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ON POTENTIAL WELLS AND GLOBAL SOLVABILITY OF THE CAUCHY PROBLEM FOR SYSTEM OF SEMI-LINEAR KLEIN-GORDON EQUATIONS WITH DISSIPATION

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Abstract. We study the Cauchy problem for the system of nonlinear Klein-Gordon equations with weak coupling and dissipative term. By introducing a family of potential wells and investigation of the invariant sets, we prove the global existence and finite time blow up of solution.

1. Introduction

Klein (1927) and Gordon (1926) derived a relativistic equation for a charged particle in an electromagnetic field, using the recently discovered ideas of quanta. In [10] the Klein-Gordon equation is reduced to

$$\frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} - \Delta\psi + \left(\frac{mc}{h}\right)^2\psi = 0$$

for the special case of a free particle in three dimensions. Later this was led to the mathematical generalization

$$\frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} - \Delta\psi + V'(\psi) = 0 \tag{1.1}$$

for some differentiable potential functions V, which is called the nonlinear Klein-Gordon equation [10]. Here V is a nonlinear function which characterizes environment. According to the available literature, there are a lot of interesting results on problem (1.1) by various analytical methods. Many results have been obtained concerning properties of blow up and global existence of solutions to the Cauchy problem of the nonlinear Klein-Gordon equations [2, 7, 8, 9, 11, 12, 14, 15, 16, 17, 18, 19, 21, 22, 23, 24, 25, 27, 29, 31, 34, 35, 36, 37, 38, 39, 40, 41]. The motion of charged mesons in an electromagnetic field can be described by the following coupled Klein-Gordon equations:

$$\begin{cases} u_{tt} - \Delta u + g|v|^{p+1}|u|^{p-1}u = 0 \\ v_{tt} - \Delta v + hg|u|^{p+1}|v|^{p-1}v = 0 \end{cases} \tag{1.2}$$

where Δ is a Laplacian operator on R^n, g and h are non-zero real constants. The system was first introduced by Segal [32].

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In this paper we consider the Cauchy problem for the system of Klein-Gordon equations with weak coupling and dissipations

$$u_{itt} - \Delta u_i + \alpha_i u_i + \gamma_i u_{it} = \sum_{j=1, j\neq i}^{m} |u_j|^{p+1} |u_i|^{p-1} u_i, \quad i = 1, \dots, m, \tag{1.3}$$

$$u_i(0,x) = u_{i0}(x), u_{it}(0,x) = u_{i1}(x), \quad i = 1, \dots, m, \tag{1.4}$$

where $u_1, \dots, u_m$ are real functions depending on $t \in [0, \infty)$, $x \in R^n$, $\alpha_i > 0$, $\gamma_i > 0$,$i = 1, \dots, m$. We suppose that $n \geq 2$ and

$$p > 0 \quad \text{if} \quad n = 2, \tag{1.5}$$

$$0 < p \leq \frac{1}{n-2} \quad \text{if} \quad n \geq 3. \tag{1.6}$$

System (1.3) determines a model of interaction between m fields with masses $\alpha_1, \dots, \alpha_m$.

We study qualitative characteristics of the family of the potential wells, the existence and nonexistence of global solutions of problem (1.3), (1.4).

Notice that, the potential well was introduced by Sattinger [31] in 1968. The potential well method was one of the most important methods for studying nonlinear evolution equations. There was a lot of investigations in this direction [27, 29, 37, 40].

In the case $m = 2$ problem (1.3),(1.4) was studied in [20, 40]. In this direction, we also note the works [2, 3, 4, 24, 26, 30, 34, 38]. The absence of global solutions for problem (1.3),(1.4) was investigated in [2]. The absence of global solutions with positive arbitrary initial energy for the system of semilinear hyperbolic equations

$$u_{itt} - \Delta u_i + \alpha_i u_i + \gamma_i u_{it} = |u_1|^{\rho_{j1}} \dots |u_m|^{\rho_{jm}},$$
$$t > 0, x \in R^n, k = 1, \dots, m$$

was investigated in [5], where $\rho_{jk} = p_j + 1$, $j \neq k$, $\rho_{kk} = p_k - 1$, $k, j = 1, \dots, m$, $p_k > 0$, $\sum\limits_{k=1}^{m} p_k + m - 2 \leq \frac{2}{n-2}$ if $n \geq 3$.

2. Preliminaries

In the sequel, by $|\cdot|_q$ we denote the usual $L_q(R^n)$- norm. For simplicity of notation, in particular, we write $|\cdot|^2$ instead of $|\cdot|_2^2$. Let $\langle \cdot, \cdot \rangle$ denote inner product in $L_2(R^n)$. Let $\|\cdot\|$ denote the norm on the Sobolev space $H^1 = H^1(R^n)$, i.e. $\|u\| = \left[|\nabla u|^2 + |u|^2 \right]^{\frac{1}{2}}$, where ∇ denotes a gradient. For simplicity, in what follows, we set $\alpha_i = \gamma_i = 1$ for $i = 1, \dots, m$ and denote the first and the second derivatives of a function u with respect to t by $\dot{u}$ and $\ddot{u}$, respectively.

We define the following functionals

$$J(\phi_1, \dots, \phi_m) = \frac{1}{2} \sum_{j=1}^{m} \|\phi_j\|^2 - \frac{1}{p+1} G,$$

and

$$I_\delta(\phi_1, \dots, \phi_m) = \delta \sum_{j=1}^{m} \|\phi_j\|^2 - 2G,$$

where $\delta > 0$ and

$$G = G(\phi_1, \ldots, \phi_m) = \sum_{\substack{i,j=1,\\ i<j}}^{m} \int_{R^n} |\phi_i(x)\phi_j(x)|^{p+1} dx.$$

Lemma 2.1. *Let* $(\phi_1, \ldots, \phi_m \in \left[H^1 \backslash \{0\}\right]^m$ *. Then*

(i) $\lim_{\lambda \to 0} J(\lambda\phi_1, \ldots, \lambda\phi_m) = 0, \lim_{\lambda \to +\infty} J(\lambda\phi_1, \ldots, \lambda\phi_m) = -\infty$*;*

(ii) *there is a single point* $\lambda^* = \lambda^*(\phi_1 \ldots, \phi_m)$ *in the interval* $0 < \lambda < +\infty$*, where*
$$\frac{d}{d\lambda} J(\lambda\phi_1, \ldots, \lambda\phi_m)|_{\lambda=\lambda^*} = 0;$$

(iii) $J(\lambda\phi_1, \ldots, \lambda\phi_m)$ *is nondecreasing on* $0 \le \lambda \le \lambda^*$*, nonincreasing on* $\lambda^* \le \lambda < +\infty$ *and it reaches its maximum at the point* $\lambda = \lambda^*$*;*

(iv) $I_1(\lambda\phi_1, \ldots, \lambda\phi_m) > 0$ *for* $0 < \lambda < \lambda^*$ *;*$I_1(\lambda\phi_1, \ldots, \lambda\phi_m) < 0$ *for* $\lambda^* < \lambda < +\infty$ *and* $I_1(\lambda^*\phi_1, \ldots, \lambda^*\phi_m) = 0$.

We define the set

$$N_\delta = \left\{(\phi_1, \ldots, \phi_m) : (\phi_1, \ldots, \phi_m) \in \left[H^1 \backslash \{0\}\right]^m, I_\delta(\phi_1, \ldots, \phi_m) = 0\right\}.$$

Suppose $(\phi_1, \ldots, \phi_m) \in N_1$, then

$$J(\phi_1, \ldots, \phi_m) = \frac{p}{2(p+1)} \sum_{j=1}^{m} \|\phi_j\|^2 > 0, \tag{2.1}$$

i.e. J is bounded from below on the set N_1 .

Consider the variation problem

$$d(\delta) = \inf_{(\phi_1,\ldots,\phi_m) \in N_\delta} J(\phi_1, \ldots, \phi_m), 0 < \delta < p+1. \tag{2.2}$$

Lemma 2.2. *There is* $(\bar{\phi}_1, \ldots, \bar{\phi}_m) \in N_1$ *such that*
$$J(\bar{\phi}_1, \ldots, \bar{\phi}_m) = \inf_{(\phi_1,\ldots,\phi_m) \in N_1} J(\phi_1, \ldots, \phi_m) = d > 0.$$

For $\delta > 0$ we define also

$$r(\delta) = r(\delta, p) = \left(\frac{\delta}{2\mu C_*^{2(p+1)}}\right)^{\frac{1}{p}},$$

where $C_* = \sup_{\|u\| \neq 0} \frac{|u|_{2(p+1)}}{\|u\|}$, μ is the lowest η number satisfy the inequality

$$\sum_{\substack{i,j=1,\\ i<j}}^{m} a_i^{p+1} a_j^{p+1} \le \eta \left[\sum_{j=1}^{m} a_i^2\right]^{p+1}.$$

Here $a_i \ge 0, i = 1, \ldots, m,$ $\sum\limits_{i=1}^{m} a_i^2 > 0$.

Lemma 2.3. *If* $(u_1, \ldots, u_m) \in \left[H^1 \backslash \{0\}\right]^m$ *and* $\sum\limits_{j=1}^{m} \|u_j\|^2 < r(\delta)$*, then* $I_\delta(u_1, \ldots, u_m) > 0$.

Lemma 2.4. *If* $(u_1,\dots,u_m)\in\left[H^1\backslash\{0\}\right]^m$ *and* $I_\delta(u_1,\dots,u_m)<0$, *then* $\sum\limits_{j=1}^m\|u_j\|^2>r(\delta)$.

Lemma 2.5. *If* $(u_1,\dots,u_m)\in\left[H^1\backslash\{0\}\right]^m$ *and* $I_\delta(u_1,\dots,u_m)=0$, *then* $\sum\limits_{j=1}^m\|u_j\|^2\ge r(\delta)$.

Lemma 2.6. *Suppose that conditions* (1.3),(1.4) *are fulfilled. Then*

$$d(\delta)\ge a(\delta)r(\delta), \tag{2.3}$$

where

$$d(\delta)=\delta^{\frac{1}{p}}\frac{p+1-\delta}{p}d, \tag{2.4}$$

$$a(\delta)=\frac{p+1-\delta}{2(p+1)}d. \tag{2.5}$$

It is obvious that

$$\lim_{\delta\to+0}d(\delta)=0, \tag{2.6}$$

$$d(p+1)=0, \tag{2.7}$$

$$d(1)=d, \tag{2.8}$$

$$d'(\delta)>0,\delta\in(0,1), \tag{2.9}$$

$$d'(\delta)<0,\delta\in(1,p+1). \tag{2.10}$$

The following theorem on the local solvability of the problem (1.3), (1.4) holds. This theorem can be proved by using Galerkin method (see[22]) or methods of nonlinear evolution equations (see[5]).

Theorem 2.1. *Let the conditions* (1.5),(1.6) *hold, then for any* $(u_{10},\dots,u_{m0})\in[H^1]^m$, $(u_{11},\dots,u_{m1})\in[L_2(R^n)]^m$ *there exists* $T'\in(0,\infty)$ *such that the problem* (1.3),(1.4) *has a weak solution* $(u_1,\dots,u_m)\in C\left([0,T'];[H^1]^m\right)\bigcap C^1\left([0,T'];[L_2(R^n)]^m\right)$. *If* $T_{\max}=\sup T'$,*i.e.* $T_{\max}$ *the length of the maximal existence interval of the solution* $(u_1(t,\cdot),\dots,u_m(t,\cdot))\in C\left([0,T_{\max});[H^1]^m\right)\bigcap C^1\left([0,T_{\max});[L_2(R^n)]^m\right)$ *then either:* $T_{\max}=+\infty$ *or*
$$\lim_{t\to T_{\max}-0}\sup\sum_{j=1}^m\left[|\dot u_j(t,\cdot)|^2+\|u_j(t,\cdot)\|\right]=+\infty$$

Remark 2.1. 1) If $(u_{10},\dots,u_{m0})\in[H^s]^m$, $(u_{11},\dots,u_{1m})\in[H^{s-1}]^m$, $s\ge1$, then $(u_1,\dots,u_m)\in C\left([0,T_{\max});[H^s]^m\right)\bigcap C^1\left([0,T_{\max});[H^{s-1}]^m\right)$

2) If $\{(u_{10_k},\dots,u_{m0_k})\}_{k=1}^\infty\subset[H^2]^m$, $\{(u_{11_k},\dots,u_{m1_k})\}_{k=1}^\infty\subset[H^1]^m$ and

$$u_{io_k}\longrightarrow u_{i0}\quad\text{in}\quad H^1,\quad u_{i1_k}\longrightarrow u_{i1}\quad\text{in}\quad L_2(R^n),i=1,\dots,m\quad\text{as}\quad k\to+\infty,$$

then for any $T^*\in(0,T_{\max})$

$$(u_{1_k}(t,\cdot),\dots,u_{m_k}(t,\cdot))\longrightarrow(u_1(t,\cdot),\dots,u_m(t,\cdot))$$

in $C\left([0,T^*];[H^1]^m\right)\bigcap C^1\left([0,T^*];[L_2(R^n)]^m\right)$.

3. Principal results

We denote by $E(t)$ the following energy function:

$$E(t) = \frac{1}{2}\sum_{j=1}^{m}\left[|\dot{u}_j(t,\cdot)|^2 + \|u_j(t,\cdot)\|^2\right] - \frac{1}{p+1}\sum_{\substack{i,j=1,\\ i<j}}^{m}|u_iu_j|_{p+1}^{p+1}$$

and we define the following sets

$$\begin{aligned} W_\delta = \big\{&(u_1,\ldots,u_m)\in[H^1]^m, I_\delta(u_1,\ldots,u_m)>0,\\ &J(u_1,\ldots,u_m)<d(\delta)\big\}\bigcup\{0,\ldots,0\}, 0<\delta<p+1,\\ V_\delta = \big\{&(u_1,\ldots,u_m)\in[H^1]^m, I_\delta(u_1,\ldots,u_m)<0,\\ &J(u_1,\ldots,u_m)<d(\delta)\big\}, 0<\delta<p+1.\end{aligned}$$

From (2.6)-(2.10) it follows that for every $e\in(0,d)$ the equation $d(\delta)=e$ has two roots δ_1,δ_2, so that $\delta_1<1<\delta_2$.

Theorem 3.1. *Suppose that* $(u_{10},\ldots,u_{m0})\in[H^1]^m$, $(u_{11},\ldots,u_{1m})\in[L_2(R^n)]^m$, *and conditions* (1.5),(1.6) *hold. If* $0<e<d$ *and* $\delta_1<\delta_2$ *are the roots of the equation* $d(\delta)=e$, *then the following assertions are valid:*

a) *if* $I_1(u_{10},\ldots,u_{m0})>0$ *or* $\|u_{10}\|=\ldots=\|u_{m0}\|=0$, *then all weak solutions* $(u_1(t,\cdot),\ldots,u_m(t,\cdot))$ *of problem* (1.3),(1.4) *with initial energy* $0<E(0)\le e$, *belong to* W_δ, *where* $\delta_1<\delta<\delta_2$;

b) *if* $I_1(u_{10},\ldots,u_{m0})<0$, *then for all weak solutions* $(u_1(t,\cdot),\ldots,u_m(t,\cdot))$ *of problem* (1.3),(1.4) *with initial energy* $0<E(0)\le e$, *belong to* V_δ, *where* $\delta_1<\delta<\delta_2$.

Proof. a) Let $(u_{10},\ldots,u_{m0})\in[H^1]^m$, $(u_{11},\ldots,u_{1m})\in[L_2(R^n)]^m$ and

$$0<E(0)\le e. \tag{3.1}$$

Let

$$I_1(u_{10},\ldots,u_{m0})>0 \quad\text{or}\quad \|u_{10}\|=\ldots=\|u_{m0}\|=0. \tag{3.2}$$

Taking into account the Remark 2.1 from (1.3),(1.4) we have the following energy equality

$$E(t)+\sum_{j=1}^{m}\int_0^t|\dot{u}_j(s,\cdot)|^2ds = E(0), 0\le t<T_{max}. \tag{3.3}$$

By virtue of (3.1) and (3.3), $J(u_1(t,\cdot),\ldots,u_m(t,\cdot))\le e$, $0\le t<T_{max}$. On the other hand for $\delta_1<\delta<\delta_2$ we have $e<d(\delta)$. Therefore

$$J(u_1(t,\cdot),\ldots,u_m(t,\cdot))<d(\delta), 0\le t<T_{max}. \tag{3.4}$$

Suppose that assertion *a*) does not hold. Then in view of (3.2) and (3.4) there exists $\bar{t}\in(0,\infty)$ such that

$$I_\delta(u_1(t,\cdot),\ldots,u_m(t,\cdot))>0, t\in(0,\bar{t}), \tag{3.5}$$

$$I_\delta(u_1(\bar{t},\cdot),\ldots,u_m(\bar{t},\cdot))=0. \tag{3.6}$$

Thus, $(u_1(\bar{t},\cdot),\ldots,u_m(\bar{t},\cdot))\in N_\delta$, therefore, by the definition of $d(\delta)$ we have

$$d(\delta)\le J(u_1(t,\cdot),\ldots,u_m(t,\cdot))$$

which contradicts (3.4).

Now we prove assertion b).

Let $(u_{10},\dots,u_{m0})\in[H^1]^m$, $(u_{11},\dots,u_{1m})\in[L_2(R^n)]^m$, $0<E(0)\le e$ and $I_1(u_{10},\dots,u_{m0})<0$. Arguing in a similar manner as in step a), we obtain the existence of $\bar t\in[0,T]$, such that for any $t\in[0,\bar t)$ the inequality

$$I_1(u_1(t,\cdot),\dots,u_m(t,\cdot))<0,$$

and the equality $I_1(u_1(\bar t,\cdot),\dots,u_m(\bar t,\cdot))=0$ are fulfilled.

We again obtain the following contradiction

$$d(\delta)\le J(u_1(\bar t,\cdot),\dots,u_m(\bar t,\cdot))\le e<d(\delta).$$

□

By Theorem 3.1 we have the following statement

Theorem 3.2. *Suppose that* $(u_{10},\dots,u_{m0})\in[H^1]^m$, $(u_{11},\dots,u_{1m})\in[L_2(R^n)]^m$, *and conditions* (1.5),(1.6) *hold. If* $0<E(0)\le e$ *and* δ_1, δ_2 *are the roots of the equation* $d(\delta)=e$, *then the sets* $W_{\delta_1\delta_2}=\bigcup\limits_{\delta_1<\delta<\delta_2}W_\delta$ *and* $V_{\delta_1\delta_2}=\bigcup\limits_{\delta_1<\delta<\delta_2}V_\delta$ *are invariant on the trajectories of the dynamical system generated by problem* (1.3),(1.4).

The following statement is a consequence of Theorem 3.2 and shows that there is a so-called vacuum zone between the two invariant sets.

Theorem 3.3. *If the assumptions of Theorem 3.2 hold, then all solutions of problem* (1.3),(1.4) *satisfy the relation* $(u_1(t,\cdot),\dots,u_m(t,\cdot))\notin N_{\delta_1\delta_2}=\bigcup\limits_{\delta_1<\delta<\delta_2}N_\delta$.

Now, consider the case $E(0)\le 0$.

Theorem 3.4. *Suppose that* $(u_{10},\dots,u_{m0})\in[H^1]^m$, $(u_{11},\dots,u_{1m})\in[L_2(R^n)]^m$, *and conditions* (1.5),(1.6) *hold. If* $E(0)\le 0$, $\|u_{10}\|+\ldots+\|u_{m0}\|\ne 0$, *then the solution of problem* (1.3),(1.4) *satisfies the inequality*

$$\sum_{j=1}^m\|u_j(t,\cdot)\|^2\ge r_0, \tag{3.7}$$

where $r_0=\left(\frac{p+1}{2\mu C_*^{2(p+1)}}\right)^{\frac1p}$.

Proof. Let $(u_1(t,\cdot),\dots,u_m(t,\cdot))$ be the solution of problem (1.1),(1.2) with initial energy $E(0)\le0$, where $\|u_{10}\|+\ldots+\|u_{m0}\|\ne0$.

Let $[0,T_{\max})$ be the maximum interval of existence of the solution $(u_1(t,\cdot),\dots,u_m(t,\cdot))$. In view of the definition of $E(t)$, we have

$$\begin{aligned}E(t)&=\frac12\sum_{j=1}^m|\dot u_j(t,\cdot)|^2+J(u_1(t,\cdot),\dots,u_m(t,\cdot))\\&=E(0)-\sum_{j=1}^m\int_0^t|\dot u_j(s,\cdot)|^2ds,t\in[0,T_{\max}),\end{aligned}\tag{3.8}$$

It follows that

$$J(u_1(t,\cdot),\dots,u_m(t,\cdot))\le0<d(\delta),t\in[0,T_{\max}) \tag{3.9}$$

and

$$\sum_{j=1}^{m} \|u_i(t,\cdot)\|^2 \leq \frac{2}{p+1} \sum_{\substack{i,j=1,\\ i<j}}^{m} |u_i u_j|_{p+1}^{p+1}. \tag{3.10}$$

On the other hand, using the Hölder's inequality and embedding $H^1 \subset L_{2(p+1)}(R^n)$, also taking into account the definition of μ we have

$$\frac{2}{p+1} \sum_{\substack{i,j=1,\\ i<j}}^{m} |u_i u_j|_{p+1}^{p+1} \leq \frac{2}{p+1} \mu C_*^{2(p+1)} \left[\sum_{j=1}^{m} \|u_j\|^2 \right]^{p+1}. \tag{3.11}$$

Since$\|u_{10}\| + \ldots + \|u_{m0}\| \neq 0$, then there exists a half-interval $[0, t_1)$, where $\|u_1(t,\cdot)\| + \ldots + \|u_m(t,\cdot)\| \neq 0$. Then from (3.10),(3.11) we obtain that

$$\sum_{j=1}^{m} \|u_j(t,\cdot)\|^2 \geq \left(\frac{p+1}{2\mu C_*^{2(p+1)}} \right)^{\frac{1}{p}} = r_0, t \in [0, t_1).$$

It follows that $\|u_1(t_1,\cdot)\| + \ldots + \|u_m(t_1,\cdot)\| \neq 0$, therefore (3.7) is also valid on the half-open interval $[t_1, t_2)$, for some $t_2 > t_1$. Thus, (3.7) is true on $[0, T_{\max})$. □

Theorem 3.5. *Suppose that* $(u_{10}, \ldots, u_{m0}) \in [H^1]^m \backslash \{0, \ldots, 0\}$, $(u_{11}, \ldots, u_{1m}) \in [L_2(R^n)]^m$, *and conditions* (1.5),(1.6) *hold. If* $E(0) < 0$ *or* $E(0) = 0$ *and* $(u_{11}, \ldots, u_{1m}) \neq (0, \ldots, 0)$, *then* $(u_1(t,\cdot), \ldots, u_m(t,\cdot)) \in V_\delta$ *for* $t \in [0, T_{\max})$, *where* $0 < \delta < p+1$.

Proof. If $E(0) < 0$, then from (3.3) we obtain

$$J(u_1(t,\cdot), \ldots, u_m(t,\cdot)) \leq E(0) < 0 < d(\delta). \tag{3.12}$$

On the other hand,

$$J(u_1(t,\cdot), \ldots, u_m(t,\cdot)) = \frac{1}{2} \left(1 - \frac{\delta}{p+1} \right) \sum_{j=1}^{m} \|u_j(t,\cdot)\|^2$$

$$+ \frac{1}{2(p+1)} I_\delta(u_1(t,\cdot), \ldots, u_m(t,\cdot)), \tag{3.13}$$

therefore,

$$I_\delta(u_1(t,\cdot), \ldots, u_m(t,\cdot)) < 0, t \in [0, T_{\max}) \quad \text{if} \quad 0 < \delta < p+1. \tag{3.14}$$

If $E(0) = 0$, then in view of Theorem 3.4, from (3.7),(3.12) we find that inequality (3.14) is also true in this case if $0 < \delta < p+1$.

Thus $(u_1(t,\cdot), \ldots, u_m(t,\cdot)) \in V_\delta$, where $0 < \delta < p+1$. □

Theorems 3.3-3.5 imply the following result.

Theorem 3.6. *If* $E(0) < d$, *then* W_1 *and* V_1 *are invariant with respect to the dynamical system generated by problem* (1.3),(1.4).

4. Existence of global solutions

Theorem 3.6 implies the following theorem on global solvability:

Theorem 4.1. *Suppose that* $(u_{10},\ldots,u_{m0}) \in [H^1]^m$, $(u_{11},\ldots,u_{1m}) \in [L_2(R^n)]^m$, $E(0) < d$ *and conditions* (1.3),(1.4) *hold. If* $(u_1(t_0,\cdot),\ldots,u_m(t_0,\cdot)) \in W_1$ *at some moment of time* $t_0 \in [0,T_{\max})$ *, then* $T_{\max} = +\infty$ *and* $(u_1(t,\cdot),\ldots,u_m(t,\cdot))$ *satisfies a priori estimate*

$$\sum_{j=1}^{m}\left[|\dot{u}_j(t,\cdot)|^2 + \|u_j(t,\cdot)\|^2\right] \le \frac{2d(p+1)}{p}, t \in [0,T_{\max}). \tag{4.1}$$

Proof. By Theorem 3.1 $(u_1(t,\cdot),\ldots,u_m(t,\cdot)) \in W_1, t \in [0,T_{\max})$, therefore $I_1(u_1(t,\cdot),\ldots,u_m(t,\cdot)) > 0, 0 \le t < T_{\max}$. Then from (3.8) and (3.13) we obtain that for $0 \le t < T_{\max}$ the apriori estimate (4.1) holds, therefore $T_{\max} = +\infty$, i.e. problem (1.3),(1.4) has a global solution. □

Theorem 4.1 implies the following statement:

Theorem 4.2. *Suppose that* $(u_{10},\ldots,u_{m0}) \in [H^1]^m$, $(u_{11},\ldots,u_{1m}) \in [L_2(R^n)]^m$, *and conditions* (1.3),(1.4) *hold. If* $0 < E(0) < d$ *and* $I_{\delta_2}(u_{10},\ldots,u_{m0}) > 0$ *or* $\|u_{10}\| = \ldots = \|u_{m0}\| = 0$, *where* $\delta_1 < \delta_2$ *are the roots of the equation* $d(\delta) = E(0)$, *then problem* (1.3),(1.4) *has a unique solution* $(u_1(\cdot),\ldots,u_m(\cdot)) \in C\left([0,\infty);[H^1]^m\right)$
$\bigcap C^1\left([0,\infty);[L_2(R^n)]^m\right)$ *and* $(u_1(t,\cdot),\ldots,u_m(t,\cdot)) \in W_\delta$, $\delta_1 < \delta < \delta_2$, $0 \le t < +\infty$.

Proof. By virtue of Theorem 3.1 and 4.1, it suffices to prove that $I_1(u_{10},\ldots,u_{m0}) > 0$. Indeed, otherwise there would exist $\bar{\delta} \in [1,\delta_2)$ such that $I_{\bar{\delta}}(u_{10},\ldots,u_{m0}) = 0$. Since $I_{\delta_2}(u_{10},\ldots,u_{m0}) > 0$, therefore $\|u_{10}\| + \ldots + \|u_{m0}\| \ne 0$. Then $J(u_{10},\ldots,u_{m0}) \ge d(\delta)$, which contradicts the inequality $J(u_{10},\ldots,u_{m0}) \le E(0) < d(\delta)$, for $\delta_1 < \delta < \delta_2$. □

5. The absence of global solutions

In this part we consider the case of $\alpha_j = 1$ and $\gamma_j = \gamma, j = 1,2,\ldots,m$

Theorem 5.1. *Suppose that* $p \ge 1$, $(u_{10},\ldots,u_{m0}) \in [H^s]^m$ *and* $(u_{11},\ldots,u_{1m}) \in [H^{s-1}]^m$ *, where* $s > \frac{n}{2}$. *Suppose also that conditions* (1.5),(1.6) *and one of the following conditions hold:*

a) $E(0) < 0$;
b) $0 \le E(0) < d$, $I(u_{10},\ldots,u_{m0}) < 0$, $\alpha_j = 1$, $\gamma_j = \gamma, j = 1,2,\ldots,m$ *and* $0 \le \gamma < \frac{p-1}{2C_*}$,

Then $T_{\max} < +\infty$ *and* $\lim\limits_{t\to T_{\max}-0} \sum\limits_{j=1}^{m} |u_j(t,\cdot)|^2 = +\infty$.

Proof. *a*) If $E(0) < 0$, then repeating the proof given in [4], we obtain the assertion of the theorem.

b) Let $0 \le E(0) < d$, $I_1(u_{10},\ldots,u_{m0}) < 0$ and $0 \le \gamma < \frac{p-1}{2C_*}$.

Let's denote

$$F(t)=\sum_{j=1}^{m}|u_j(t,\cdot)|^2, t\in[0,T_{\max}).$$

Then we obtain

$$\dot F(t)=2\sum_{j=1}^{m}\langle u_j(t,\cdot),\dot u_j(t,\cdot)\rangle, t\in[0,T_{\max}). \tag{5.1}$$

Assume that the assertion of Theorem 4.1 is not true, i.e. $T_{\max}=+\infty$. Since $(u_{10},\dots,u_{m0})\in[H^s]^m$ and $(u_{11},\dots,u_{1m})\in[H^{s-1}]^m$, where $s>\frac{n}{2}$,
then $(u_1(t,x),\dots,u_m(t,x))\in C\left([0,\infty);[H^s]^m\right)\bigcap C^1\left([0,\infty);[H^{s-1}]^m\right)$ and obviously $\ddot F(t)\in C[0,\infty)$. Taking into account (1.3), by a simple calculation we obtain

$$\frac{d}{dt}[e^{\gamma t}\dot F(t)]=$$

$$=2e^{\gamma t}\sum_{j=1}^{m}|\dot u_j(t,\cdot)|^2+2(\delta-1)e^{\gamma t}\sum_{j=1}^{m}\|u_j(t_j)\|^2$$

$$-2e^{\gamma t}I_\delta(u_1(t,\cdot),\dots,u_m(t,\cdot)). \tag{5.2}$$

Since $E(0)<d$, therefore there exists δ_1,δ_2 such that $\delta_1<1<\delta_2$ and

$$d(\delta_i)=E(0), i=1,2.$$

In (3.4), we put $\delta_1=\delta_2$. According to Theorem 3.5

$$I_{\delta_2}(u_1(t,\cdot),\dots,u_m(t,\cdot))\le 0, \tag{5.3}$$

therefore, from (5.2),(5.3) we get

$$\frac{d}{dt}[e^{\gamma t}\dot F(t)]\ge 2(\delta-1)e^{\gamma t}\sum_{j=1}^{m}\|u_j(t,\cdot)\|^2. \tag{5.4}$$

On the other hand, applying Lemma 2.4, we have the following estimation

$$\sum_{j=1}^{m}\|u_j(t,\cdot)\|^2>r(\delta_2). \tag{5.5}$$

From (5.3) and (5.5) it follows that

$$\frac{d}{dt}[e^{\gamma t}\dot F(t)]\ge e^{\gamma t}c(\delta_2), \tag{5.6}$$

where $c(\delta_2)=2(\delta_2-1)r(\delta_2)$. From (5.6), we find that for sufficiently large t_0

$$\dot F(t)\ge\frac{c(\delta_2)}{2\lambda}, t\ge t_0. \tag{5.7}$$

Thus,

$$\lim_{t\to+\infty}F(t)=+\infty. \tag{5.8}$$

On the other hand,

$$\ddot F(t)=2\sum_{j=1}^{m}\left[|\dot u_j(t,\cdot)|^2-\|u_j(t,\cdot)\|^2\right]-2\gamma\sum_{j=1}^{m}\langle u_j(t,\cdot),\dot u_j(t,\cdot)\rangle$$

$$+2\cdot\sum_{\substack{i,j=1,\\ i<j}}^{m}|u_iu_j|_{p+1}^{p+1}=(p+3)\sum_{j=1}^{m}|\dot u_j(t,\cdot)|^2+(p-1)\sum_{j=1}^{m}\|u_j(t,\cdot)\|^2$$

$$-2\gamma\left\langle u_j(t,\cdot),\dot u_j(t,\cdot)\right\rangle+2\gamma(p+1)\sum_{j=1}^{m}\int_0^t|\dot u_j(s,\cdot)|^2ds-2(p+1)E(0). \qquad (5.9)$$

Using the Hölder and Young inequalities, we have

$$\left|2\gamma\sum_{j=1}^{m}\left\langle u_j(t,\cdot),\dot u_j(t,\cdot)\right\rangle\right|\le(p-1-\varepsilon)\sum_{j=1}^{m}|\dot u_j(t,\cdot)|^2$$

$$+\frac{4\gamma^2}{p-1-\varepsilon}\sum_{j=1}^{m}|u_j(t,\cdot)|^2. \qquad (5.10)$$

When using the imbedding $H^1\subset L_2(R^n)$ from (5.8) and (5.10), we get

$$\ddot F(t)\ge(4+\varepsilon)\sum_{j=1}^{m}|\dot u_j(t,\cdot)|^2+\psi(t), \qquad (5.11)$$

where

$$\psi(t)=\left(p-1-\frac{4\gamma^2C_*^2}{p-1-\varepsilon}\right)\sum_{j=1}^{m}\|u_j(t,\cdot)\|^2-2(p+1)E(0).$$

By virtue of (5.8), for sufficiently small $\varepsilon>0$,there exists t_1 such that for any $t\ge t_1$ we have the estimation

$$\ddot F(t)\ge(4+\varepsilon)\sum_{j=1}^{m}|\dot u_j(t,\cdot)|^2. \qquad (5.12)$$

It follows from (5.1) and (5.12)

$$\ddot F(t)F(t)-\left(1+\frac{\varepsilon}{4}\right)\dot F^2(t)\ge(4+\varepsilon)\sum_{j=1}^{m}|\dot u_j(t,\cdot)|^2\sum_{j=1}^{m}|u_j(t,\cdot)|^2$$

$$-4\left(1+\frac{\varepsilon}{4}\right)\left[\sum_{j=1}^{m}\left\langle u_j(t,\cdot),\dot u_j(t,\cdot)\right\rangle\right]^2,\quad t\ge t_1.$$

Using Hölder's inequality, we obtain

$$\ddot F(t)F(t)-\left(1+\frac{\varepsilon}{4}\right)\dot F^2(t)\ge 0,\quad t\ge t_1. \qquad (5.13)$$

From (5.11) and (5.13) we have the following inequality

$$\left(F^{-\frac{\varepsilon}{4}}(t)\right)''\le 0,\quad t\ge t_1, \qquad (5.14)$$

From (5.7) it follows that

$$\left(F^{-\frac{\varepsilon}{4}}(t)\right)'=-\frac{\varepsilon F'(t)}{4\cdot F^{1+\frac{\varepsilon}{4}}(t)}<0,\quad t\ge t_1. \qquad (5.15)$$

In view of (5.14) and (5.15) there exists $t^* \in (0, t_1)$ such that $\lim_{t \to t^*} F^{-1}(t) = 0$, i.e. $\lim_{t \to t^*} F(t) = +\infty$. This contradiction shows that our assumption is not true. □

Remark 5.1. Under the assumptions of Theorem 5.1

$$\lim_{t \to T_{\max}-0} \sum_{j=1}^{m} \left[\|u_j(t,\cdot)\|^2 + |\dot{u}_j(t,\cdot)|^2 \right] = +\infty.$$

6. Proofs of Lemmas

Proof of Lemma 2.1. Properties of (i) follow directly from

$$J(\lambda\phi_1, \dots, \lambda\phi_m) = \lambda^2 \frac{1}{2} \sum_{j=1}^{m} \left(|\nabla\phi_j|^2 + |\phi_j|^2 \right)$$

$$-\lambda^{2p+2} \frac{1}{p+1} \sum_{\substack{i,j=1,\\ i<j}}^{m} |\phi_i\phi_j|_{p+1}^{p+1};$$

(ii) Elementary computation shows that

$$\frac{d}{d\lambda} J(\lambda\phi_1, \dots, \lambda\phi_m)$$

$$= \lambda \sum_{j=1}^{m} \|\phi_j\|^2 - 2\lambda^{2p+1} \sum_{\substack{i,j=1,\\ i<j}}^{m} \int_{R^n} |\phi_i\phi_j|^{p+1} dx \tag{6.1}$$

Hence, it is evident that at the point

$$\lambda^* = \left| \frac{\sum\limits_{j=1}^{m} \|\phi_j\|^2}{2 \sum\limits_{\substack{i,j=1,\\ i<j}}^{m} \int\limits_{R^n} |\phi_i\phi_j|^{p+1} dx} \right|^{\frac{1}{2p}}$$

the following equality holds

$$\frac{d}{d\lambda} J(\lambda\phi_1, \dots, \lambda\phi_m)|_{\lambda=\lambda^*} = 0.$$

(iii) From (6.1) it is clear that

$$\frac{d}{d\lambda} J(\lambda\phi_1, \dots, \lambda\phi_m) > 0 \quad \text{for} \quad 0 < \lambda < \lambda^*,$$

and

$$\frac{d}{d\lambda} J(\lambda\phi_1, \dots, \lambda\phi_m) < 0 \quad \text{for} \quad \lambda^* < \lambda < +\infty,$$

i.e. the assertion (iii) is true.

(iv) From definitions of the functionals J and I_1 and also from (6.1) it follows that

$$I_1(\lambda\phi_1, \dots, \lambda\phi_m) = \lambda \frac{d}{d\lambda} J(\lambda\phi_1, \dots, \lambda\phi_m).$$

Proof of Lemma 2.2. From (2.1) it follows that if $(u_1,\dots,u_m)\in N_1$, then

$$J(u_1,\dots,u_m)=\frac{p}{2(p+1)}\sum_{j=1}^{m}\|u_j\|^2. \tag{6.2}$$

Let $(u_{1k},\dots,u_{mk})$ be a minimizing sequence, i.e.

$$\lim_{k\to\infty}J(u_{1k},\dots,u_{mk})=\inf_{(u_1,\dots,u_m)\in N_1}J(u_1,\dots,u_m)=d.$$

Let us denote $u_{j\lambda}=\lambda u_j$, $j=1,\dots,m$ and denote by $\nu_{jk}=(u^*_{jk})_{\mu k}$ the Schwartz symmetrization [6, 13, 28] of the function $y_{jk}=\mu_k u_{jk}$ with respect to the variable x, where μ_k is chosen so that $(\nu_{1k},\dots,\nu_{mk})\in N_1$. By virtue of (6.2)

$$J(\nu_{1k},\dots,\nu_{mk})=\frac{p}{2(p+1)}\sum_{j=1}^{m}\|\nu_{jk}\|^2. \tag{6.3}$$

On the other hand,

$$\int\limits_{R^n}\|\nabla\nu_{jk}\|^2dx=\int\limits_{R^n}\|\nabla(u^*_{jk})_{\mu k}\|^2dx$$

$$=\int\limits_{R^n}\|(\nabla(u_{jk})_{\mu k})^*\|^2dx\le\int\limits_{R^n}\|\nabla(u_{jk})_{\mu k}\|^2dx \tag{6.4}$$

(see [6, 13, 28]).

From (6.3), (6.4) it follows that

$$J(\nu_{1k},\dots,\nu_{mk})\le J((u_{1k})_{\mu k},\dots,(u_{mk})_{\mu k}). \tag{6.5}$$

On the other hand, according to the choice of

$$J((u_{1k})_{\mu k},\dots,(u_{mk})_{\mu k})\le J(u_{1k},\dots,u_{mk}). \tag{6.6}$$

Consequently, $\lim\limits_{k\to\infty}J(\nu_{1k},\dots,\nu_{mk})=d$. It follows that

$$\|\nabla\nu_{jk}\|\le c, \tag{6.7}$$

where $c>0$ is a constant not dependent on $k=1,2,\dots$. Then we conclude that there exists such a $(\nu_{1\infty},\dots,\nu_{m\infty})\in[H^1]^m$ that, possibly taking along a subsequence,

$$\nu_{jk}\to\nu_{j\infty}\quad\text{weakly in}\quad H^1\quad\text{as}\quad k\to+\infty, j=1,\dots,m. \tag{6.8}$$

Then by virtue of the compactness of embedding $H^1_{radial}\subset L_q(R^n)$ [33], where $q\le\frac{2n}{n-2}$, it follows that

$$\nu_{jk}\to\nu_{j\infty}\quad\text{in}\quad L_q(R^n)\quad\text{as}\quad k\to+\infty, j=1,\dots,m. \tag{6.9}$$

Let us prove that $(\nu_{1\infty},\dots,\nu_{m\infty})\ne(0,\dots,0)$. Assume the opposite, i.e.

$$(\nu_{1\infty},\dots,\nu_{m\infty})=(0,\dots,0). \tag{6.10}$$

Using the Hölder inequality

$$\sum_{j=1}^{m}\|\nu_{jk}\|^2=$$

$$= 2 \sum_{\substack{i,j=1, \\ i<j}}^{m} \int_{R^n} |\nu_{ik}|^{p+1} \cdot |\nu_{jk}|^{p+1} dx \leq 2 \sum_{\substack{i,j=1, \\ i<j}}^{m} \|\nu_{ik}\|^{p+1}_{L_{2(p+1)}} \cdot \|\nu_{jk}\|^{p+1}_{L_{2(p+1)}}, \tag{6.11}$$

from (1.6), (6.9), (6.10) we obtain

$$G(\nu_{1k}, \dots, \nu_{mk}) \to 0, \quad \text{as} \;\; k \to +\infty.$$

On the other hand, $I(\nu_{1k}, \dots, \nu_{mk}) = 0$, so from (6.9) it follows that

$$\nu_{jk} \to 0 \quad \text{star in} \quad H^1 \;\; \text{as} \;\; k \to +\infty, j = 1, \dots, m. \tag{6.12}$$

By the multiplicative inequality of Gagliardo-Nirenberg type [1], we have

$$\|\nu_{jk}\|^{p+1}_{L_{2(p+1)}(R^n)} \leq |\nabla \nu_{jk}|^{(p+1)\theta} |\nu_{jk}|^{(p+1)(1-\theta)}, \tag{6.13}$$

where

$$\theta = n \left(\frac{1}{2} - \frac{1}{2(p+1)} \right) = \frac{np}{2(p+1)}, j = 1, \dots, m.$$

From (6.7) and (6.13) we have

$$\|\nu_{jk}\|^{p+1}_{L_{2(p+1)}(R^n)} \leq c |\nabla \nu_{jk}|^{(p+1)\theta}, j = 1, \dots, m.$$

Therefore by virtue of (6.11) and (6.14) we have

$$\sum_{j=1}^{m} \|\nu_{jk}\|^2 \leq C \left(\sum_{j=1}^{m} \|\nu_{jk}\|^2 \right)^{\frac{np}{2}},$$

where $C > 0$ is a constant not dependent on $k = 1, 2, \dots$. It follows that

$$\sum_{j=1}^{m} \|\nu_{jk}\|^2 \geq c_1 > 0,$$

where $c_1 = C^{\frac{-2}{np-2}}$. Therefore, our assumption isn't correct. Thus $d > 0$.

Proof of Lemma 2.3. It follows from (3.11) that

$$G \leq \sum_{\substack{i,j=1, \\ i<j}}^{m} \|u_i\|^{p+1}_{2(p+1)} \cdot \|u_j\|^{p+1}_{2(p+1)} \leq \mu C_*^{2(p+1)} \left[\sum_{j=1}^{m} \|u_j\|^2 \right]^{p+1}.$$

If $\sum\limits_{j=1}^{m} \|u_j\|^2 < r(\delta)$, then we get $2G \leq \delta \sum\limits_{j=1}^{m} \|u_j\|^2$. From the definition of $I_\delta(u_1, \dots, u_m)$, we have $I_\delta(u_1, \dots, u_m) > 0$.

Proof of Lemma 2.4. If $(u_1, \dots, u_m) \in \left[H^1\right]^m$, $\|u_{10}\| \neq 0, \dots, \|u_{m0}\| \neq 0$ and $I_\delta(u_1, \dots, u_m) < 0$, then we have the following inequality

$$\delta \sum_{j=1}^{m} \|u_j\|^2 < 2 \sum_{\substack{i,j=1, \\ i<j}}^{m} |u_i u_j|^{p+1}_{p+1} \leq 2\mu C_*^{2(p+1)} \left[\sum_{j=1}^{m} \|u_j\|^2 \right]^{p+1}$$

from which the required inequality is obtained.

Proof of Lemma 2.5. If $\|u_{10}\| + \ldots + \|u_{m0}\| \neq 0$, then from $I_\delta(u_1, \ldots, u_m) = 0$ we get

$$\delta \sum_{j=1}^{m} \|u_j\|^2 = 2 \sum_{\substack{i,j=1,\\ i<j}} |u_i u_j|_{(p+1)}^{p+1} \leq 2\mu C_*^{2(p+1)} \left[\sum_{j=1}^{m} \|u_j\|^2\right]^{p+1}$$

Thus,

$$\sum_{j=1}^{m} \|u_j\|^2 \geq r(\delta) = \left(\frac{\delta}{2\mu C_*^{2(p+1)}}\right)^{\frac{1}{p}}.$$

Proof of Lemma 2.6. In view of Lemma 2.5, for each $(u_1, \ldots, u_m) \in N_1$ we have

$$\sum_{j=1}^{m} \|u_j\|^2 \geq r(\delta).$$

Therefore,

$$J(u_1, \ldots, u_m) = \frac{1}{2}\left(1 - \frac{\delta}{p+1}\right) \sum_{j=1}^{m} \|u_j\|^2 \geq a(\delta) r(\delta),$$

where $0 < \delta < p+1$, $a(\delta) = \frac{p+1-\delta}{2(p+1)} d$. It follows that $d(\delta) = a(\delta) r(\delta)$. Suppose that $(\bar{u}_1, \ldots, \bar{u}_m) \in N_1$ is a minimizing element, i.e. $d = J(\bar{u}_1, \ldots, \bar{u}_m)$.

For any $\delta > 0$, $\lambda = \lambda(\delta)$ is chosen so that

$$\delta \sum_{j=1}^{m} \|\lambda \bar{u}_j\|^2 = 2 \sum_{\substack{i,j=1,\\ i<j}}^{m} |\lambda \bar{u}_i \lambda \bar{u}_j|_{p+1}^{p+1}. \tag{6.14}$$

Hence we obtain that

$$\lambda(\delta) = \left[\frac{\delta \sum\limits_{j=1}^{m} \|\bar{u}_j\|^2}{2 \sum\limits_{\substack{i,j=1,\\ i<j}} |\bar{u}_i \bar{u}_j|_{p+1}^{p+1}}\right]^{\frac{1}{2p}} = \delta^{\frac{1}{2p}}.$$

In view of (6.14), $(\lambda(\delta)\bar{u}_1, \ldots, \lambda(\delta)\bar{u}_m) \in N_\delta$, therefore, by definition of $d(\delta)$, we have the following inequality

$$d(\delta) \leq J(\lambda(\delta)\bar{u}_1, \ldots, \lambda(\delta)\bar{u}_m) = \frac{1}{2}\delta^{\frac{1}{p}} \sum_{j=1}^{m} \|\bar{u}_j\|^2$$

$$-\frac{1}{p+1}\delta^{\frac{p+1}{p}} \sum_{\substack{i,j=1,\\ i<j}}^{m} |\bar{u}_i \bar{u}_j|_{p+1}^{p+1}. \tag{6.15}$$

On the other hand,

$$(\bar{u}_1, \ldots, \bar{u}_m) \in N_1. \tag{6.16}$$

Therefore,

$$2\sum_{\substack{i,j=1,\\ i<j}}^{m} |\bar{u}_i\bar{u}_j|_{p+1}^{p+1} = \sum_{j=1}^{m} \|\bar{u}_j\|^2. \tag{6.17}$$

It follows from (6.15) and (6.17) that

$$d(\delta) \le \frac{1}{2}\delta^{\frac{1}{p}}\left(1 - \frac{\delta}{p+1}\right)\sum_{j=1}^{m} \|\bar{u}_j\|^2. \tag{6.18}$$

Since $(\bar{u}_1, \dots, \bar{u}_m)$ is the minimizing element, we have

$$d = J(\bar{u}_1, \dots, \bar{u}_m) = \frac{p}{2(p+1)}\sum_{j=1}^{m} \|\bar{u}_j\|^2$$

i.e.

$$\sum_{j=1}^{m} \|\bar{u}_j\|^2 = \frac{2(p+1)}{p} d. \tag{6.19}$$

It follows from (6.18) and (6.19) that

$$d(\delta) \le \frac{1}{2}\delta^{\frac{1}{p}}\left(1 - \frac{\delta}{p+1}\right)\frac{2(p+1)}{p} d = \frac{p+1-\delta}{p}\delta^{\frac{1}{p}} d. \tag{6.20}$$

Let $(\bar{\nu}_1, \dots, \bar{\nu}_m) \in N_\delta$ be the minimizing element of the functional $J(u_1, \dots, u_m)$, i.e.

$$J(\bar{\nu}_1, \dots, \bar{\nu}_m) = \min_{(\nu_1,\dots,\nu_m)\in N_\delta} J(\nu_1, \dots, \nu_m) = d(\delta).$$

The parameter $\mu = \mu(\delta)$ is chosen so that

$$(\mu\bar{\nu}_1, \dots, \mu\bar{\nu}_m) \in N_1, \quad \text{i.e.} \quad I_1(\mu\bar{\nu}_1, \dots, \mu\bar{\nu}_m) = 0 \tag{6.21}$$

Then,

$$\mu = \mu(\delta) = \left[\frac{\sum\limits_{j=1}^{m} \|\bar{\nu}_j\|^2}{2\sum\limits_{\substack{i,j=1,\\ i<j}}^{m} |\bar{\nu}_i\bar{\nu}_j|_{(p+1)}^{p+1}}\right]^{\frac{1}{2p}} = \left(\frac{1}{\delta}\right)^{\frac{1}{2p}}.$$

By the definition of d, we have

$$d \le J(\mu\bar{\nu}_1, \dots, \mu\bar{\nu}_m) = \frac{1}{2}\left(\frac{1}{\delta}\right)^{\frac{1}{p}}\sum_{j=1}^{m} \|\bar{\nu}_j\|^2$$

$$-\frac{1}{p+1}\left(\frac{1}{\delta}\right)^{\frac{p+1}{p}}\sum_{\substack{i,j=1,\\ i<j}}^{m} |\bar{\nu}_i\bar{\nu}_j|_{(p+1)}^{p+1} = \left(\frac{1}{\delta}\right)^{\frac{1}{p}}\frac{p}{2(p+1)}\sum_{j=1}^{m} \|\bar{\nu}_j\|^2. \tag{6.22}$$

On the other hand,

$$J(\bar{\nu}_1, \dots, \bar{\nu}_m) = \left(\frac{1}{2} - \frac{\delta}{2(p+1)}\right)\sum_{j=1}^{m} \|\bar{\nu}_j\|^2.$$

Hence we have

$$\sum_{j=1}^{m} \|\bar{\nu}_j\|^2 = \frac{2(p+1)}{p+1-\delta} J(\bar{\nu}_1, \dots, \bar{\nu}_m) = \frac{2(p+1)}{p+1-\delta} \cdot d(\delta). \tag{6.23}$$

From (6.22) and (6.23) it follows that

$$d \le \left(\frac{1}{\delta}\right)^{\frac{1}{p}} \frac{p}{p+1-\delta} \cdot d(\delta),$$

i.e.

$$d(\delta) \ge \frac{p+1-\delta}{p} \delta^{\frac{1}{p}} d. \tag{6.24}$$

Comparing (6.20) and (6.24), we obtain that

$$d(\delta) = \frac{p+1-\delta}{p} \delta^{\frac{1}{p}} d.$$

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References

[1] R. A. Adams, *Sobolev Spaces*, Academic Press, New York, 1975.

[2] A. B. Aliev, A. A. Kazimov, Nonexistence of global solutions of the Cauchy Problem for systems of Klein-Gordon equations with positive initial energy, *Difererential Equations* **51** (2015), no.12, 1563–1568.

[3] A. B. Aliev, A. A. Kazimov, The Existence and Nonexistence of Global Solutions of the Cauchy Problem for Klein-Gordon Systems, *Dokl. Math.* **87** (2013), no.1, 39–41.

[4] A. B. Aliev, A. A. Kazimov, V. F. Guliyeva, Global existence and nonexistence results for a class of semilinear hyperbolic systems, *Math. Meth. Appl. Sci* **36** (2013), no.9, 1133–1144.

[5] A. B. Aliev, G. I. Yusifova, Nonexistence of global solutions of Cauchy problems for systems of semilinear hyperbolic equations with positive initial energy, *Electronic Journal of Differential Equations* 2017, Paper No. 211, 10 pp.

[6] A. Alvino, P. L. Lions, G. Trombetti, Comparison results for elliptic and parabolic via Schwarz symmetrization, *Annales de I'I.H.P., section C* **7** (1990), no.2, 37–65.

[7] B. A. Bilgin, V. K. Kalantarov, Blow up of solutions to the initial boundary value problem for quasilinear strongly damped wave equations. *J. Math. Anal. Appl.* **403** (2013), 89–94.

[8] T. Cazenave, Uniform estimates for solutions of nonlinear Klein-Gordon equations. *J. Funct. Anal.* **60** (1985), 36–55.

[9] S. Cocco, M. Barbu, M. Peyard, Vector nonlinear Klein-Gordon pattices: General derivation os small amplitude envelope solution solutions. *Physics Letters A.* **253** (1999), 161–167.

[10] P. J. Drazin, R. S. Johnson, *Solitons: An Introduction*, Cambridge University Press, **XII**, 1989, 226 pp.

[11] F. Gazzola, M. Squassina, Global solutions and finite time blow up for damped semilinear wave equations. *Ann. Inst. Henri Poincare, Anal. Non Lincaire* **23** (2006), 185–207.

[12] B. R. Ikehata, Y. Miataka, Y. Nakataka, Decay estimates of solutions of dissipative wave equations in R^n with lower power nonlinearities. *J. Math. Soc. Japan* **56** (2004), no.2, 365–373.

[13] S. Kesavan, *Symmetrization and applications*, Series in Analysis **3** (2006), 160pp.

[14] M. Kirane, M. Qafqasou, Fujita's Exponent for a Semilinear Wave Equation with Linear Damping, *Advanced Nonlinear Studies* **2** (2002), 41–49.

[15] M. Kirane, N. Tatar, Non-existence results for a semilinear hyperbolic problem with boundary condition of memory type, *Journal for Analysis and its Applications* **19** (2000), no.2, 453–468.

[16] M. O. Korpusov, Non-existence of global solutions to generalized dissipative Klein - Gordon equations with positive energy, *Electron. J. Differential Equations* **2012** (2012), no. 119, 1–10.

[17] H. A. Levine, J. Serrin, Global nonexistence theorems for quasilinear evolution equation with dissipation, *Arch. Ration Mech. Anal.* **137** (1997), 341–361.

[18] H. A. Levine, G. Todorova, Blow up of solutions of the Cauchy problem for a wave equation with nonlinear damping and source terms and positive initial energy in four space - time dimensions, *Proc. Amer. Math. Soc.* **129** (2001), no.3, 793–805.

[19] T. T. Li, Y. Zhou, Breakdown of solutions to $U_{tt} - \Delta U + U_t =\mid U \mid^{1+\alpha}$, *Discrete Contin. Dyn. Syst.* **1** (1995), 503–520.

[20] W. Liu, Global Existence, asymptotic behavior and blow - up of solutions for Coupled Klein - Gordon equations with damping terms, *Nonlinear Anal.* **73** (2010), 244–255.

[21] V. G. Makhankov, *Dynamics of classical solutions (in non - integrable systems)*, Physics Reports **35** (1978), 1–128.

[22] L. A. Medeiros, M. M. Miranda, Weak solutions for a system of nonlinear Klein-Gordon equations, *Annali di Mathematica, Pura ed Applicata* **146** (1986), no.1, 173–183.

[23] L. A. Medeiros, G. Perla Menzala, On a mixed problem for a class of nonlinear Klein - Gordon equations, *Acta Math., Hungar.* **52** (1988), 61–69.

[24] S. A. Messaoudi, B. Said-Houari, Global nonexistence of positive initial-energy solutions of a system of nonlinear viscoelastic wave equations with damping and source terms, *J. Math. Anal. Appl.* **365** (2010), no.1, 277–287.

[25] S. A. Messaoudi, Blow up in a nonlinearly damped wave equation, *Math. Nachr.* **231** (2001), 105–111.

[26] E. Mitidieri, S. I. Pokhozhaev, *Apriori estimates and the absence of solutions to nonlinear partial differential equations and inequalities*, Tr. Mat. Inst. Steklova **234** (2001), 384pp.

[27] L. E. Payne, D. H. Sattinger, Sadle points and instability of nonlinear hyperbolic equations, *Israel of Math.* **22** (1975), no.3, 273–303.

[28] G. Polya, G. Szegö, Isoperimetric inequalities in mathematical physics, *Ann. Math. Stud. Princeton, University Press* **27** (1970), no.195, 645–651.

[29] Xu Runzhang, Global existence blow up and asymptotic behaviour of solutions for nonlinear Klein - Gordon equation with dissipative term, *Math. Methods, Appl. Sci.* **33** (2010), 831–844.

[30] B. Said-Houari, Global nonexistence of positive initial-energy solutions of a system of nonlinear wave equations with damping and source terms, *Differential and Integral Equations* **23** (2010), no.1-2, 79–92.

[31] D. H. Sattinger, On global solution of nonlinear hyperbolic equations, *Arch. Rational Mech. Anal.* **30** (1968), 148–172.

[32] I. Segal, Non-linear semi-group,*Ann. of Math.* **78** (1963), no.2, 339–364.

[33] W. A. Straus, On weak solutions of semi - linear hyperbolic equations, *An. Acad. Brazil Ci.* **42** (1970), 645–651.

[34] G. Todorova, E. Vitillaro, Blow-up for nonlinear dissipative wave equations in Rn, *J. Math. Anal. Appl.* **303** (2005), 242–257.

[35] G. Todorova, B. Yordanov, Critical exponent for a nonlinear wave equation with damping, *C. R. Acad. Sci., Paris* **330** (2000), no.1, 557–562.

[36] Y. Wang, Non-existence of global solutions of a class of coupled non-linear Klein-Gordon with non - negative potential and arbitrary initial energy, *IMA J. Appl. Math.* **74** (2009), 392–415.

[37] Liu Yacheng, On potential wells and vacuum isolating of solutions for semilinear wave equations, *J. Differential Equations* 192 (2003), 155–169.

[38] Y. Ye, Global existence and nonexistence of solutions for coupled nonlinear wave equations with damping and source terms, *Bull. Korean Math. Soc.* **51** (2014), no.6, 1697–1710.

[39] Q. S. Zhag, A blow - up result for a nonlinear wave equation with damping: The critical case, *C. R. Acad. Sci., Paris, Serie I*, **333** (2001), 109–114.

[40] J. Zhang, On the standing wave in coupled non-linear Klein - Gordon equations, *Math. Methods, Appl. Sci.* **26** (2003), 11–25.

[41] Y. Zhou, Global existence and nonexistence for a nonlinear wave equation with damping and source terms, *Math. Nachr.* **278** (2005), 1341-1358.

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