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GENERALIZED DIRICHLET PROBLEMS FOR MAGNETIC SCHRÖDINGER OPERATOR

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Abstract. In the paper, we consider different generalizations of the Dirichlet problem in an arbitrary domain of n-dimensional Euclidean space R_n for a magnetic Schrödinger operator. Their equivalence under different conditions on magnetic and electric potentials are proved. Magnetic Sobolev space of first order is introduced and it is proved that this space is topologically equivalent to the ordinary Sobolev space of first order. The interval of a real axis for a spectral parameter, where the Dirichlet first generalized problem has a unique solution, is shown. The Green operator for the Dirichlet first and second generalized problems is introduced and its boundedness from the first order conjugated Sobolev space to the first order Sobolev space is proved.

1. Notation and Problem Statement

By x we will denote points of n-dimensional space R_n. Let $x_k,\ k=1,2,...,n$ stand for the coordinates of x. The element of n-dimensional volume is denoted by dx. Besides, in the sequel, we denote the scalar product by $(.,.)$ and the value of a distribution f at $\varphi(x)$ by $< f,\varphi >$.

Assume

$$H_{a,V}=\sum_{k=1}^{n}\left(\frac{1}{i}\frac{\partial}{\partial x_k}+a_k(x)\right)^2+V(x),$$

where $V(x)$ is an electric potential, $a(x)=(a_1(x),a_2(x),...,a_n(x))$ is a magnetic potential, i is an imaginary unit;

$$A_k=\sup_{x\in G}|a_k(x)|,\quad k=1,2,...,n,$$

where G is an arbitrary domain in R_n;

$$A_0=\max\{A_1,A_2,...,A_n\};$$

$$B_k=\sup_{x\in G}\left|\frac{\partial a_k(x)}{\partial x_k}\right|,\qquad k=1,2,...,n;$$

$$B_0=\max\{B_1,B_2,...,B_n\};$$

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$$a_0 = \inf_{x\in G}\left\{\sum_{k=1}^{n} a_k^2(x)\right\};$$
$$V_0 = \inf_{x\in G} V(x).$$

Let $C_0^\infty(G)$ be the set of all infinitely differentiable functions on G with a compact support and $D'(G)$ be the space of all distributions on G. We denote the functions having in G quadratically integrable generalized derivatives up to order k, by $W_2^k(G)$. The norm in this space is defined as follows:

$$\|f\|^2_{W_2^k(G)} = \sum_{|\alpha|\le k} \|D^\alpha f(x)\|^2_{L_2(G)},$$

where $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n)$ is a multi-index, $|\alpha| = \alpha_1 + \alpha_2 + ... + \alpha_n$,

$$D^\alpha f(x) = \frac{\partial^{|\alpha|} f(x)}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}...\partial x_n^{\alpha_n}}.$$

Set $L_2(G) = W_2^0(G)$. Denote by $\overset{0}{W}{}_2^k(G)$ the closure of the space $C_0^\infty(G)$ in $W_2^k(G)$. More precisely, $f(x) \in \overset{0}{W}{}_2^k(G)$, if there exists a sequence of functions $\{\varphi_n(x)\}_{n=1}^\infty$ from $C_0^\infty(G)$ such that $\varphi_n(x) \to f$ in $W_2^k(G)$. Denote the space associated to $\overset{0}{W}{}_2^k(G)$ by $\overset{0}{W}{}_2^{\prime,k}(G)$.

Let $B^m(G)$ stand for the space of functions $f(x)$ with continuous and bounded in G partial derivatives to order m inclusively.

Definition 1.1. (Dirichlet's first generalized problem). Let G be an arbitrary domain in R_n and $f \in \overset{0}{W}{}_2^{\prime,1}(G)$. The problem of finding of the solution $u(x)$ from the class $\overset{0}{W}{}_2^1(G)$, of the equation

$$\sum_{k=1}^{n}\left(\frac{1}{i}\frac{\partial}{\partial x_k} + a_k(x)\right)^2 u(x) + V(x)u(x) - \lambda u(x) = f(x), \tag{1.1}$$

where λ is a spectral parameter in the sense distributions in the domain G is called *Dirichlet's first generalized problem for the magnetic Schrödinger operator.*

Note that in this problem there are no conditions on behavior of the boundary of the domain G, i.e. on the boundedness of the domain G or smoothness of its boundary.

Definition 1.2. (Dirichlet's second generalized problem). Let

$$f(x) \in \overset{0}{W}{}_2^{\prime,1}(G).$$

The problem of finding of a function from the space $\overset{0}{W}{}_2^1(G)$, satisfying the equality

$$\sum_{k=1}^{n}\int_G\left(\frac{\partial u(x)}{\partial x_k} + ia_k(x)u(x)\right)\left(\frac{\partial\overline{\varphi(x)}}{\partial x_k} - ia_k(x)\overline{\varphi(x)}\right)dx +$$
$$+\int_G (V(x)-\lambda)u(x)\overline{\varphi(x)}dx = < f, \overline{\varphi} > \tag{1.2}$$

for any function $\varphi(x) \in \overset{0}{W}{}_2^1(G)$, is called the *Dirichlet second generalized problem for the magnetic Schrödinger operator.*

Note that the Dirichlet problem for the Laplace equation was first formulated by K. Gauss in 1828. The first studies on solvability of this problem belongs to P. Dirichlet (see [6]). At the early XX century Fredholm [5] proved that for domains with rather smooth boundaries the Dirichlet problem has a unique solution.

The space of functions differentiable in generalized sense was introduced by S.L. Sobolev (see [8]). The imbedding theorems proved by him enabled to study different generalized boundary value problems for differential equations. A full review of results concerning boundary value problems in domains with nonsmooth boundaries, may be found in [6] and [7].

In spite of the fact that the magnetic Schrödinger operator in the whole space R_n is intensively studied (see i.e. [1], [2] and [3]), the Dirichlet generalized problem for magnetic Schrödinger operator arising from variational principles and convenient in use, unlike the ordinary Schrödinger operator was not studied enough. Recently, there appear classical boundary value problems for this operator (see e.g. [4]).

In this paper, we study the existence and uniqueness of solutions of the generalized Dirichlet problems in an arbitrary domain of n-dimensional space R_n for the magnetic Schrödinger operator.

2. On equivalence of both generalized Dirichlet problems for the magnetic Schrödinger operator

Throughout this paper we suppose that the electric and magnetic potentials satisfy the following conditions:

a) the functions $a_k(x)$, $k = 1, 2, ..., n$, are real and belong to the space $B^1(G)$;

b) the function $V(x)$ is real and belongs to the space $B^0(G) \equiv B(G)$.

Theorem 2.1. *Under the conditions a) and b) the first and second generalized Dirichlet problems are equivalent.*

Proof. Let a function $u(x) \in \overset{0}{W}{}_2^1(G)$ be the solution of the first generalized Dirichlet problem. Then for any function $\varphi(x)$ from $C_0^\infty(G)$ the following equality is valid:

$$< \sum_{k=1}^{n} \left(\frac{1}{i} \frac{\partial}{\partial x_k} + a_k(x) \right)^2 u(x) + V(x)\, u(x) - \lambda u(x), \overline{\varphi(x)} >=< f, \overline{\varphi} > . \tag{2.1}$$

Hence, by the definition of generalized derivatives (see e.g. [9]) we have

$$\sum_{k=1}^{n} \int_G \left(\frac{\partial u(x)}{\partial x_k} + i a_k(x)\, u(x) \right) \left(\frac{\partial \overline{\varphi(x)}}{\partial x_k} - i a_k(x)\, \overline{\varphi(x)} \right) dx +$$

$$+ \int_G (V(x) - \lambda)\, u(x)\, \overline{\varphi(x)} dx =< f, \overline{\varphi} > . \tag{2.2}$$

Prove that equality (2.2) is valid for any function $\varphi(x)$ from the space $\overset{0}{W}{}_2^1(G)$ as well. Indeed, by the definition of the space $\overset{0}{W}{}_2^1(G)$ there exists a sequence of functions $\{\varphi_n(x)\}_{n=1}^{\infty}$ from $C_0^{\infty}(G)$ such that $\varphi_n(x) \to \varphi(x)$ in the norm $\|\cdot\|_{W_2^1(G)}$. As $\varphi_n(x) \in C_0^{\infty}(G)$, $n = 1, 2, ...$, then equality (2.2) is valid for them, i.e.

$$\sum_{k=1}^{n} \int_G \left(\frac{\partial u(x)}{\partial x_k} + i a_k(x) u(x) \right) \left(\frac{\partial \overline{\varphi_n(x)}}{\partial x_k} - i a_k(x) \overline{\varphi_n(x)} \right) dx +$$

$$+ \int_G (V(x) - \lambda) u(x) \overline{\varphi_n(x)} dx = < f, \overline{\varphi_n(x)} >, \qquad n = 1, 2, ... \; . \tag{2.3}$$

Taking into account $u(x) \in L_2(G)$, $\frac{\partial u(x)}{\partial x_k} \in L_2(G)$, $k = 1, 2, ..., n$, and passing to the limit $n \to \infty$ in the equality (2.3) we obtain that equality (1.2) is valid for any function $\varphi(x)$ from $\overset{0}{W}{}_2^1(G)$. Conversely, let $u(x) \in \overset{0}{W}{}_2^1(G)$ be the solution of the second generalized Dirichlet problem, i.e. equality (1.2) be fulfilled for any function from $\overset{0}{W}{}_2^1(G)$. Again, using the definition of generalized derivatives we are convinced that the function $u(x)$ is the solution of the first generalized Dirichlet problem. □

3. Magnetic Sobolev space

Assume $\lambda_0 = a_0 - nB_0 - 2nA_0^2 + V_0$ and consider in $C_0^{\infty}(G)$ the functional

$$b_{a,V,\mu}(u) = h_{a,V}(u) - \mu \|u\|_{L_2(G)}^2, \tag{3.1}$$

where $\mu \in (-\infty, \ \lambda_0)$,

$$h_{a,V}(u) = \sum_{k=1}^{n} \int_G \left| \frac{\partial u(x)}{\partial x_k} + i a_k(x) u(x) \right|^2 dx + \int_G V(x) |u(x)|^2 dx.$$

Theorem 3.1. *A quadratic form determined by the formula*

$$\|u\|_{a,V,\mu} = \sqrt{b_{a,V,\mu}(u)},$$

is a norm in $C_0^{\infty}(G)$ and this norm is equivalent to the norm $\|u\|_{W_2^1(G)}$.

Proof. At first we estimate the following functional from below

$$h_{a,0}(u) = \sum_{k=1}^{n} \int_G \left| \frac{\partial u(x)}{\partial x_k} + i a_k(x) u(x) \right|^2 dx, \qquad u \in C_0^{\infty}(G).$$

Let $u(x) = \sigma(x) + i\tau(x) \in C_0^{\infty}(G)$, where $\sigma(x) = Re u(x)$, $\tau(x) = Im u(x)$. Then we have:

$$\left| \tfrac{\partial u(x)}{\partial x_k} + i a_k(x) u(x) \right|^2 = \left(\tfrac{\partial \sigma(x)}{\partial x_k} - a_k(x)\tau(x) \right)^2 + \left(\tfrac{\partial \tau(x)}{\partial x_k} + a_k(x)\sigma(x) \right)^2 =$$

$$= \left| \tfrac{\partial u(x)}{\partial x_k} \right|^2 + a_k^2(x) |u(x)|^2 + 2a_k(x) \left(\sigma(x) \tfrac{\partial \tau(x)}{\partial x_k} - \tau(x) \tfrac{\partial \sigma(x)}{\partial x_k} \right). \tag{3.2}$$

Using equality (3.2), we obtain

$$\left|\frac{\partial u\,(x)}{\partial x_k}+ia_k(x)u(x)\right|^2=\left|\frac{\partial u\,(x)}{\partial x_k}\right|^2+a_k^2(x)\left|u(x)\right|^2+$$
$$+2a_k(x)\frac{\partial\left(\sigma\,(x)\,\tau(x)\right)}{\partial x_k}-4a_k(x)\tau(x)\frac{\partial\sigma\,(x)}{\partial x_k} \tag{3.3}$$

or

$$\left|\frac{\partial u\,(x)}{\partial x_k}+ia_k(x)u(x)\right|^2=\left|\frac{\partial u\,(x)}{\partial x_k}\right|^2+a_k^2(x)\left|u(x)\right|^2-$$
$$-2a_k(x)\frac{\partial\left(\sigma\,(x)\,\tau(x)\right)}{\partial x_k}+4a_k(x)\sigma(x)\frac{\partial\tau\,(x)}{\partial x_k}\,. \tag{3.4}$$

From (3.3) and (3.4) we have

$$\int_G\left|\frac{\partial u\,(x)}{\partial x_k}+ia_k(x)u(x)\right|^2dx=\int_G\left|\frac{\partial u\,(x)}{\partial x_k}\right|^2dx+\int_G a_k^2(x)\left|u(x)\right|^2dx-$$
$$-2\int_G\frac{\partial a_k\,(x)}{\partial x_k}\sigma(x)\tau(x)dx-4\int_G\frac{\partial\sigma(x)}{\partial x_k}a_k\,(x)\,\tau(x)dx \tag{3.5}$$

or

$$\int_G\left|\frac{\partial u\,(x)}{\partial x_k}+ia_k(x)u(x)\right|^2dx=\int_G\left|\frac{\partial u\,(x)}{\partial x_k}\right|^2dx+\int_G a_k^2(x)\left|u(x)\right|^2dx+$$
$$+2\int_G\frac{\partial a_k(x)}{\partial x_k}\sigma(x)\tau(x)dx+4\int_G\frac{\partial\tau\,(x)}{\partial x_k}a_k(x)\sigma(x)dx. \tag{3.6}$$

Using the inequalities

$$\int_G\frac{\partial a_k(x)}{\partial x_k}\sigma(x)\tau(x)dx\le\frac{1}{2}\sup_{x\in G}\left|\frac{\partial a_k(x)}{\partial x_k}\right|\int_G\left|u\,(x)\right|^2dx,$$

$$\int_G a_k(x)\sigma(x)\frac{\partial\tau(x)}{\partial x_k}dx\le\frac{1}{2}\sup_{x\in G}\left|a_k(x)\right|\int_G\left[\frac{1}{\varepsilon}\sigma^2(x)+\varepsilon\left(\frac{\partial\tau(x)}{\partial x_k}\right)^2\right]dx,$$

$$\int_G a_k(x)\tau(x)\frac{\partial\sigma(x)}{\partial x_k}dx\le\frac{1}{2}\sup_{x\in G}\left|a_k(x)\right|\int_G\left[\frac{1}{\varepsilon}\tau^2(x)+\varepsilon\left(\frac{\partial\sigma(x)}{\partial x_k}\right)^2\right]dx,$$

where ε is any positive number, from(3.5) and (3.6) we obtain

$$\int_G\left|\frac{\partial u\,(x)}{\partial x_k}+ia_k(x)u(x)\right|^2dx\ge\int_G\left|\frac{\partial u\,(x)}{\partial x_k}\right|^2dx+\int_G a_k^2(x)\left|u(x)\right|^2dx-$$
$$-\sup_{x\in G}\left|\frac{\partial a_k\,(x)}{\partial x_k}\right|\int_G\left|u(x)\right|^2dx-2\sup_{x\in G}\left|a_k\,(x)\right|\int_G\left[\frac{1}{\varepsilon}\tau^2(x)+\varepsilon\left(\frac{\partial\sigma(x)}{\partial x_k}\right)^2\right]dx \tag{3.7}$$

or

$$\int_G\left|\frac{\partial u\,(x)}{\partial x_k}+ia_k(x)u(x)\right|^2dx\ge\int_G\left|\frac{\partial u\,(x)}{\partial x_k}\right|^2dx+\int_G a_k^2(x)\left|u(x)\right|^2dx-$$
$$-\sup_{x\in G}\left|\frac{\partial a_k\,(x)}{\partial x_k}\right|\int_G\left|u(x)\right|^2dx-2\sup_{x\in G}\left|a_k\,(x)\right|\int_G\left[\frac{1}{\varepsilon}\sigma^2(x)+\varepsilon\left(\frac{\partial\tau(x)}{\partial x_k}\right)^2\right]dx. \tag{3.8}$$

From inequalities (3.7) and (3.8) we have

$$\int_G \left| \frac{\partial u\,(x)}{\partial x_k} + ia_k(x)u(x) \right|^2 dx \geq \int_G \left| \frac{\partial u\,(x)}{\partial x_k} \right|^2 dx + \int_G a_k^2(x)\,|u(x)|^2\,dx-$$

$$-\sup_{x\in G}\left| \frac{\partial a_k\,(x)}{\partial x_k} \right| \int_G |u(x)|^2\,dx - \sup_{x\in G}|a_k\,(x)| \int_G \left[\frac{1}{\varepsilon}\,(u(x))^2 + \varepsilon \left| \frac{\partial u(x)}{\partial x_k} \right|^2 \right] dx. \quad (3.9)$$

Summing the inequality (3.9) over k in the range from 1 to n, we obtain the estimation

$$h_{a,0}(u) \geq \sum_{k=1}^{n} \int_G \left| \frac{\partial u\,(x)}{\partial x_k} \right|^2 dx+$$

$$+\int_G \left(\sum_{k=1}^{n} a_k^2\,(x) \right) |u\,(x)|^2\,dx - \sum_{k=1}^{n} \left\{ \sup_{x\in G} \left| \frac{\partial a_k\,(x)}{\partial x_k} \right| \right\} \int_G |u(x)|^2\,dx-$$

$$-\sum_{k=1}^{n} \left\{ \sup_{x\in G} |a_k\,(x)| \right\} \frac{1}{\varepsilon} \int_G |u(x)|^2\,dx - \varepsilon \sum_{k=1}^{n} \left\{ \sup_{x\in G} |a_k\,(x)| \cdot \int_G \left| \frac{\partial u\,(x)}{\partial x_k} \right|^2 dx \right\}. \quad (3.10)$$

By means of inequality (3.10), we obtain the lower bound for the functional $b_{a,V,\mu}\,(u)$:

$$b_{a,V,\mu}\,(u) \geq (1-\varepsilon A_0) \sum_{k=1}^{n} \int_G \left| \frac{\partial u\,(x)}{\partial x_k} \right|^2 dx+$$

$$+\left(a_0 - nB_0 - nA_0\frac{1}{\varepsilon} + V_0 - \mu \right) \cdot \int_G |u\,(x)|^2\,dx. \quad (3.11)$$

Choosing $\varepsilon = \frac{1}{2A_0}$ in the estimation (3.11), we have

$$b_{a,V,\mu}\,(u) \geq \frac{1}{2} \sum_{k=1}^{n} \int_G \left| \frac{\partial u\,(x)}{\partial x_k} \right|^2 dx + \left(a_0 - nB_0 - 2A_0^2 n + V_0 - \mu \right) \cdot \int_G |u\,(x)|^2\,dx. \quad (3.12)$$

Assuming

$$m_\mu = \min\left\{ \frac{1}{2},\quad a_0 - nB_0 - 2A_0^2 n + V_0 - \mu \right\}$$

in the inequality (3.12), we obtain that for any $u\,(x) \in C_0^\infty(G)$ there is the inequality

$$b_{a,V,\mu}\,(u) \geq m_\mu \,\|u\|^2_{W_2^1(G)}\;. \quad (3.13)$$

From the boundedness of the functions $a_k(x),\ k = 1, 2, ..., n,\quad V\,(x)$ in the domain G it follows that there exists such a positive number M_μ that

$$b_{a,V,\mu}\,(u) \leq M_\mu \,\|u\|^2_{W_2^1(G)}\,,\quad u \in C_0^\infty(G). \quad (3.14)$$

From inequalities (3.13) and (3.14) it follows that the norms $\|u\|_{a,V,\mu} = \sqrt{b_{a,V,\mu}(u)}$ and $\|u\|^2_{W_2^1(G)}$ are equivalent. □

Denote by $W_{a,V,\mu}$ the closure of the space $C_0^{\infty}(G)$ in the norm $\|\cdot\|_{a,V,\mu}$. The Hilbert space $W_{a,V,\mu}$ is called a magnetic Sobolev space of first order.

According to Theorem 3.1, the norms $\|\cdot\|_{a,V,\mu}$ and $\|\cdot\|_{W_2^1(G)}$ are equivalent, therefore the space $W_{a,V,\mu}$ as a topological space coincides with the space $W_2^1(G)$. However, we think they are different as they have various scalar products. But in spite of this, their associated spaces $W^*_{a,V,\mu}$ and $\overset{0}{W}{}_2^{\prime,1}(G)$, as functional spaces, coincide.

4. Existence and uniqueness of the solution of the generalized Dirichlet problem for the magnetic Schrödinger operator for $\mu \in (-\infty,\ \lambda_0)$

Let$f \in W^*_{a,V,\mu}$. By the Riesz theorem (see e.g. [7]) there exists a unique element from the space $W_{a,V,\mu}$ such that $< f,\ \overline{\varphi(x)} >= (u,\ \varphi)_{W_{a,V,\mu}}$ for all $\varphi(x) \in W_{a,V,\mu}$ and

$$\|u\|_{W_{a,V,\mu}} = \|f\|_{W^*_{a,V,\mu}}\,. \tag{4.1}$$

On the other hand, as noted above, $f \in \overset{0}{W}{}_2^{\prime,1}(G)$. Therefore

$$< f,\ \overline{\varphi(x)} >= (u,\ \varphi)_{\overset{0}{W}{}_2^1(G)}\,.$$

Taking into account the definition of the scalar product in $W_{a,V,\mu}$ we obtain that for any $f \in \overset{0}{W}{}_2^{\prime,1}(G)$ there exists a unique element $u\,(x)$ from the space $W_{a,V,\mu}$ such that for any $\varphi\,(x) \in \overset{0}{W}{}_2^1(G)$ the following equality is valid:

$$\begin{gathered}\sum_{k=1}^{n} \int_G \left(\frac{\partial u(x)}{\partial x_k} + i a_k(x) u(x)\right)\left(\frac{\partial \overline{\varphi(x)}}{\partial x} - i a_k(x)\overline{\varphi(x)}\right) dx + \\ + \int_G \left(V\,(x) - \mu\right) u(x)\overline{\varphi(x)} dx = < f,\ \overline{\varphi}\ >,\end{gathered} \tag{4.2}$$

i.e. for any μ from the interval $(-\infty, \lambda_0)$ the following equality is valid

$$< \sum_{k=1}^{n}\left(\frac{1}{i}\frac{\partial}{\partial x_k} + a_k(x)\right)^2 u(x) + V(x)u(x) - \mu\, u(x),\ \ \overline{\varphi\,(x)}\ \ >= < f,\ \overline{\varphi} > . \tag{4.3}$$

According to Theorem 2.1, in view of equality (4.3) we obtain the following statement.

Theorem 4.1. *Let G be an arbitrary domain in R_n and the conditions a) and b) be fulfilled. Then for any number from the interval $(-\infty, \lambda_0)$ the first generalized Dirichlet problem for the magnetic Schrödinger operator has a unique solution.*

From inequality (4.1) and Theorem 3.1 we obtain that if $\mu \in (-\infty, \lambda_0)$, then there exists a positive number L_μ such that for any $f \in \overset{0}{W}{}_2^{\prime,1}(G)$ there exists a unique element $u(x) \in \overset{0}{W}{}_2^1(G)$ such that the following inequality is fulfilled

$$\|u\|_{\overset{0}{W}{}_2^1(G)} \le L_\mu \,\|f\|_{\overset{0}{W}{}_2^{\prime,1}(G)}\,. \tag{4.4}$$

The operator that associates to every functional f from $\overset{0}{W}{}_2^{\prime,1}(G)$ a unique element from the space $\overset{0}{W}{}_2^1(G)$ is denoted by G_μ. The operator G_μ is said to be the Green operator of the first generalized Dirichlet problem for the magnetic Schrödinger operator.

From estimation (4.4) it follows the following theorem.

Theorem 4.2. *Under the conditions of Theorem 4.1, the linear Green operator G_μ is a continuous operator from the space $\overset{0}{W}{}_2^{\prime,1}(G)$ to $\overset{0}{W}{}_2^1(G)$.*

References

[1] A.R. Aliev, E.H. Eyvazov, On the discreteness of the spectrum of the magnetic Schrödinger operator, *Funct. Anal. Appl.* **46** (2012), no. 4, 305-307 (translated from *Funktsional. Anal. i Prilozhen.* **46** (2012), no. 4, 83-85.

[2] A.R. Aliev, E.H. Eyvazov, On the essential spectrum of electromagnetic Schrödinger operator with singular electric potential, *Complex Variables and Elliptic Equations* **59** (2014), no. 1, 18-27.

[3] J. Avron, I. Herbst, B. Simon, Schrödinger operators with magnetic fields. I. General interactions, *Duke Mathematical Journal*, **45** (1978), no. 4, 846-883.

[4] L. Erdös, M. Loss, V. Vougalter, Diamagnetic behavior of sums Dirichlet eigenvalues, *Ann. Inst. Fourier (Grenoble)*, **50** (2000), no. 3, 891-907.

[5] E.I. Fredholm, Sur une nouvelle methode pour la resolution du probleme de Dirichlet, *Kong. Vetenskaps-Akademiens Forth. Stockholm*, 1900, 39-46.

[6] A.K. Gushchin, On the Dirichlet problem for an elliptic equation, *Vestnik Samara State Tech. University*, ser. phys.-math. science **19** (2015), no. 1, 19-43. (in Russian)

[7] S. Mizohata, *The theory of partial differential equations*, Mir, Moscow, 1977, 504 pp. (in Russian)

[8] S.L. Sobolev, *Some applications of functional analysis in mathematical physics*, Nauka, Moscow, 1979, 334 pp. (in Russian)

[9] V.S. Vladimirov, *Generalized functions in mathematical physics*, Nauka, Moscow, 1979, 319 pp. (in Russian)

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