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A REVIEW ON RECENT EXTENSIONS OF FRAMES AND WOVEN FRAMES IN HILBERT SPACES

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Abstract. The last two decades have seen tremendous activity in the development of frame theory and many generalizations of frames have come into existence. In this manuscript, we present a short review on some of the newest extensions and generalizations of frames in Hilbert spaces.

1. Introduction

The Hilbert space is the natural framework for the mathematical description of many areas of physics: certainly for quantum mechanics, signal, image analysis and etc. In each cases, there arises the problem of representing an arbitrary vector in terms of simpler ones, i.e., in terms of the elements of some basis $\{f_j\}_{j\in\mathbb{N}}$. The most economical solution, and the one advocated by mathematicians, is of course to use an orthonormal basis, which gives in addition the uniqueness of the decomposition,

$$f = \sum_{j\in\mathbb{N}} \langle f_j, f\rangle f_j$$

for any f in the underlying Hilbert space. Unfortunately, orthonormal bases are often difficult to find and sometimes hard to work with. One way to give up orthogonality of the basis vectors and uniqueness of the decomposition- while maintaining its other useful properties is using the notion of frames. The notion of frames in Hilbert spaces was introduced by Duffin and Schaeffer during their study of nonharmonic Fourier series in 1952.

Discrete frames in Hilbert spaces has been introduced by Duffin and Schaeffer [22] and popularized by Daubechies, Grossmann and Meyer [21]. A discrete frame is a countable family of elements in a separable Hilbert space which allows stable and not necessarily unique decompositions of arbitrary elements in an expansion of frame elements.

The last two decades have seen tremendous activity in the development of frame theory and many generalizations of frames have come into existence. The first author summarized some of that extensions in [38]. In this manuscript,

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we present some of the new extensions and generalizations of frames in Hilbert spaces.

Let $\mathcal{H}$ be a separable Hilbert space with the inner product $\langle\cdot,\cdot\rangle$. Recall that a countable family of elements $\{f_j\}_{j\in J}$ in $\mathcal{H}$ is a frame for $\mathcal{H}$ if there exist constants $A, B > 0$ such that

$$A\|f\|^2 \leq \sum_{j\in J} |\langle f_j, f\rangle|^2 \leq B\|f\|^2, \quad \forall f \in \mathcal{H}. \tag{1.1}$$

The constants A and B are called lower and upper frame bounds, respectively. In case $A = B$, it called a tight frame and if $A = B = 1$ it is known Parseval frame. If the second inequality in (1.1) holds, it is called a Bessel sequence. For a frame $\{f_j\}_{j\in J}$ in $\mathcal{H}$, the operator $Sf = \sum_{j\in J}\langle f_j, f\rangle f_j, f \in \mathcal{H}$ called the frame operator. This operator is a bounded, self-adjoint, invertible, positive operator and any $f \in \mathcal{H}$ has an expansion

$$f = \sum_{j\in J}\langle S^{-1}f_j, f\rangle f_j = \sum_{j\in J}\langle f_j, f\rangle S^{-1}f_j. \tag{1.2}$$

The family $\{S^{-1}f_j\}_{j\in J}$ is also a frame with bounds B^{-1}, A^{-1}, this frame is called the canonical dual or reciprocal frame of $\{f_j\}_{j\in J}$.

For a more complete treatment of frame theory, we recommend the excellent book of Christensen [18], the tutorials of Casazza [11, 12] and the memoir of Han and Larson [31].

We denote by $\mathcal{B}(\mathcal{H})$ the algebra of all bounded linear operators. A bounded operator $T \in \mathcal{B}(\mathcal{H})$ is called positive (respectively, non- negative), if $\langle Tf, f\rangle > 0$ for all $f \neq 0$ (respectively, $\langle Tf, f\rangle \geq 0$ for all f). Every non-negative operator is clearly self-adjoint. If $T \in \mathcal{B}(\mathcal{H})$ is non-negative then there exists a unique non-negative operator S such that $S^2 = T$. This will be denoted by $S = T^{\frac{1}{2}}$. Moreover, if an operator D commutes with T then D commutes with every operator in the C^*-algebra generated by T and I, specially D commutes with $T^{\frac{1}{2}}$. Let $\mathcal{B}^+(\mathcal{H})$ be the set of positive operators on $\mathcal{H}$. For self-adjoint operators T_1 and T_2, the notation $T_1 \leq T_2$ or $T_2 - T_1 \geq 0$ means

$$\langle T_1 f, f\rangle \leq \langle T_2 f, f\rangle, \quad \forall f \in \mathcal{H}.$$

We denote by $\mathcal{GL}(\mathcal{H})$ the set of all bounded linear operators which have bounded inverse. It is easy to see that if $S, T \in \mathcal{GL}(\mathcal{H})$ then T^*, T^{-1} and ST are also in $\mathcal{GL}(\mathcal{H})$. Let $\mathcal{GL}^+(\mathcal{H})$ be the set of all positive operators in $\mathcal{GL}(\mathcal{H})$. For $U \in \mathcal{B}(\mathcal{H})$, $U \in \mathcal{GL}^+(\mathcal{H})$ if and only if there exists positive constants $0 < m \leq M < \infty$ such that

$$mI \leq U \leq MI.$$

For U^{-1},

$$M^{-1}I \leq U^{-1} \leq m^{-1}I.$$

Throughout this paper, $\mathcal{K}_1$, $\mathcal{K}_2$ and $\mathcal{H}$ are complex separable Hilbert spaces, $K \in \mathcal{B}(\mathcal{H})$, $C, C' \in \mathcal{GL}^+(\mathcal{H})$ and $\{\mathcal{H}_j\}_{j=1}^{\infty} \subset \mathcal{K}_1$ and $\{\mathcal{W}_k\}_{k=1}^{\infty} \subset \mathcal{K}_2$ are sequences of closed subspaces.

The following theorem can be found in [37].

Theorem 1.1. *Let $T_1, T_2, T_3 \in \mathcal{L}(\mathcal{H})$ and $T_1 \leq T_2$. Suppose $T_3 \geq 0$ commutes with T_1 and T_2 then $T_1T_3 \leq T_2T_3$.*

In the study of frames, the underlying Hilbert spaces are usually separable Hilbert spaces, however this notion investigated also in non-separable Hilbert spaces (see [9]). In the current manuscript, we consider just separable Hilbert spaces.

2. Recent generalizations of frames

In the last decade, motivated by new applications of frame theory, many generalizations and extensions of the concept of frames introduced in Hilbert and Banach spaces: fusion frames, continuous frames, Banach frames, g-frames, K-frames, controlled frames, operator valued frames, p-frames, pg-frames, frames for Hilbert C^*-modules, Hilbert-Schmidt frames, and etc. Also, by composing some of these generalizations, new families introduced and found its suitable place in both theoretical and applications of frames: continuous fusion frames, continuous g-frames, continuous K-frames, K-fusion frames, g-fusion frames, continuous controlled frames, controlled g-frames, controlled K-frames, t-frames [8] and etc. The authors believe that new extensions of frames will be investigated and will be used in applications.

At the following subsections, we beefily present and review some of the recent extensions and generalizations of the notion of frames in Hilbert spaces.

2.1. Fusion frames. In the modern life's using of frames, many of the applications can not be modeled by one single frame system. They require distributed processing such as sensor networks [35]. To handle such emerging applications of frames, new methods developed. One starting point was to first build frames "locally" and then piece them together to obtain frames for the whole space. So we can first construct frames or choose already known frames for smaller spaces and in the second step one would construct a frame for the whole space from them. Another construction uses subspaces which are quasi-orthogonal to construct local frames and piece them together to get global frames[24]. An elegant approach was introduced in [16] that formulates a general method for piecing together local frames to get global frames. This powerful construction was introduced by Casazza and Kutyniok in [16], named *frames of subspaces* which thereafter they agree on a terminology of *fusion frames.* This notion provides a useful framework in modeling sensor networks [17].

Fusion frames can be regarded as a generalization of conventional frame theory. It turns out that the fusion frame theory is in fact more delicate due to complicated relationships between the structure of the sequence of weighted subspaces and the local frames in the subspaces and due to sensitivity with respect to change of the weights.

Definition 2.1. Let $\mathcal{H}$ be a Hilbert space and I be a (finite or infinite) countable index set. Assume that $\{W_i\}_{i\in I}$ be a sequence of closed subspaces in $\mathcal{H}$ and $\{v_i\}_{i\in I}$ be a family of weights, i.e., $v_i > 0$ for all $i \in I$. We say that the family $\mathcal{W} = \{(W_i, v_i)\}_{i\in I}$ is a *fusion frame* or a *frame of subspaces* with respect to $\{v_i\}_{i\in I}$ for $\mathcal{H}$ if there exist constants $0 < A \leq B < \infty$ such that

$$A\|x\|^2 \leq \sum_{i\in I} v_i^2 \|P_{W_i}(x)\|^2 \leq B\|x\|^2 \quad \forall x \in \mathcal{H},$$

where P_{W_i} denotes the orthogonal projection onto W_i, for each $i \in I$. The fusion frame $\mathcal{W} = \{(W_i, v_i)\}_{i\in I}$ is called *tight* if $A = B$ and *Parseval* if $A = B = 1$. If all v_i*'s* take the same value v, then $\mathcal{W}$ is called *v-uniform*. Moreover, $\mathcal{W}$ is called an *orthonormal fusion basis* for $\mathcal{H}$ if $\mathcal{H} = \bigoplus_{i\in I} W_i$. If $\mathcal{W} = \{(W_i, v_i)\}_{i\in I}$ possesses an upper fusion frame bound but not necessarily a lower bound, we call it a *Bessel fusion sequence* with Bessel fusion bound B. The normalized version of $\mathcal{W}$ is obtained when we choose $v_i = 1$ for all $i \in I$. Note that, we use this term merely when $\{(W_i, 1)\}_{i\in I}$ formes a fusion frame for $\mathcal{H}$.

Without loss of generality, we may assume that the family of weights $\{v_i\}_{i\in I}$ belongs to $\ell^{\infty}_{+}(I)$.

Notation: For any family $\{\mathcal{H}_i\}_{i\in I}$ of Hilbert spaces, we use

$$(\sum_{i\in I} \oplus \mathcal{H}_i)_{\ell_2} = \left\{ \{f_i\}_{i\in I} : f_i \in \mathcal{H}_i, \sum_{i\in I} \|f_i\|^2 < \infty \right\}$$

with inner product

$$\langle \{f_i\}_{i\in I}, \{g_i\}_{i\in I} \rangle = \sum_{i\in I} \langle f_i, g_i \rangle, \quad \{f_i\}_{i\in I}, \{g_i\}_{i\in I} \in (\sum_{i\in I} \oplus \mathcal{H}_i)_{\ell_2}$$

and

$$\|\{f_i\}_{i\in I}\| := \sqrt{\sum_{i\in I} \|f_i\|^2}.$$

It is easy to show that $(\sum_{i\in I} \oplus \mathcal{H}_i)_{\ell_2}$ is a Hilbert space.

Let us state some definitions and propositions needed in the studying of fusion frames.

Definition 2.2. Let $\mathcal{W} = \{(W_i, v_i)\}_{i\in I}$ be a fusion frame for $\mathcal{H}$. The *synthesis operator* $T_{\mathcal{W}} : (\sum_{i\in I} \oplus W_i)_{\ell_2} \to \mathcal{H}$ is defined by

$$T_{\mathcal{W}}(\{f_i\}_{i\in I}) = \sum_{i\in I} v_i f_i, \quad \{f_i\}_{i\in I} \in (\sum_{i\in I} \oplus W_i)_{\ell_2}.$$

In order to map a signal to the representation space, i.e., to analyze it, the *analysis operator* $T^*_{\mathcal{W}}$ is employed which is defined by

$$T^*_{\mathcal{W}} : \mathcal{H} \to (\sum_{i\in I} \oplus W_i)_{\ell_2} \quad with \quad T^*_{\mathcal{W}}(f) = \{v_i P_{W_i}(f)\}_{i\in I},$$

for any $f \in \mathcal{H}$. The *fusion frame operator* $S_{\mathcal{W}}$ for $\mathcal{W}$ is defined by

$$S_{\mathcal{W}}(f) = T_{\mathcal{W}} T^*_{\mathcal{W}}(f) = \sum_{i\in I} v_i^2 P_{W_i}(f), \quad f \in \mathcal{H}.$$

It follows from [16] that for each fusion frame, the operator $S_{\mathcal{W}}$ is invertible, positive and $AI \leq S_{\mathcal{W}} \leq BI$. Any $f \in \mathcal{H}$ has the representation $f = \sum_{i\in I} v_i^2 S^{-1}_{\mathcal{W}} P_{W_i}(f)$.

Proposition 2.1. [16] *Let* $\{W_i\}_{i\in I}$ *be a family of subspaces for* $\mathcal{H}$*. Then the following conditions are equivalent.*

(1) $\{W_i\}_{i\in I}$ *is an orthonormal fusion basis for* $\mathcal{H}$*;*

(2) $\{W_i\}_{i\in I}$ *is a 1-uniform Parseval fusion frame for* $\mathcal{H}$.

Like the concept of Riesz bases, this notion proposed on fusion setting.

Definition 2.3. [16] We call a fusion frame $\mathcal{W} = \{(W_i, v_i)\}_{i\in I}$ for $\mathcal{H}$ a *Riesz decomposition* of $\mathcal{H}$, if every $f \in \mathcal{H}$ has a unique representation $f = \sum_{i\in I} f_i$, $f_i \in W_i$.

Proposition 2.2. [16] *If* $\mathcal{W} = \{(W_i, v_i)\}_{i\in I}$ *is an orthonormal fusion basis for* $\mathcal{H}$*, then it is also a Riesz decomposition of* $\mathcal{H}$.

Definition 2.4. [16] A family of subspaces $\{W_i\}_{i\in I}$ of $\mathcal{H}$ is called *minimal* if for each $i \in I$,

$$W_i \cap \overline{\operatorname{span}}_{j\neq i}\{W_j\}_{j\in I} = \{0\}.$$

Proposition 2.3. [16] *Let* $\mathcal{W} = \{(W_i, v_i)\}_{i\in I}$ *be a fusion frame for* $\mathcal{H}$*. Then the following conditions are equivalent.*

(1) $\mathcal{W} = \{(W_i, v_i)\}_{i\in I}$ *is a Riesz decomposition of* $\mathcal{H}$*;*
(2) $\{W_i\}_{i\in I}$ *is minimal;*
(3) *the synthesis operator is one to one;*
(4) *the analysis operator is onto.*

2.2. g-frames. Sun [45] raised the concept of g-frame and a g-Riesz bases in a Hilbert space and obtained some results for g-frames and g-Riesz bases. He also observed that fusion frames is a particular case of g-frame in a Hilbert space. Also, a system of bounded quasi-projectors introduced by Fornasier [24] is a particular case of g-frame in a Hilbert space. Thought this subsection $\{\mathcal{H}_i, i \in I\}$ is a family of Hilbert spaces.

Definition 2.5. We call $\Lambda = \{\Lambda_i \in \mathcal{B}(\mathcal{H}, \mathcal{H}_i) \ : \ i \in I\}$ a g-frame for $\mathcal{H}$ with respect to $\{\mathcal{H}_i\}_{i\in I}$, or simply, a g-frame for $\mathcal{H}$, if there exist two positive constants A, B such that

$$A\|f\|^2 \le \sum_{i\in I} \|\Lambda_i f\|^2 \le B\|f\|^2, \quad f \in \mathcal{H}.$$

The positive numbers A and B are called the lower and upper g-frame bounds, respectively. We call Λ a tight g-frame if $A = B$ and we call it a Parseval g-frame if $A = B = 1$. If only the second inequality holds, we call it a g-Bessel sequence. If Λ is a g-frame, then the g-frame operator S_Λ is defined by

$$S_\Lambda f = \sum_{i\in I} \Lambda_i^* \Lambda_i f, \quad f \in \mathcal{H}$$

which is a bounded, positive and invertible operator such that

$$AI \le S_\Lambda \le BI$$

and for each $f \in \mathcal{H}$, we have

$$f = S_\Lambda S_\Lambda^{-1} f = S_\Lambda^{-1} S_\Lambda f = \sum_{i\in I} S_\Lambda^{-1} \Lambda_i^* \Lambda_i f = \sum_{i\in I} \Lambda_i^* \Lambda_i S_\Lambda^{-1} f.$$

The canonical dual g-frame for Λ is defined by $\{\Lambda_i S_\Lambda^{-1}\}_{i\in I}$ with bounds $\frac{1}{B}, \frac{1}{A}$. In other words, $\{\Lambda_i S_\Lambda^{-1}\}_{i\in I}$ and $\{\Lambda_i\}_{i\in\Lambda}$ are dual g-frames with respect to each other.

It is easy to show that by letting $\mathcal{H}_i = W_i$, $\Lambda_i = P_{W_i}$ and $v_i = 1$, a fusion is a g-frame.

2.3. t-frames. In [8], B. Bilalov and F. Guliyeva considered the tensor product of Hilbert spaces and the bilinear mapping generated by this product. They introduced the concept of t-frame using the Hilbert-valued scalar product. Theoretically, some facts about t-frames can be established using earlier results for G-frames obtained in [44, 45]. But, the concept of t-frame allows many facts relating to ordinary frames to be extended to the case of t-frame. Also, they studied and proved the stability of t-frameness with respect to quadratic closeness.

Let X and Y be some Hilbert spaces and $Z = X \bigotimes Y$ be their tensor product. Modifying the notations of [8], we have the following definition.

Definition 2.6. System $\{y_n\} \subseteq Y$ is called a t-frame in Z if there exist positive numbers A and B such that

$$A\|z\|_Z^2 \leq \sum_{n=1}^{\infty} \|\langle y_n, z\rangle\|_X^2 \leq B\|z\|_Z^2, \quad z \in Z.$$

Constants A and B are called the bounds of t-frame. A t-frame is called tight if $A = B$.

For detailed studies on t-frames and its stability, we recommend the extensive paper [8].

2.4. K-frames. Atomic systems for subspaces were first introduced by Feichtinger and Werther in [23] based on examples arising in sampling theory. In 2011, Gavruta [28] introduced K-frames in Hilbert spaces to study atomic decomposition systems, and discussed some properties of them. Let K be a linear and bounded operator on $\mathcal{H}$. A family of elements $\{f_j\}_{j\in J}$ in $\mathcal{H}$ is a K-frame for $\mathcal{H}$ if there exist constants $A, B > 0$ such that

$$A\|K^*f\|^2 \leq \sum_{j\in J} |\langle f_j, f\rangle|^2 \leq B\|f\|^2, \quad \forall f \in \mathcal{H}. \tag{2.1}$$

K-frames are limited to the range of a bounded linear operator in Hilbert spaces, that is, they replace the lower bound condition $A\|f\|^2$ of classical frames (2.1) by new lower condition $A\|K^*f\|^2$. In recent years, K-frames have been widely studied in [29] and [49] with a paramount field in frame theory.

2.5. K-g-frames. After introducing g-frames and K-frames, a natural generalization is K-g-frames. In [50], Y. Zhou and Y. Zhu put forward the concept of K-g-frames, which are more general than ordinary g-frames in Hilbert spaces. Naturally, K-g-frames have become one of the most active fields in frame theory in recent years. Like K-frames, K-g-frames are limited to the range of a bounded linear operator in Hilbert spaces and have gained greater flexibility in practical applications relative to g-frames. In [50, 51], several properties and characterizations of K-g-frames were obtained. In [32], the interchangeability of two g-Bessel sequences with respect to a K-g-frame studied.

Definition 2.7. Let $K \in \mathcal{B}(\mathcal{H})$, we call $\Lambda = \{\Lambda_i \in B(\mathcal{H}, \mathcal{H}_i) \ : \ i \in I\}$ a K-g-frame for $\mathcal{H}$ with respect to $\{\mathcal{H}_i\}_{i\in I}$ for $\mathcal{H}$, if there exist two positive constants A, B such that

$$A\|K^* f\|^2 \leq \sum_{i\in I} \|\Lambda_i f\|^2 \leq B\|f\|^2, \quad f \in \mathcal{H}.$$

The positive numbers A and B are called the lower and upper g-frame bounds, respectively. We call Λ a tight K-g-frame. If only the second inequality holds, we call it a K-g-Bessel sequence.

It is immediately follows from the above definition that: Every K-g-frame is a g-Bessel sequence for $\mathcal{H}$ with respect to $\{\mathcal{H}_i\}$ and when $K = I$, K-g-frame is the g-frame. For more details and properties of K-g-frames, see [50, 51, 32, 33].

2.6. Continuous frames. The concept of generalization of frames was proposed by G. Kaiser [34] and independently by Ali, Antoine and Gazeau [2] to a family indexed by some locally compact space endowed with a Radon measure. These frames are known as *continuous frames*. Gabardo and Han in [27] called these frames *frames associated with measurable spaces*, Askari-Hemmat, Dehghan and Radjabalipour in [5] called these frames *generalized frames* and in mathematical physics are referred to *Coherent states*[2]. The strong motivation to study of continuous frames is that the windowed Fourier transform and the continuous wavelet transform are both special cases. The reader is referred to [1, 26, 30] for a detailed account of windowed Fourier transform and wavelet transform. For more studies on continuous frames and its applications, the interested reader can refer to [1, 2, 5, 19, 25, 27, 39]. In this paper, we focus on positive measures and separable complex Hilbert spaces.

Wavelet and Gabor frames are used very often in signal processing algorithms. Both systems are derived from a continuous transform, which can be seen as a continuous frame [1, 26, 30].

Definition 2.8. Let $\mathcal{H}$ be a complex Hilbert space and (Ω, μ) be a measure space with positive measure μ. The mapping $F : \Omega \to \mathcal{H}$ is called a continuous frame with respect to (Ω, μ), if

(1) F is weakly-measurable, i.e., for all $f \in \mathcal{H}$, the function $\omega \to \langle f, F(\omega)\rangle$ is a measurable function on Ω;

(2) there exist constants $A, B > 0$ such that

$$A\|f\|^2 \leq \int_\Omega |\langle f, F(\omega)\rangle|^2 \, d\mu(\omega) \leq B\|f\|^2, \quad f \in \mathcal{H}. \tag{2.2}$$

The constants A and B are called continuous frame bounds. F is called a tight continuous frame if $A = B$ and Parseval if $A = B = 1$. The mapping F is called *Bessel* if the second inequality in (2.2) holds. In this case, B is called the *Bessel constant.*

If μ is counting measure and $\Omega = \mathbb{N}$ then F is a discrete frame. In this sense continuous frames are the more general setting.

The first inequality in (2.2), shows that F is complete, i.e.,

$$\overline{\text{span}}\{F(\omega)\}_{\omega\in\Omega} = \mathcal{H}.$$

Let F be a continuous frame with respect to (Ω, μ), then the mapping

$$S_F f = \int_\Omega \langle f, F(\omega)\rangle F(\omega)\, d\mu(\omega) \quad (f \in \mathcal{H}),$$

which is valid in the weak sense, called the continuous frame operator. The identity $\langle S_F f, f\rangle = \int_\Omega |\langle f, F(\omega)\rangle|^2\, d\mu(\omega)$, shoes that S_F is positive, invertible and $AI \leq S_F \leq BI$, also $\frac{1}{B} I \leq S_F^{-1} \leq \frac{1}{A} I$.

Thus, every $f \in \mathcal{H}$ has the representations

$$f = S_F^{-1} S_F f = \int_\Omega \langle f, F(\omega)\rangle S_F^{-1} F(\omega)\, d\mu(\omega)$$

$$f = S_F S_F^{-1} f = \int_\Omega \langle f, S_F^{-1} F(\omega)\rangle F(\omega)\, d\mu(\omega).$$

Theorem 2.1. [40] *Let (Ω, μ) be a measure space and let F be a Bessel mapping from Ω to $\mathcal{H}$. Then the operator $T_F : L^2(\Omega, \mu) \to \mathcal{H}$ weakly defined by*

$$\langle T_F \varphi, h\rangle = \int_\Omega \varphi(\omega)\langle F(\omega), h\rangle\, d\mu(\omega), \quad h \in \mathcal{H}$$

is well defined, linear, bounded and its adjoint is given by

$$T_F^* : \mathcal{H} \to L^2(\Omega, \mu), \quad (T_F^* h)(\omega) = \langle h, F(\omega)\rangle, \quad \omega \in \Omega.$$

The operator T_F is called the pre-frame operator or synthesis operator and T_F^* is called the analysis operator of F.

It is well known that discrete Bessel sequences in a Hilbert space are norm bounded above: if

$$\sum_n |\langle f, f_n\rangle|^2 \leq B \parallel f \parallel^2$$

for all $f \in \mathcal{H}$, then $\parallel f_n \parallel \leq \sqrt{B}$ for all n. For continuous Bessel mappings, the following example shows that, it is possible to make a continuous Bessel mapping which is unbounded.

Example 2.1. [39] Take an (essentially) unbounded (Lebesgue) measurable function $a : \mathbb{R} \to \mathbb{C}$ such such that $a \in L^2(\mathbb{R}) \setminus L^\infty(\mathbb{R})$. It is easy to see that such functions indeed exist; consider for example the function

$$b(x) := \frac{1}{\sqrt{|x|}}, 0 < |x| < 1, \quad b(x) = \frac{1}{|x|^2}, |x| \geq 1 \quad \textit{and} \quad b(x) = 0, x = 0.$$

This function is clearly in $L^1(\mathbb{R}) \setminus L^\infty(\mathbb{R})$ and furthermore, $b(x) \geq 0$ for all $x \in \mathbb{R}$. Now take $a(x) = \sqrt{b(x)}$. Choose a fixed vector $h \in \mathcal{H}, h \neq 0$. Then, the mapping

$$F : \mathbb{R} \to \mathcal{H}, \omega \mapsto F(\omega) = a(\omega)h$$

is weakly (Lebesgue) measurable and a continuous Bessel mapping, but $\|F(\omega)\|$ is unbounded.

Also, even continuous frames need not necessarily norm bounded, see an example in [39].

For any separable Hilbert space there exists a frame and more generally any separable Banach space can be equipped with a Banach frame with respect to an appropriately chosen sequence space [13]. Concerning the existence of continuous frames, it is natural to ask: dose there exist continuous frames for any Hilbert space and any measure space? The existence of continuous frame depends on the dimension of space and the measure of Ω which is studied in [39]. In that paper, they considered four cases:

- $\mu(\Omega) = \infty$ and $dim\mathcal{H} = \infty$;
- $\mu(\Omega) < \infty$ and $dim\mathcal{H} < \infty$;
- $\mu(\Omega) = \infty$ and $dim\mathcal{H} < \infty$;
- $\mu(\Omega) < \infty$ and $dim\mathcal{H} = \infty$.

2.7. Continuous K-g-frames. The continuous version of K-g-frames have been introduced in [3] in the following way.

Definition 2.9. A family $\Lambda = \{\Lambda_\omega \in B(\mathcal{H}, \mathcal{H}_\omega) : \ \omega \in \Omega\}$ is called a continuous K-g-frame or c-K-g-frame for $\mathcal{H}$ with respect to $\{\mathcal{H}_\omega\}_{\omega\in\Omega}$, if

(i) $\{\Lambda_\omega f\}_{\omega\in\Omega}$ is strongly measurable for each $f \in \mathcal{H}$;
(ii) there exist constants $0 < A \leq B < \infty$ such that

$$A\|K^*f\|^2 \leq \int_\Omega \|\Lambda_\omega f\|^2 \, d\mu(\omega) \leq B\|f\|^2, \ f \in \mathcal{H}. \tag{2.3}$$

The constants A, B are called lower and upper c-K-g-frame bounds, respectively. If A, B can be chosen such that $A = B$, then $\{\Lambda_\omega\}_{\omega\in\Omega}$ is called a tight c-K-g-frame and if $A = B = 1$, it is called Parseval c-K-g-frame. The family $\{\Lambda_\omega\}_{\omega\in\Omega}$ is called a c-g-Bessel family if the right hand inequality in (2.3) holds. In this case, B is called the Bessel constant.

Now, suppose that $\{\Lambda_\omega\}_{\omega\in\Omega}$ is a c-K-g-frame for $\mathcal{H}$ with respect to $\{H_\omega\}_{\omega\in\Omega}$ with frame bounds A, B. The c-K-g-frame operator is defined by

$$S : \mathcal{H} \longrightarrow \mathcal{H}$$

$$\langle Sf, g\rangle = \int_\Omega \langle f, \Lambda_\omega^*\Lambda_\omega g\rangle \, d\mu(\omega), \qquad f, g \in \mathcal{H}.$$

Therefore,

$$AKK^* \leq S \leq BI.$$

Example 2.2. Suppose that $\mathcal{H}$ is an infinite dimensional separable Hilbert space and $\{e_n\}_{n=1}^\infty$ is an orthonormal basis for $\mathcal{H}$. Define the operator $K \in \mathcal{B}(\mathcal{H})$ as follow:

$$Ke_{2n} = e_{2n} + e_{2n-1}; \ Ke_{2n-1} = 0, \ n = 1, 2, \dots .$$

For each $f \in \mathcal{H}$, we have

$$Kf = \sum_{n=1}^{\infty} \langle f, e_{2n}\rangle(e_{2n} + e_{2n-1})$$

and

$$K^*f = \sum_{n=1}^{\infty} \langle f, e_{2n} + e_{2n-1}\rangle e_{2n}.$$

Also,

$$\|K^*f\|^2 \leq \sum_{n=1}^{\infty} |\langle f, e_{2n} + e_{2n-1}\rangle|^2 \leq 4\|f\|^2,$$

that is, $\{f_n\}_{n=1}^{\infty} = \{e_{2n} + e_{2n-1}\}_{n=1}^{\infty}$ is a K-frame for $\mathcal{H}$. Now, let (Ω, μ) be a σ-finite measure space with infinite measure and $\{\mathcal{H}_\omega\}_{\omega\in\Omega}$ be a family of Hilbert spaces. Since Ω is σ-finite, it can be written as a disjoint union $\Omega = \bigcup \Omega_k$ of countably many subsets $\Omega_k \subseteq \Omega$ such that $\mu(\Omega_k) < \infty$ for all $k \in \mathbb{N}$. Without less of generality, assume that $\mu(\Omega_k) > 0$ for all $k \in \mathbb{N}$. For each $\omega \in \Omega$, define the operator $\Lambda_\omega : \mathcal{H} \longrightarrow H_\omega$ by

$$\Lambda_\omega(f) = \frac{1}{\mu(\Omega_k)}\langle f, f_k\rangle h_\omega, \;\; f \in \mathcal{H},$$

where k is such that $\omega \in \Omega_k$ and h_ω is an arbitrary element of $\mathcal{H}_\omega$ such that $\|h_\omega\| = 1$. For each $f \in \mathcal{H}$, $\{\Lambda_\omega f\}_{\omega\in\Omega}$ is strongly measurable (since h_ω's are fixed) and

$$\int_\Omega \|\Lambda_\omega f\|^2 d\mu(\omega) = \sum_{n=1}^{\infty} |\langle f, f_n\rangle|^2.$$

Therefore

$$\|K^*f\|^2 \leq \int_\Omega \|\Lambda_\omega f\|^2 d\mu(\omega) = \sum_{n=1}^{\infty} |\langle f, f_n\rangle|^2 \leq 4\|f\|^2,$$

that is, $\{\Lambda_\omega f\}_{\omega\in\Omega}$ is a c-K-g-frame for $\mathcal{H}$ with respect to $\{\mathcal{H}_\omega\}_{\omega\in\Omega}$.

Remark 2.1. Like K-frame operator, the c-K-g-frame operator is not invertible. In general if K has closed range, then S is invertible on $\mathcal{R}(K)$ and we have (see [49])

$$B^{-1}\|f\|^2 \leq \langle (S|_{\mathcal{R}(K)})^{-1} f, f\rangle \leq A^{-1}\|K^\dagger\|^2\|f\|^2, \quad f \in \mathcal{H}.$$

Theorem 2.2. [3] *Let $K \in \mathcal{B}(\mathcal{H})$. Then the following statements are equivalent.*

(i) *$\{\Lambda_\omega\}_{\omega\in\Omega}$ is a c-K-g-frame for $\mathcal{H}$ with respect to $\{\mathcal{H}_\omega\}_{\omega\in\Omega}$.*

(ii) *$\{\Lambda_\omega\}_{\omega\in\Omega}$ is a c-g-Bessel family for $\mathcal{H}$ with respect to $\{\mathcal{H}_\omega\}_{\omega\in\Omega}$ and there exists a c-g-Bessel family $\{\Gamma_\omega\}_{\omega\in\Omega}$ for $\mathcal{H}$ with respect to $\{\mathcal{H}_\omega\}_{\omega\in\Omega}$ such that*

$$\langle Kf, h\rangle = \int_\Omega \langle \Lambda_\omega^*\Gamma_\omega f, h\rangle d\mu(\omega) \;,\;\; f, h \in \mathcal{H}. \tag{2.4}$$

2.8. Controlled frames. Controlled frames, as one of the newest generalizations of frames, have been introduced to improve the numerical efficiency of iterative algorithms for inverting the frame operator on abstract Hilbert spaces [6], however, they are used earlier just as a tool for spherical wavelets [10]. Since then, controlled frames have been widely studied. In 2016, Hua and Huang [33] introduced (C, C')-controlled K-g-frame as follows: Assume that K, C, C' be linear and bounded operators on $\mathcal{H}$ such that C and C' are positive and have bounded inverse. The family $\{\Lambda_j : \mathcal{H} \to \mathcal{K}_j\}_{j\in J}$ is called (C, C')-controlled K-g-frame for $\mathcal{H}$ with respect to $\{\mathcal{K}_j\}_{j\in J}$, if there exist constants $0 < A \leq B < \infty$ such that

$$A\|K^*f\|^2 \leq \sum_{j\in J} \langle \Lambda_j Cf, \Lambda_j C'f\rangle \leq B\|f\|^2, \;\; f \in \mathcal{H}.$$

2.9. Controlled continuous K-g-frames. In this subsection, the notions of continuous, controlled, K-frames and g-frames composition used under the name continuous (C, C')-controlled K-g-frames. This notion introduced in [42] and some of its properties obtained.

Definition 2.10. [42] Assume that $K \in \mathcal{B}(\mathcal{H})$ and $C, C' \in \mathcal{GL}^+(\mathcal{H})$. We say that $\Lambda = \{\Lambda_\omega \in \mathcal{B}(\mathcal{H}, \mathcal{K}_\omega) : \omega \in \Omega\}$ is a continuous (C, C')-controlled K-g-frame for $\mathcal{H}$ with respect to $\{\mathcal{K}_\omega\}_{\omega\in\Omega}$ if

(1) for each $f \in \mathcal{H}, \{\Lambda_\omega f\}_{\omega\in\Omega}$ is strongly measurable,

(2) there are two constants $0 < A \leq B < \infty$ such that

$$A\|K^* f\|^2 \leq \int_\Omega \langle \Lambda_\omega C f, \Lambda_\omega C' f\rangle \, d\mu_\omega \leq B\|f\|^2, \quad f \in \mathcal{H}. \tag{2.5}$$

We call A, B lower and upper frame bounds for continuous (C, C')-controlled K-g-frame, respectively. If the right-hand side of (2.5) holds then we call $\Lambda = \{\Lambda_\omega \in \mathcal{B}(\mathcal{H}, \mathcal{K}_\omega) : \omega \in \Omega\}$ a continuous (C, C')-controlled g-Bessel family for $\mathcal{H}$ with respect to $\{\mathcal{K}_\omega\}_{\omega\in\Omega}$.
If $K = I$ then we call $\Lambda = \{\Lambda_\omega \in \mathcal{B}(\mathcal{H}, \mathcal{K}_\omega) : \omega \in \Omega\}$ a continuous (C, C')-controlled g-frame for $\mathcal{H}$ with respect to $\{\mathcal{K}_\omega\}_{\omega\in\Omega}$.
If $C' = I$ then we call $\Lambda = \{\Lambda_\omega \in \mathcal{B}(\mathcal{H}, \mathcal{K}_\omega) : \omega \in \Omega\}$ a continuous C-controlled K-g-frame for $\mathcal{H}$ with respect to $\{\mathcal{K}_\omega\}_{\omega\in\Omega}$.
If $C = C' = I$ then we call $\Lambda = \{\Lambda_\omega \in \mathcal{B}(\mathcal{H}, \mathcal{K}_\omega) : \omega \in \Omega\}$ a continuous K-g-frame for $\mathcal{H}$ with respect to $\{\mathcal{K}_\omega\}_{\omega\in\Omega}$.

3. Woven frames

Woven frames in Hilbert spaces were introduced by Bemrose et al. [7, 14, 15] in 2015. After that, Vashisht, Deepshikha, Arefijamaal and etc. have done more studies over the past few years [4, 20, 46, 47]. The frame related operators for this families have been defined [41] and some of the properties of woven frames have been characterized in terms of synthesis, analysis and frame operators of woven frames. For $m \in \mathbb{N}$, we use $[m] := \{1, 2, 3, ..., m\}$.

In the following, we briefly mention definition of woven frames by presenting an example.

Definition 3.1. Let $F = \{f_{ij}\}_{i\in\mathbb{I}}$ for $j \in [m]$ be a family of frames for the separable Hilbert space $\mathcal{H}$. If there exist universal constants C and D, such that for every partition $\{\sigma_j\}_{j\in[m]}$ of $\mathbb{I}$ and for every $j \in [m]$, the family $F_j = \{f_{ij}\}_{i\in\sigma_j}$ is a frame for $\mathcal{H}$ with bounds C and D, then F is said a woven frames. For every $j \in [m]$, the frames $F_j = \{f_{ij}\}_{i\in\sigma_j}$ are said weaving frames.

The constants C and D are called the lower and upper woven frame bounds. If $C = D$, then $F = \{f_{ij}\}_{i\in\mathbb{I},j\in[m]}$ is said a tight woven frame and if for every $j \in [m]$, the family $F_j = \{f_{ij}\}_{i\in\sigma_j}$ is a Bessel sequence, then the family $F = \{f_{ij}\}_{i\in\mathbb{I},j\in[m]}$ is said a Bessel woven.

In the continuation, we dissect woven and weaving frames for $m = 2$.

Example 3.1. Let $\{e_i\}_{i=1}^2$ be an orthonormal basis for the two dimensional vector space $V = \overline{\text{span}}\,\{e_i\}_{i=1}^2$ with inner product and suppose that F and G are the sets:

$$F = \{2e_1, 2e_2 - e_1, 3e_2\} \quad , \quad G = \{2e_1, 2e_1 + e_2, 2e_2\}\,.$$

Since both of F and G span the space V, then those are frames. For obtaining their bounds, we have:

$$\sum_{i=1}^{3} |\langle f, f_i\rangle|^2 \;=\; |\langle f, 2e_1\rangle|^2 + |\langle f, 2e_2 - e_1\rangle|^2 + |\langle f, 3e_2\rangle|^2\,,$$

then

$$4\|f\|^2 \le \sum_{i=1}^{3} |\langle f, f_i\rangle|^2 \le 17\|f\|^2,$$

thus $F = \{f_i\}_{i=1}^3$ is a frame for V with lower bound 4 and upper bound 17. Similarly, $G = \{g_i\}_{i=1}^3$ is a frame for V with frame bounds 4 and 9. The sets F and G are woven frames for V. For example, if $\sigma_1 = \{1, 2\}$, then for every $f \in V$, we have

$$4\,\|f\|^2 \le \sum_{i\in\sigma_1} |\langle f, f_i\rangle|^2 + \sum_{i\in\sigma_1^c} |\langle f, g_i\rangle|^2 \le 12\,\|f\|^2\,.$$

So $\{f_i\}_{i\in\sigma_1} \bigcup \{g_i\}_{i\in\sigma_1^c}$ is a weaving frame with bounds $C_1 = 4$ and $D_1 = 12$. Now, if we take:

$$C = \min\{C_i; \quad 1 \le i \le 8\} \quad , \quad D = \max\{D_i; \quad 1 \le i \le 8\}\,,$$

then F and G form a woven frames for V with universal bounds C and D.

For each family of subspaces $\left\{\left(\ell^2(\mathbb{I})\right)_j\right\}_{j\in[m]}$ of $\ell^2(\mathbb{I})$, we have

$$\left(\ell^2(\mathbb{I})\right)_j = \left\{ \{c_{ij}\}_{i\in\sigma_j} \mid c_{ij} \in \mathbb{C}\ ,\ \sigma_j \subset \mathbb{I},\ \sum_{i\in\sigma_j} |c_i|^2 < \infty \right\} \quad , \quad \forall j \in [m].$$

We define the space:

$$\left(\sum_{j\in[m]} \bigoplus \left(\ell^2(\mathbb{I})\right)_j\right)_{\ell_2} = \left\{ \{c_{ij}\}_{i\in\mathbb{I}, j\in[m]} \mid \{c_{ij}\}_{i\in\mathbb{I}} \in \left(\ell^2(\mathbb{I})\right)_j, \forall j \in [m] \right\},$$

with the inner product

$$\left\langle \{c_{ij}\}_{i\in\mathbb{I}, j\in[m]}\,, \{c'_{ij}\}_{i\in\mathbb{I}, j\in[m]} \right\rangle = \sum_{i\in\mathbb{I}, j\in[m]} \left| c_{ij}\overline{c'_{ij}} \right|,$$

it is easy to show that this space is a Hilbert space.

Theorem 3.1. *The family* $\{f_{ij}\}_{i\in\mathbb{I}, j\in[m]}$ *is a Bessel woven if and only if the operator*

$$T_F : \left(\sum_{j\in[m]} \bigoplus \left(\ell^2(\mathbb{I})\right)_j\right)_{\ell_2} \longrightarrow \mathcal{H} \quad , \quad T_F\,\{c_{ij}\}_{i\in\mathbb{I}, j\in[m]} = \sum_{i\in\mathbb{I}, j\in[m]} c_{ij} f_{ij}$$

is well defined, linear and bounded.

Definition 3.2. Let $F=\{f_{ij}\}_{i\in\mathbb{I},j\in[m]}$ be a woven Bessel. Then for every partition $\{\sigma_j\}_{j\in[m]}$, the family $F_j=\{f_{ij}\}_{i\in\sigma_j}$ for $j\in[m]$ is a Bessel sequence. Therefore, we define the analysis operator of F_j by

$$U_{\sigma_j}:\mathcal{H}\longrightarrow\left(\ell^2(\mathbb{I})\right)_j,\quad U_{\sigma_j}(f)=\{\langle f,f_{ij}\rangle\}_{i\in\sigma_j},\quad \forall j\in[m],f\in\mathcal{H},$$

also $Ran\left(U_{\sigma_j}\right)\subseteq\left(\ell^2(\mathbb{I})\right)_j\subseteq\ell^2(\mathbb{I})$. The adjoint of U_{σ_j} is called the synthesis operator and in this paper, we denote by $T_{\sigma_j}=U^*_{\sigma_j}$. By elementary calculation, for every $j\in[m]$, we have:

$$T_{\sigma_j}:\left(\ell^2(\mathbb{I})\right)_j\longrightarrow\mathcal{H},\quad T_{\sigma_j}\{c_{ij}\}_i=\sum_{i\in\sigma_j}c_{ij}f_{ij},\ \forall\{c_{ij}\}_i\in\left(\ell^2(\mathbb{I})\right)_j.$$

The frame operator of a weaving Bessel is obtained by combination of analysis and synthesis operators. For every $f\in\mathcal{H}$ and $j\in[m]$:

$$S_{\sigma_j}f=T_{\sigma_j}U_{\sigma_j}f=T_{\sigma_j}\{\langle f,f_{ij}\rangle\}_{i\in\sigma_j}=\sum_{i\in\sigma_j}\langle f,f_{ij}\rangle f_{ij}.$$

The operator S_{σ_j} is bounded, self-adjoint and invertible. We call the family $\left\{S^{-1}_{\sigma_j}f_{ij}\right\}_{i\in\sigma_j}$ standard dual weaving frame of F_j. Now, we define the analysis and synthesis operators for the Bessel woven $F=\{f_{ij}\}_{i\in\mathbb{I},j\in[m]}$:

$$U_F:\mathcal{H}\longrightarrow\left(\sum_{j\in[m]}\bigoplus\left(\ell^2(\mathbb{I})\right)_j\right)_{\ell_2},\quad U_F(f)=\{\langle f,f_{ij}\rangle\}_{i\in\mathbb{I},j\in[m]},$$

and

$$T_F:\left(\sum_{j\in[m]}\bigoplus\left(\ell^2(\mathbb{I})\right)_j\right)_{\ell_2}\longrightarrow\mathcal{H},\quad T_F\{c_{ij}\}_{i\in\mathbb{I},j\in[m]}=\sum_{i\in\mathbb{I},j\in[m]}c_{ij}f_{ij}.$$

The operators U_F and T_F are well defined and bounded, they are called analysis and synthesis operators, respectively. Also, by combination of U_F and T_F, the woven frame operator S_F, for all $f\in\mathcal{H}$, is defined by

$$S_F:\mathcal{H}\longrightarrow\mathcal{H},\quad S_Ff=T_FU_Ff=\sum_{i\in\mathbb{I},j\in[m]}\langle f,f_{ij}\rangle f_{ij}.$$

The operator S_F is bounded, linear and self-adjoint operator. Also every $f\in\mathcal{H}$ can be represented as

$$f=\sum_{i\in\mathbb{I},j\in[m]}\left\langle f,S_F^{-1}f_{ij}\right\rangle f_{ij}=\sum_{i\in\mathbb{I},j\in[m]}\langle f,f_{ij}\rangle S_F^{-1}f_{ij}.$$

The family $\left\{S_F^{-1}f_{ij}\right\}_{i\in\mathbb{I},j\in[m]}$ is said the standard dual woven of F.

In the next theorem, we demonstrate that the woven frames are equivalent to boundedness of woven frame operator.

Theorem 3.2. *Let $\{f_{ij}\}_{i\in\mathbb{I},j\in[m]}$ be finite family of Bessel sequences in $\mathcal{H}$. Then the following conditions are equivalent:*

(i) $\{f_{ij}\}_{i\in\mathbb{I},j\in[m]}$ *is woven frames with universal woven frame bounds C and D.*

(ii) *for the operator* $S_F f = \sum_{i\in\mathbb{I},j\in[m]} \langle f, f_{ij}\rangle f_{ij}$, *we have* $CI_{\mathcal{H}} \leq S_F \leq DI_{\mathcal{H}}$.

The next result shows that, we can constitute tight woven frames from every woven frames by weaving operators.

Theorem 3.3. *Let* $F = \{f_{ij}\}_{i\in\mathbb{I},j\in[m]}$ *be woven frame for* $\mathcal{H}$ *with universal woven bounds* C *and* D *and the woven frame operator* S_F. *Then* $\left\{S_F^{-\frac{1}{2}} f_{ij}\right\}_{i\in\mathbb{I},j\in[m]}$ *is a tight woven frame and*

$$f = \sum_{i\in\mathbb{I},j\in[m]} \left\langle f, S_F^{-\frac{1}{2}} f_{ij}\right\rangle S_F^{-\frac{1}{2}} f_{ij} \quad f \in \mathcal{H}.$$

In the following theorem, we investigate the effect of a bounded and invertible operator on woven frames.

Theorem 3.4. *Let* $F = \{f_{ij}\}_{i\in\mathbb{I},j\in[m]}$ *be a woven frame for* $\mathcal{H}$ *with woven frame operator* S_F *and universal bounds* C *and* D *and* $E : \mathcal{H} \longrightarrow \mathcal{H}$ *be a bounded operator. Then the operator* E *is invertible if and only if* $\{Ef_{ij}\}_{i\in\mathbb{I},j\in[m]}$ *is woven frame for* $\mathcal{H}$. *In this case, the universal bounds of* F *are* $C\left\|E^{-1}\right\|^{-2}, D\|E\|^2$ *and the woven frame operator is* ES_FE^*.

From the previous theorem, we can obtain the next result.

Theorem 3.5. *Let* $F = \{f_{ij}\}_{i\in\mathbb{I},j\in[m]}$ *be woven frame with universal woven bounds* C *and* D *for* $\mathcal{H}$ *and woven frame operator* S_F. *Then for every* $\sigma_j \subset \mathbb{I}$, $j \in [m]$, *we have:*

(i) *The sequence* $\left\{S_{\sigma_j} f_{ij}\right\}_{i\in\sigma_j}$ *for every* $j \in [m]$ *is a weaving frame i.e. the family* $\{S_F f_{ij}\}_{i\in\mathbb{I},j\in[m]}$ *is woven frame for* $\mathcal{H}$, *with universal lower and upper woven bounds* C^3 *and* D^3, *respectively.*

(ii) *The sequence* $\left\{S_{\sigma_j}^{-1} f_{ij}\right\}_{i\in\sigma_j}$ *for every* $j \in [m]$ *is a weaving frame i.e. the family* $\left\{S_F^{-1} f_{ij}\right\}_{i\in\mathbb{I},j\in[m]}$ *is a woven frame for* $\mathcal{H}$, *with universal lower and upper woven bounds* $\frac{C}{D^2}$ *and* $\frac{D}{C^2}$, *respectively.*

3.1. Woven-weaving fusion frames. As, we mentioned fusion frames have a lot of applications and all of papers published in this area. Extension of woven-weaving to fusion frames have been investigated in [41]. In this subsection, we review this notion.

Definition 3.3. A family of fusion frames $\{W_{ij}\}_{i=1}^{\infty}$, for $j \in [m]$, with respect to weights $\{\nu_{ij}\}_{i\in\mathbb{I},j\in[m]}$, is said woven fusion frames if there are universal constant $\mathcal{A}$ and $\mathcal{B}$, such that for every partition $\{\sigma_j\}_{j\in[m]}$ of $\mathbb{I}$, the family $\{W_{ij}\}_{i\in\sigma_j,j\in[m]}$ is a fusion frame for $\mathcal{H}$ with lower and upper frame bounds $\mathcal{A}$ and $\mathcal{B}$. Each family $\{W_{ij}\}_{i\in\sigma_j,j\in[m]}$ is called a weaving fusion frame.

For abbreviation, we use W.F.F instead of the statement of woven fusion frame. The following theorem states the equivalence conditions between woven frames and woven fusion frames (W.F.F).

Theorem 3.6. *Suppose for every $i \in \mathbb{I}$, $\mathbb{J}_i$ is a subset of the index set $\mathbb{I}$ and $\nu_i, \mu_i > 0$. Let $\{f_{i,j}\}_{j\in\mathbb{J}_i}$ and $\{g_{i,j}\}_{j\in\mathbb{J}_i}$ be frame sequences in $\mathcal{H}$ with frame bounds $(\mathcal{A}_{f_i}, \mathcal{B}_{f_i})$ and $(\mathcal{A}_{g_i}, \mathcal{B}_{g_i})$ respectively. Define*

$$W_i = \overline{\operatorname{span}}\{f_{i,j}\}_{j\in\mathbb{J}_i}, \quad V_i = \overline{\operatorname{span}}\{g_{i,j}\}_{j\in\mathbb{J}_i}, \quad \forall i \in \mathbb{I},$$

and choose orthonormal bases $\{e_{i,j}\}_{j\in\mathbb{J}_i}$ and $\{e'_{i,j}\}_{j\in\mathbb{J}_i}$ for each subspaces W_i and V_i, respectively. Suppose that

$$0 < \mathcal{A}_f = \inf_{i\in\mathbb{I}} \mathcal{A}_{f_i} \leq \mathcal{B}_f = \sup_{i\in\mathbb{I}} \mathcal{B}_{g_i} < \infty$$

and

$$0 < \mathcal{A}_g = \inf_{i\in\mathbb{I}} \mathcal{A}_{f_i} \leq \mathcal{B}_g = \sup_{i\in\mathbb{I}} \mathcal{B}_{g_i} < \infty.$$

Then the following conditions are equivalent:

(i) $\{\nu_i f_{i,j}\}_{i\in\mathbb{I}, j\in\mathbb{J}_i}$ *and* $\{\mu_i g_{i,j}\}_{i\in\mathbb{I}, j\in\mathbb{J}_i}$ *are woven frames in* $\mathcal{H}$.
(ii) $\{\nu_i e_{i,j}\}_{i\in\mathbb{I}, j\in\mathbb{J}_i}$ *and* $\left\{\mu_i e'_{i,j}\right\}_{i\in\mathbb{I}, j\in\mathbb{J}_i}$ *are woven frames in* $\mathcal{H}$.
(iii) $\{W_i\}_{i\in\mathbb{I}}$ *and* $\{V_i\}_{i\in\mathbb{I}}$ *are* W.F.F *in* $\mathcal{H}$ *with respect to weights* $\{\nu_i\}_{i\in\mathbb{I}}$, $\{\mu_i\}_{i\in\mathbb{I}}$, *respectively.*

Proof. Since for every $i \in \mathbb{I}$, $\{f_{i,j}\}_{j\in\mathbb{J}_i}$ and $\{g_{i,j}\}_{j\in\mathbb{J}_i}$ are frames for W_i and V_i with frame bounds $(\mathcal{A}_{f_i}, \mathcal{B}_{f_i})$ and $(\mathcal{A}_{g_i}, \mathcal{B}_{g_i})$, then for $\sigma \subset \mathbb{I}$;

$$\begin{aligned}
& \mathcal{A}_f \sum_{i\in\sigma} \nu_i^2 \|P_{W_i}(f)\|^2 + \mathcal{A}_g \sum_{i\in\sigma^c} \mu_i^2 \|P_{V_i}(f)\|^2 \\
\leq & \sum_{i\in\sigma} \mathcal{A}_{f_i} \nu_i^2 \|P_{W_i}(f)\|^2 + \sum_{i\in\sigma^c} \mathcal{A}_{g_i} \mu_i^2 \|P_{V_i}(f)\|^2 \\
= & \sum_{i\in\sigma} \mathcal{A}_{f_i} \|\nu_i P_{W_i}(f)\|^2 + \sum_{i\in\sigma^c} \mathcal{A}_{g_i} \|\mu_i P_{V_i}(f)\|^2 \\
\leq & \sum_{i\in\sigma} \sum_{j\in\mathbb{J}_i} |\langle \nu_i P_{W_i}(f), f_{i,j}\rangle|^2 + \sum_{i\in\sigma^c} \sum_{j\in\mathbb{J}_i} |\langle \mu_i P_{V_i}(f), g_{i,j}\rangle|^2 \\
\leq & \sum_{i\in\sigma} \mathcal{B}_{f_i} \|\nu_i P_{W_i}(f)\|^2 + \sum_{i\in\sigma^c} \mathcal{B}_{g_i} \|\mu_i P_{V_i}(f)\|^2 \\
\leq & \mathcal{B}_f \sum_{i\in\sigma} \|\nu_i P_{W_i}(f)\|^2 + \mathcal{B}_g \sum_{i\in\sigma^c} \|\mu_i P_{V_i}(f)\|^2.
\end{aligned}$$

(i) $\Rightarrow$ (iii): Let $\{\nu_i f_{i,j}\}_{i\in\mathbb{I}, j\in\mathbb{J}_i}$ and $\{\mu_i g_{i,j}\}_{i\in\mathbb{I}, j\in\mathbb{J}_i}$ be woven frame for $\mathcal{H}$, with universal frame bounds $\mathcal{C}$ and $\mathcal{D}$. The above calculation shows that for every

$f \in \mathcal{H}$,

$$\begin{aligned} & \sum_{i\in\sigma} \nu_i^2 \|P_{W_i}(f)\|^2 + \sum_{i\in\sigma^c} \mu_i^2 \|P_{V_i}(f)\|^2 \\ \leq\ & \frac{1}{\mathcal{A}} \left(\sum_{i\in\sigma} \sum_{j\in\mathbb{J}_i} |\langle P_{W_i}(f), \nu_i f_{i,j}\rangle|^2 + \sum_{i\in\sigma^c} \sum_{j\in\mathbb{J}_i} |\langle P_{V_i}(f), \mu_i g_{i,j}\rangle|^2 \right) \\ =\ & \frac{1}{\mathcal{A}} \left(\sum_{i\in\sigma} \sum_{j\in\mathbb{J}_i} |\langle f, \nu_i f_{i,j}\rangle|^2 + \sum_{i\in\sigma^c} \sum_{j\in\mathbb{J}_i} |\langle f, \mu_i g_{i,j}\rangle|^2 \right) \\ \leq\ & \frac{\mathcal{D}}{\mathcal{A}} \|f\|^2, \end{aligned}$$

where $\mathcal{A} = \min\{\mathcal{A}_f, \mathcal{A}_g\}$. For lower frame bound,

$$\begin{aligned} & \sum_{i\in\sigma} \nu_i^2 \|P_{W_i}(f)\|^2 + \sum_{i\in\sigma^c} \mu_i^2 \|P_{V_i}(f)\|^2 \\ \geq\ & \frac{1}{\mathcal{B}} \left(\sum_{i\in\sigma} \sum_{j\in\mathbb{J}_i} |\langle P_{W_i}(f), \nu_i f_{i,j}\rangle|^2 + \sum_{i\in\sigma^c} \sum_{j\in\mathbb{J}_i} |\langle P_{V_i}(f), \mu_i g_{i,j}\rangle|^2 \right) \\ =\ & \frac{1}{\mathcal{B}} \left(\sum_{i\in\sigma} \sum_{j\in\mathbb{J}_i} |\langle f, \nu_i f_{i,j}\rangle|^2 + \sum_{i\in\sigma^c} \sum_{j\in\mathbb{J}_i} |\langle f, \mu_i g_{i,j}\rangle|^2 \right) \\ \geq\ & \frac{\mathcal{C}}{\mathcal{B}} \|f\|^2, \end{aligned}$$

for every $f \in \mathcal{H}$, $\mathcal{B} = \max\{\mathcal{B}_f, \mathcal{B}_g\}$. This calculations consequences (iii).

(iii) $\Rightarrow$ (i): Let $\{W_i\}_{i\in\mathbb{I}}$ and $\{V_i\}_{i\in\mathbb{I}}$ be W.F.F with universal frame bounds $\mathcal{C}$ and $\mathcal{D}$. Then for every $f \in \mathcal{H}$, we have

$$\begin{aligned} & \sum_{i\in\sigma} \sum_{j\in\mathbb{J}_i} |\langle f, \nu_i f_{i,j}\rangle|^2 + \sum_{i\in\sigma^c} \sum_{j\in\mathbb{J}_i} |\langle f, \mu_i g_{i,j}\rangle|^2 \\ =\ & \sum_{i\in\sigma} \sum_{j\in\mathbb{J}_i} |\langle \nu_i P_{W_i}(f), f_{i,j}\rangle|^2 + \sum_{i\in\sigma^c} \sum_{j\in\mathbb{J}_i} |\langle \mu_i P_{V_i}(f), g_{i,j}\rangle|^2 \\ \geq\ & \sum_{i\in\sigma} \mathcal{A}_{f_i} \nu_i^2 \|P_{W_i}(f)\|^2 + \sum_{i\in\sigma^c} \mathcal{A}_{g_i} \mu_i^2 \|P_{V_i}(f)\|^2 \\ \geq\ & \mathcal{A} \left(\sum_{i\in\sigma} \nu_i^2 \|P_{W_i}(f)\|^2 + \sum_{i\in\sigma^c} \mu_i^2 \|P_{V_i}(f)\|^2 \right) \\ \geq\ & \mathcal{A}\mathcal{C} \|f\|^2, \end{aligned}$$

and similarly

$$\sum_{i\in\sigma} \sum_{j\in\mathbb{J}_i} |\langle f, \nu_i f_{i,j}\rangle|^2 + \sum_{i\in\sigma^c} \sum_{j\in\mathbb{J}_i} |\langle f, \mu_i g_{i,j}\rangle|^2 \leq \mathcal{B}\mathcal{D} \|f\|^2.$$

So (i) holds.

(ii) ⇔ (iii): Since $\{e_{i,j}\}_{j\in\mathbb{J}_i}$ and $\left\{e'_{i,j}\right\}_{j\in\mathbb{J}_i}$ are orthonormal bases for subspaces W_i and V_i, respectively, then for any $f\in\mathcal{H}$, we have:

$$\begin{aligned}
&\sum_{i\in\sigma}\nu_i^2\|P_{W_i}(f)\|^2+\sum_{i\in\sigma^c}\mu_i^2\|P_{V_i}(f)\|^2\\
=\ &\sum_{i\in\sigma}\nu_i^2\|\sum_{j\in\mathbb{J}}\langle f,e_{i,j}\rangle e_{i,j}\|^2+\sum_{i\in\sigma^c}\mu_i^2\|\sum_{j\in\mathbb{J}}\langle f,e'_{i,j}\rangle e'_{i,j}\|^2\\
=\ &\sum_{i\in\sigma}\nu_i^2\sum_{j\in\mathbb{J}}|\langle f,e_{i,j}\rangle|^2+\sum_{i\in\sigma^c}\mu_i^2\sum_{j\in\mathbb{J}}|\langle f,e'_{i,j}\rangle|^2\\
=\ &\sum_{i\in\sigma}\sum_{j\in\mathbb{J}}|\langle f,\nu_i e_{i,j}\rangle|^2+\sum_{i\in\sigma^c}\sum_{j\in\mathbb{J}}|\langle f,\mu_i e'_{i,j}\rangle|^2.
\end{aligned}$$

So (ii) is equivalent with (iii). □

Combining of Theorem 3.4 and Theorem 3.6, we get the following result.

Theorem 3.7. *Assume that $\{W_i\}_{i\in\mathbb{I}}$ and $\{V_i\}_{i\in\mathbb{I}}$ are fusion frames with weights $\{\mu_i\}_{i\in\mathbb{I}}$ and $\{\nu_i\}_{i\in\mathbb{I}}$ respectively. Also, if $\{W_i\}_{i\in\mathbb{I}}$ and $\{V_i\}_{i\in\mathbb{I}}$ are* W.F.F *and E is a self-adjoint and invertible operator on $\mathcal{H}$, such that $E^*E(W)\subset W$, for every closed subspace W of $\mathcal{H}$. Then for every $\sigma\subset\mathbb{I}$, the sequence $\{EW_i\}_{i\in\sigma}\bigcup\{EV_i\}_{i\in\sigma^c}$ is a fusion frame with frame operator $ES_\sigma E^{-1}$ where S_σ is frame operator of $\{EW_i\}_{i\in\sigma}\bigcup\{EV_i\}_{i\in\sigma^c}$, i.e. $\{EW_i\}_{i\in\mathbb{I}}$ and $\{EV_i\}_{i\in\mathbb{I}}$ are* W.F.F.

3.2. Weaving g-frames. Recently the notions of woven and weaving frames extend to weaving g-frames for Hilbert spaces in [36]. They developed some of the fundamental properties of weaving g-frames for their own sake.

Definition 3.4. A family of g-frames $\{\Lambda_{i,j}\}_{i\in I,j\in[m]}$ for a Hilbert space $\mathcal{H}$ is said to be woven if there are universal constants A and B so that for every partition $\{\sigma_j\}_{j\in[m]}$ of I, the family $\{\Lambda_{i,j}\}_{i\in\sigma_j,j\in[m]}$ is a g-frame for $\mathcal{H}$ with lower and upper frame bounds A and B , respectively.

Also, they showed that:

Proposition 3.1. *Let $\{\Lambda_{i,j}\}_{i\in I,j\in[m]}$ be a woven family of g-frames for $\mathcal{H}$ with common frame bounds A and B. If the operator T has a closed range on $\mathcal{H}$, then $\{\Lambda_{i,j}T\}_{i\in I,j\in[m]}$ is also woven with bounds $A\|T^\dagger\|^{-2}$ and $B\|T\|^2$.*

and

Theorem 3.8. *Let $\{\Lambda_i\}_{i\in\mathbb{N}}$ and $\{\Gamma_i\}_{i\in\mathbb{N}}$ be two g-Riesz bases for which there are common constants $0<A\le B<\infty$ so that for every σ of $\mathbb{N}$, the family $\{\Lambda_i\}_{i\in\sigma}\bigcup\{\Gamma_i\}_{i\in\sigma^c}$ is a g-Riesz sequence with Riesz bounds A and B. Then for every partition σ of $\mathbb{N}$ the family $\{\Lambda_i\}_{i\in\sigma}\bigcup\{\Gamma_i\}_{i\in\sigma^c}$ is actually a g-Riesz basis, that is, the two g-Riesz bases are woven.*

Theorem 3.9. *Let $\Lambda:=\{\Lambda_i\}_{i\in\mathbb{N}}$ be a g-Riesz basis and let $\Gamma:=\{\Gamma_i\}_{i\in\mathbb{N}}$ be a g-frame for $\mathcal{H}$. If Λ and Γ are woven, then Γ must actually be a g-Riesz basis.*

3.3. Controlled weaving K-g-frames. Motivated and inspired by the above mentioned extensions, the concept of controlled weaving K-g-frame introduced in [43]. This notion includes ordinary frame, K-frame, g-frame, controlled frame and weaving frame.

Definition 3.5. Two (C, C')-controlled K-g-frames $\Lambda = \{\Lambda_j\}_{j=1}^{\infty}$ and $\Omega = \{\Omega_k\}_{k=1}^{\infty}$ for $\mathcal{H}$ with respect to $\{\mathcal{H}_j\}_{j=1}^{\infty}$ and $\{\mathcal{W}_k\}_{k=1}^{\infty}$ (respectively) are (C, C')-controlled K-g-woven if there are universal constants $0 < A \leq B < \infty$ so that for every subset σ of $\mathbb{N}$, the family $\{\Lambda_j\}_{j\in\sigma} \cup \{\Omega_k\}_{k\in\sigma^c}$ is a (C, C')-controlled K-g-frame for $\mathcal{H}$ (with respect to $\{\mathcal{H}_j\}_{j\in\sigma} \cup \{\mathcal{W}_k\}_{k\in\sigma^c}$) with lower and upper K-g-frame bounds A and B, respectively.

The following Theorem gives some equivalent conditions which can be useful to characterization of these families.

Theorem 3.10. [43] *Let $\Lambda = \{\Lambda_j\}_{j=1}^{\infty}$ and $\Omega = \{\Omega_j\}_{j=1}^{\infty}$ be sequences of operators such that $\Lambda_j \in \mathcal{B}(\mathcal{H},\ \mathcal{H}_j)$ and $\Omega_j \in \mathcal{B}(\mathcal{H},\ \mathcal{W}_j)$ for all $j \in \mathbb{N}$. Then the following conditions are equivalent.*

(i) *Λ and Ω are (C, C')-controlled K-g-woven frames for $\mathcal{H}$.*

(ii) (a) *There exists $A > 0$ such that for any subset σ of $\mathbb{N}$ there exists a bounded linear operator $U_\sigma : \left(\sum_{n=1}^{\infty} \oplus \mathcal{Z}_n^\sigma\right)_{\ell^2} \to \mathcal{H}$ (here, $\mathcal{Z}_n^\sigma = \mathcal{H}_n$ for $n \in \sigma$ and $\mathcal{Z}_n^\sigma = \mathcal{W}_n$ for $n \in \sigma^c$) such that*

$$U_\sigma\left(\Xi_j^*\Xi_j\{g_n\}_{n=1}^{\infty}\right) = \begin{cases} (CC')^{\frac{1}{2}}\Lambda_j^*(g_j)\ ,\ j \in \sigma \\ \\ (CC')^{\frac{1}{2}}\Omega_j^*(g_j)\ ,\ j \in \sigma^c \end{cases}$$

for all

$$\{g_n\}_{n=1}^{\infty} \in \left(\sum_{n=1}^{\infty} \oplus \mathcal{Z}_n^\sigma\right)_{\ell^2},$$

where $\{\Xi_n\}_{n=1}^{\infty}$ is the standard g-orthonormal basis for

$$\left(\sum_{n=1}^{\infty} \oplus \mathcal{Z}_n^\sigma\right)_{\ell^2}.$$

(b) *$AKK^* \leq U_\sigma U_\sigma^*$.*

An arbitrary (C, C')-controlled K-g-Bessel sequence in $\mathcal{H}$ need not be a (C, C')-controlled K-g- frame for $\mathcal{H}$. The following theorem gives sufficient conditions for (C, C')-controlled K-g-Bessel sequences to constitute woven (C, C')-controlled K-g-frames for the underlying space.

Theorem 3.11. [43] *Let $\Lambda \equiv \{\Lambda_j\}_{j=1}^{\infty}$ and $\Omega \equiv \{\Omega_j\}_{j=1}^{\infty}$ be (C, C')-controlled K-g- Bessel sequences for $\mathcal{H}$ with respect to $\{\mathcal{H}_j\}_{j=1}^{\infty}$ such that for each $f \in \mathcal{H}$,*

$$K(f) = \sum_{i\in\mathbb{N}} C\Lambda_i^*\Omega_i C'(f) \quad \text{and} \quad C\Lambda_i^*\Omega_i C' = C\Omega_i^*\Lambda_i C' \quad i \in \mathbb{N}.$$

Then, Λ and Ω are (C, C')-controlled K-g-woven frames for $\mathcal{H}$.

The next theorem provides a necessary and sufficient condition for (C, C')-controlled K-g-woven frames which connects to ordinary weaving K-frames. More precisely, if frame bounds of frames associated with atomic spaces are positively confined, then (C, C')-controlled K-g-woven frames give ordinary weaving K-frames and vice-versa.

Theorem 3.12. [43] *Suppose that* $\Lambda \equiv \{\Lambda_j\}_{j=1}^{\infty}$ *and* $\Omega \equiv \{\Omega_j\}_{j=1}^{\infty}$ *are* (C, C')*-controlled* K*-g-frames for* $\mathcal{H}$ *with respect to* $\{\mathcal{H}_j\}_{j=1}^{\infty}$ *and* $\{\mathcal{W}_j\}_{j=1}^{\infty}$*, respectively. Let* $\{f_{jk}\}_{k\in I_j\subset\mathbb{N}}$ *and* $\{g_{jk}\}_{k\in Q_j\subset\mathbb{N}}$ *be frames for* $\mathcal{H}_j$ *and* $\mathcal{W}_j$*, respectively* $(j \in \mathbb{N})$ *with frame bounds* α, β *and* α', β'*, respectively. Then the following conditions are equivalent.*

(i) Λ *and* Ω *are* (C, C')*-controlled* K*-g-woven.*

(ii) $\{(CC')^{\frac{1}{2}}\Lambda_j^* f_{jk}\}_{j\in\mathbb{N},k\in I_j}$ *and* $\{(CC')^{\frac{1}{2}}\Omega_j^* g_{jk}\}_{j\in\mathbb{N},k\in Q_j}$ *are woven* K*-frames for* $\mathcal{H}$*.*

The following theorem gives a sufficient condition for weaving K-g-frames in terms of positive operators associated with given K-g frames.

Theorem 3.13. [43] *Let* $\Lambda \equiv \{\Lambda_j\}_{j=1}^{\infty}$ *and* $\Omega \equiv \{\Omega_j\}_{j=1}^{\infty}$ *be* K*-g-frames for* $\mathcal{H}$ *with respect to* $\{\mathcal{H}_j\}_{j=1}^{\infty}$ *and* $\{\mathcal{W}_j\}_{j=1}^{\infty}$*, respectively. For any* $\mathrm{J} \subseteq \mathbb{N}$*, suppose that the operator* $U_{\mathrm{J}} : \mathcal{H} \to \mathcal{H}$ *defined by*

$$U_{\mathrm{J}}(f) = \sum_{i\in\mathrm{J}} [\Omega_i^*\Omega_i(f) - \Lambda_i^*\Lambda_i(f)], \ f \in \mathcal{H},$$

is a positive linear operator. Then Λ *and* Ω *are* K*-g-woven.*

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