphi(t))) - \int_t^{\infty}\left[q_1(s)x(s-\sigma_1(s)) + f_1(s)\right]ds\right)\\&\geq \frac{-L_1}{p_{1_0}}\left(\alpha + p_2K_2N_8 - \int_t^{\infty}\left[q_1(s)N_8 + f_1(s)\right]ds\right) \geq N_7.\end{aligned}$$

This means that $S\Omega \subset \Omega$. To apply the contraction mapping principle, the remaining is to show that S is a contraction mapping on Ω. Thus, if $x_1, x_2 \in \Omega$ and $t \geq t_1$, by using (2.34), we can obtain

$$\begin{aligned}|(Sx_1)(t) - (Sx_2)(t)| &\leq \frac{-L_2}{p_1}\left(1 - p_2K_2 + \int_{\varphi(t)}^{\infty}\left[q_1(s) + q_2(s)\right]ds\right)\|x_1 - x_2\|\\&\leq \frac{-L_2}{p_1}\left(1 - p_2K_2 + \int_t^{\infty}\left[q_1(s) + q_2(s)\right]ds\right)\|x_1 - x_2\|\\&\leq \lambda_8\|x_1 - x_2\|,\end{aligned}$$

where $\lambda_8 = \left(1 - \frac{p_{1_0}L_2N_7}{p_1L_1N_8}\right)$. This implies that

$$\|Sx_1 - Sx_2\| \leq \lambda_8\|x_1 - x_2\|.$$

Since $\lambda_8 < 1$, S is a contraction mapping on Ω. Thus S has a unique fixed point which is a positive and bounded solution of (1.1). $\square$

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