ON POTENTIAL WELLS AND GLOBAL SOLVABILITY OF THE CAUCHY PROBLEM FOR SYSTEM OF SEMI-LINEAR KLEIN-GORDON EQUATIONS WITH DISSIPATION

AKBAR B. ALIEV AND GULSHAN KH. SHAFIYEVA

Abstract. We study the Cauchy problem for the system of nonlinear Klein-Gordon equations with weak coupling and dissipative term. By introducing a family of potential wells and investigation of the invariant sets, we prove the global existence and finite time blow up of solution.

1. Introduction

Klein (1927) and Gordon (1926) derived a relativistic equation for a charged particle in an electromagnetic field, using the recently discovered ideas of quanta. In [10] the Klein-Gordon equation is reduced to

$$\frac{1}{c^2} \frac{\partial^2 \psi}{\partial t^2} - \Delta \psi + \left( \frac{mc}{h} \right)^2 \psi = 0$$

for the special case of a free particle in three dimensions. Later this was led to the mathematical generalization

$$\frac{1}{c^2} \frac{\partial^2 \psi}{\partial t^2} - \Delta \psi + V'(\psi) = 0 \quad (1.1)$$

for some differentiable potential functions $V$, which is called the nonlinear Klein-Gordon equation [10]. Here $V$ is a nonlinear function which characterizes environment. According to the available literature, there are a lot of interesting results on problem (1.1) by various analytical methods. Many results have been obtained concerning properties of blow up and global existence of solutions to the Cauchy problem of the nonlinear Klein-Gordon equations [2, 7, 8, 9, 11, 12, 14, 15, 16, 17, 18, 19, 21, 22, 23, 24, 25, 27, 29, 31, 34, 35, 36, 37, 38, 39, 40, 41]. The motion of charged mesons in an electromagnetic field can be described by the following coupled Klein-Gordon equations:

$$\begin{cases} u_{tt} - \Delta u + g|v|^{p+1}|u|^{p-1}u = 0 \\ v_{tt} - \Delta v + h|u|^{p+1}|v|^{p-1}v = 0 \end{cases} \quad (1.2)$$

where $\Delta$ is a Laplacian operator on $\mathbb{R}^n$, $g$ and $h$ are non-zero real constants. The system was first introduced by Segal [32].
In this paper we consider the Cauchy problem for the system of Klein-Gordon equations with weak coupling and dissipations

\[ u_{itt} - \Delta u_i + \alpha_i u_i + \gamma_i u_{it} = \sum_{j=1, j \neq i}^{m} |u_j|^{p+1} |u_i|^{p-1} u_i, \quad i = 1, \ldots, m, \quad (1.3) \]

\[ u_i(0, x) = u_{i0}(x), u_{it}(0, x) = u_{i1}(x), \quad i = 1, \ldots, m, \quad (1.4) \]

where \( u_1, \ldots, u_m \) are real functions depending on \( t \in [0, \infty), x \in \mathbb{R}^n, \alpha_i > 0, \gamma_i > 0, i = 1, \ldots, m. \) We suppose that \( n \geq 2 \) and

\[ p > 0 \quad \text{if} \quad n = 2, \quad (1.5) \]

\[ 0 < p \leq \frac{1}{n-2} \quad \text{if} \quad n \geq 3. \quad (1.6) \]

System (1.3) determines a model of interaction between \( m \) fields with masses \( \alpha_1, \ldots, \alpha_m. \)

We study qualitative characteristics of the family of the potential wells, the existence and nonexistence of global solutions of problem (1.3), (1.4).

Notice that, the potential well was introduced by Sattinger [31] in 1968. The potential well method was one of the most important methods for studying nonlinear evolution equations. There was a lot of investigations in this direction [27, 29, 37, 40].

In the case \( m = 2 \) problem (1.3),(1.4) was studied in [20, 40]. In this direction, we also note the works [2, 3, 4, 24, 26, 30, 34, 38]. The absence of global solutions for problem (1.3),(1.4) was investigated in [2]. The absence of global solutions with positive arbitrary initial energy for the system of semilinear hyperbolic equations

\[ u_{ttt} - \Delta u_i + \alpha_i u_i + \gamma_i u_{tt} = |u_1|^{\rho_{j1}} \ldots |u_m|^{\rho_{jm}}, \quad t > 0, x \in \mathbb{R}^n, k = 1, \ldots, m, \]

was investigated in [5], where \( \rho_{jk} = p_j + 1, j \neq k, \rho_{kk} = p_k - 1, k, j = 1, \ldots, m, \)

\[ p_k > 0, \sum_{k=1}^{m} p_k + m - 2 \leq \frac{2}{n-2} \quad \text{if} \quad n \geq 3. \]

2. Preliminaries

In the sequel, by \( | \cdot |_q \) we denote the usual \( L_q(\mathbb{R}^n) \)- norm. For simplicity of notation, in particular, we write \( | \cdot |^2 \) instead of \( | \cdot |^2_2. \) Let \( \langle \cdot, \cdot \rangle \) denote inner product in \( L_2(\mathbb{R}^n). \) Let \( \| \cdot \| \) denote the norm on the Sobolev space \( H^1 = H^1(\mathbb{R}^n), \) i.e. \( \| u \| = [\| \nabla u \|^2 + |u|^2]^{\frac{1}{2}} \), where \( \nabla \) denotes a gradient. For simplicity, in what follows, we set \( \alpha_i = \gamma_i = 1 \) for \( i = 1, \ldots, m \) and denote the first and the second derivatives of a function \( u \) with respect to \( t \) by \( \dot{u} \) and \( \ddot{u}, \) respectively.

We define the following functionals

\[ J(\phi_1, \ldots, \phi_m) = \frac{1}{2} \sum_{j=1}^{m} \| \phi_j \|^2 - \frac{1}{p+1} G; \]

and

\[ I_\delta(\phi_1, \ldots, \phi_m) = \delta \sum_{j=1}^{m} \| \phi_j \|^2 - 2G, \]
where $\delta > 0$ and
\[
G = G(\phi_1, \ldots, \phi_m) = \sum_{i,j=1, i\neq j}^{m} \int_{\mathbb{R}^n} |\phi_i(x)\phi_j(x)|^{p+1} dx.
\]

**Lemma 2.1.** Let $(\phi_1, \ldots, \phi_m) \in \left[H^1 \setminus \{0\}\right]^m$. Then
(i) $\lim_{\lambda \to 0} J(\lambda \phi_1, \ldots, \lambda \phi_m) = 0$, $\lim_{\lambda \to +\infty} J(\lambda \phi_1, \ldots, \lambda \phi_m) = -\infty$;
(ii) there is a single point $\lambda^* = \lambda^*(\phi_1, \ldots, \phi_m)$ in the interval $0 < \lambda < +\infty$, where
\[
\frac{d}{d\lambda} J(\lambda \phi_1, \ldots, \lambda \phi_m) |_{\lambda = \lambda^*} = 0;
\]
(iii) $J(\lambda \phi_1, \ldots, \lambda \phi_m)$ is nondecreasing on $0 \leq \lambda \leq \lambda^*$, nonincreasing on $\lambda^* \leq \lambda < +\infty$ and it reaches its maximum at the point $\lambda = \lambda^*$;
(iv) $I_1(\lambda \phi_1, \ldots, \lambda \phi_m) > 0$ for $0 < \lambda < \lambda^*$; $I_1(\lambda^* \phi_1, \ldots, \lambda^* \phi_m) < 0$ for $\lambda^* < \lambda < +\infty$ and $I_1(\lambda^* \phi_1, \ldots, \lambda^* \phi_m) = 0$.

We define the set
\[
N_\delta = \{ (\phi_1, \ldots, \phi_m) : (\phi_1, \ldots, \phi_m) \in \left[H^1 \setminus \{0\}\right]^m, I_\delta(\phi_1, \ldots, \phi_m) = 0 \}.
\]
Suppose $(\phi_1, \ldots, \phi_m) \in N_1$, then
\[
J(\phi_1, \ldots, \phi_m) = \frac{p}{2(p+1)} \sum_{j=1}^{m} \|\phi_j\|^2 > 0,
\]
i.e. $J$ is bounded from below on the set $N_1$.

Consider the variation problem
\[
d(\delta) = \inf_{(\phi_1, \ldots, \phi_m) \in N_\delta} J(\phi_1, \ldots, \phi_m), 0 < \delta < p+1.
\]

**Lemma 2.2.** There is $(\tilde{\phi}_1, \ldots, \tilde{\phi}_m) \in N_1$ such that
\[
J(\tilde{\phi}_1, \ldots, \tilde{\phi}_m) = \inf_{(\phi_1, \ldots, \phi_m) \in N_1} J(\phi_1, \ldots, \phi_m) = d > 0.
\]
For $\delta > 0$ we define also
\[
r(\delta) = r(\delta, p) = \left( \frac{\delta}{2\mu C_*^{2(p+1)}} \right)^{\frac{1}{p}},
\]
where $C_* = \sup_{\|u\| \neq 0} \frac{|u|_2^{2(p+1)}}{\|u\|^p}$, $\mu$ is the lowest $\eta$ number satisfy the inequality
\[
\sum_{i,j=1, i<j}^{m} a_i^{p+1} a_j^{p+1} \leq \eta \left[ \sum_{j=1}^{m} a_j^2 \right]^{p+1}.
\]
Here $a_i \geq 0, i = 1, \ldots, m$, $\sum_{i=1}^{m} a_i^2 > 0$.

**Lemma 2.3.** If $(u_1, \ldots, u_m) \in \left[H^1 \setminus \{0\}\right]^m$ and $\sum_{j=1}^{m} \|u_j\|^2 < r(\delta)$, then $I_\delta(u_1, \ldots, u_m) > 0$. 

Lemma 2.4. If \((u_1, \ldots, u_m) \in [H^1 \setminus \{0\}]^m\) and \(I_\delta(u_1, \ldots, u_m) < 0\), then 
\[\sum_{j=1}^m \|u_j\|^2 > r(\delta).\]

Lemma 2.5. If \((u_1, \ldots, u_m) \in [H^1 \setminus \{0\}]^m\) and \(I_\delta(u_1, \ldots, u_m) = 0\), then 
\[\sum_{j=1}^m \|u_j\|^2 \geq r(\delta).\]

Lemma 2.6. Suppose that conditions (1.3), (1.4) are fulfilled. Then 
\[d(\delta) \geq a(\delta) r(\delta),\]  
where 
\[d(\delta) = \delta^{\frac{1}{p} \frac{p+1-\delta}{p}} d,\]  
\[a(\delta) = \frac{p+1-\delta}{2(p+1)} d.\]  
It is obvious that 
\[\lim_{\delta \to +0} d(\delta) = 0,\]  
\[d(p+1) = 0,\]  
\[d(1) = d,\]  
\[d'(\delta) > 0, \delta \in (0, 1),\]  
\[d'(\delta) < 0, \delta \in (1, p+1).\]  

The following theorem on the local solvability of the problem (1.3), (1.4) holds. This theorem can be proved by using Galerkin method (see [22]) or methods of nonlinear evolution equations (see [5]).

Theorem 2.1. Let the conditions (1.5), (1.6) hold, then for any \((u_{10}, \ldots, u_{m0}) \in [H^1]^m, (u_{11}, \ldots, u_{m1}) \in [L_2(R^n)]^m\) there exists \(T' \in (0, \infty)\) such that the problem (1.3), (1.4) has a weak solution \((u_1, \ldots, u_m) \in C([0, T']; [H^1]^m) \cap C^1([0, T']; [L_2(R^n)]^m).\) If \(T_{\text{max}} = \sup T', i.e. T_{\text{max}} the length of the maximal existence interval of the solution \((u_1(t, \cdot), \ldots, u_m(t, \cdot)) \in C([0, T_{\text{max}}]; [H^1]^m) \cap C^1([0, T_{\text{max}}]; [L_2(R^n)]^m)\) then either: \(T_{\text{max}} = +\infty\) or
\[\lim_{t \to T_{\text{max}} - 0} \sup_{j=1}^m \sum_{j=1}^m [\|u_j(t, \cdot)\|^2 + \|u_j(t, \cdot)\|] = +\infty.\]

Remark 2.1. 1) If \((u_{10}, \ldots, u_{m0}) \in [H^s]^m, (u_{11}, \ldots, u_{m1}) \in [H^{s-1}]^m, s \geq 1,\) then \((u_1, \ldots, u_m) \in C([0, T_{\text{max}}]; [H^s]^m) \cap C^1([0, T_{\text{max}}]; [H^{s-1}]^m)\)
2) If \(\{(u_{10k}, \ldots, u_{m0k})\}_{k=1}^\infty \subset [H^2]^m, \{(u_{11k}, \ldots, u_{m1k})\}_{k=1}^\infty \subset [H^1]^m\) and \(u_{0k} \to u_0\) in \(H^1, u_{1k} \to u_1\) in \(L_2(R^n), i = 1, \ldots, m\) as \(k \to +\infty,\) 
then for any \(T^* \in (0, T_{\text{max}})\)
\[(u_{1k}(t, \cdot), \ldots, u_{mk}(t, \cdot)) \to (u_1(t, \cdot), \ldots, u_m(t, \cdot))\]
in \(C([0, T^*]; [H^1]^m) \cap C^1([0, T^*]; [L_2(R^n)]^m).\)
and we define the following sets

\[ W_\delta = \{ (u_1, \ldots, u_m) \in [H^1]^m, I_\delta(u_1, \ldots, u_m) > 0, \]
\[ J(u_1, \ldots, u_m) = \{ 0 \}, 0 < \delta < p + 1, \]
\[ V_\delta = \{ (u_1, \ldots, u_m) \in [H^1]^m, I_\delta(u_1, \ldots, u_m) < 0, \]
\[ J(u_1, \ldots, u_m) < d(\delta), 0 < \delta < p + 1. \]

From (2.6)-(2.10) it follows that for every \( e \in (0, d) \) the equation \( d(\delta) = e \) has two roots \( \delta_1, \delta_2 \), so that \( \delta_1 < 1 < \delta_2 \).

**Theorem 3.1.** Suppose that \((u_{10}, \ldots, u_{m0}) \in [H^1]^m, (u_{11}, \ldots, u_{1m}) \in [L_2(R^n)]^m, and conditions (1.5),(1.6) hold. If \( 0 < e < d \) and \( \delta_1 < \delta_2 \) are the roots of the equation \( d(\delta) = e \), then the following assertions are valid:

a) if \( I_1(u_{10}, \ldots, u_{m0}) > 0 \) or \( \|u_{10}\| = \ldots = \|u_{m0}\| = 0 \), then all weak solutions \((u_1(t, \cdot), \ldots, u_m(t, \cdot))\) of problem (1.3),(1.4) with initial energy \( 0 < E(0) \leq e \), belong to \( W_\delta \), where \( \delta_1 < \delta < \delta_2 \);

b) if \( I_1(u_{10}, \ldots, u_{m0}) < 0 \), then for all weak solutions \((u_1(t, \cdot), \ldots, u_m(t, \cdot))\) of problem (1.3),(1.4) with initial energy \( 0 < E(0) \leq e \), belong to \( V_\delta \), where \( \delta_1 < \delta < \delta_2 \).

**Proof.** a) Let \((u_{10}, \ldots, u_{m0}) \in [H^1]^m, (u_{11}, \ldots, u_{1m}) \in [L_2(R^n)]^m \) and

\[ 0 < E(0) \leq e. \]  

(3.1)

Let

\[ I_1(u_{10}, \ldots, u_{m0}) > 0 \quad \text{or} \quad \|u_{10}\| = \ldots = \|u_{m0}\| = 0. \]  

(3.2)

Taking into account the Remark 2.1 from (1.3),(1.4) we have the following energy equality

\[ E(t) + \sum_{j=1}^{m} \int_{0}^{t} |\dot{u}_j(s, \cdot)|^2 ds = E(0), 0 \leq t < T_{\text{max}}. \]  

(3.3)

By virtue of (3.1) and (3.3), \( J(u_1(t, \cdot), \ldots, u_m(t, \cdot)) \leq e, 0 \leq t < T_{\text{max}}. \) On the other hand for \( \delta_1 < \delta < \delta_2 \) we have \( e < d(\delta) \). Therefore

\[ J(u_1(t, \cdot), \ldots, u_m(t, \cdot)) < d(\delta), 0 \leq t < T_{\text{max}}. \]  

(3.4)

Suppose that assertion a) does not hold. Then in view of (3.2) and (3.4) there exists \( \bar{t} \in (0, \infty) \) such that

\[ I_\delta(u_1(t, \cdot), \ldots, u_m(t, \cdot)) > 0, t \in (0, \bar{t}), \]  

(3.5)

\[ I_\delta(u_1(\bar{t}, \cdot), \ldots, u_m(\bar{t}, \cdot)) = 0. \]  

(3.6)

Thus, \((u_1(\bar{t}, \cdot), \ldots, u_m(\bar{t}, \cdot)) \in N_\delta\), therefore, by the definition of \( d(\delta) \) we have

\[ d(\delta) \leq J(u_1(t, \cdot), \ldots, u_m(t, \cdot)) \]

which contradicts (3.4).
Now we prove assertion b).
Let \((u_{10}, \ldots, u_{n0}) \in [H^1]^{m}, (u_{11}, \ldots, u_{1m}) \in [L_2(R^n)]^{m}\), \(0 < E(0) \leq \varepsilon\) and \(I_j(u_{10}, \ldots, u_{n0}) < 0\). Arguing in a similar manner as in step a), we obtain the existence of \(\ell \in [0, T]\), such that for any \(t \in [0, \ell]\) the inequality
\[
I_1(u_1(t, \cdot), \ldots, u_m(t, \cdot)) < 0,
\]
and the equality \(I_1(u_1(\ell, \cdot), \ldots, u_m(\ell, \cdot)) = 0\) are fulfilled.
We again obtain the following contradiction
\[
d(\delta) \leq J(u_1(\ell, \cdot), \ldots, u_m(\ell, \cdot)) \leq \varepsilon < d(\delta). \quad \square
\]

By Theorem 3.1 we have the following statement

**Theorem 3.2.** Suppose that \((u_{10}, \ldots, u_{n0}) \in [H^1]^{m}, (u_{11}, \ldots, u_{1m}) \in [L_2(R^n)]^{m}\), and conditions (1.5),(1.6) hold. If \(0 < E(0) \leq \varepsilon\) and \(\delta_1, \delta_2\) are the roots of the equation \(d(\delta) = \varepsilon\), then the sets \(W_{\delta_1\delta_2} = \bigcup_{\delta_1<\delta<\delta_2} W_\delta\) and \(V_{\delta_1\delta_2} = \bigcup_{\delta_1<\delta<\delta_2} V_\delta\) are invariant on the trajectories of the dynamical system generated by problem (1.3),(1.4).

The following statement is a consequence of Theorem 3.2 and shows that there is a so-called vacuum zone between the two invariant sets.

**Theorem 3.3.** If the assumptions of Theorem 3.2 hold, then all solutions of problem (1.3),(1.4) satisfy the relation \((u_1(t, \cdot), \ldots, u_m(t, \cdot)) \notin N_{\delta_1\delta_2}\) are invariant on the trajectories of the dynamical system generated by problem (1.3),(1.4).

Now, consider the case \(E(0) \leq 0\).

**Theorem 3.4.** Suppose that \((u_{10}, \ldots, u_{n0}) \in [H^1]^{m}, (u_{11}, \ldots, u_{1m}) \in [L_2(R^n)]^{m}\), and conditions (1.5),(1.6) hold. If \(E(0) \leq 0\), \(\|u_{10}\| + \ldots + \|u_{n0}\| \neq 0\), then the solution of problem (1.3),(1.4) satisfies the inequality
\[
\sum_{j=1}^{m} \|u_j(t, \cdot)\|^2 \geq r_0, \quad (3.7)
\]
where \(r_0 = \left(\frac{p+1}{2\mu C_{2(p+1)}}\right)^{\frac{1}{p}}\).

**Proof.** Let \((u_1(t, \cdot), \ldots, u_m(t, \cdot))\) be the solution of problem (1.1),(1.2) with initial energy \(E(0) \leq 0\), where \(\|u_{10}\| + \ldots + \|u_{n0}\| \neq 0\).

Let \([0, T_{\text{max}}]\) be the maximum interval of existence of the solution \((u_1(t, \cdot), \ldots, u_m(t, \cdot))\). In view of the definition of \(E(t)\), we have
\[
E(t) = \frac{1}{2} \sum_{j=1}^{m} \|\dot{u}_j(t, \cdot)\|^2 + J(u_1(t, \cdot), \ldots, u_m(t, \cdot))
\]
\[
= E(0) - \sum_{j=1}^{m} \int_0^t |\dot{u}_j(s, \cdot)|^2 ds, t \in [0, T_{\text{max}}], \quad (3.8)
\]
It follows that
\[
J(u_1(t, \cdot), \ldots, u_m(t, \cdot)) \leq 0 < d(\delta), t \in [0, T_{\text{max}}) \quad (3.9)
\]
On the other hand,

\[ \sum_{j=1}^{m} \|u_i(t, \cdot)\|^2 \leq \frac{2}{p+1} \sum_{i,j=1, i<j}^{m} |u_i u_j|^{p+1}_{p+1}. \]  

(3.10)

On the other hand, using the Hölder’s inequality and embedding \( H^1 \subset L_{2(p+1)}(\mathbb{R}^n) \), also taking into account the definition of \( \mu \) we have

\[ \frac{2}{p+1} \sum_{i,j=1, i<j}^{m} |u_i u_j|^{p+1}_{p+1} \leq \frac{2}{p+1} \mu C^2_{s(p+1)} \left[ \sum_{j=1}^{m} \|u_j\|^2 \right]^{p+1}. \]  

(3.11)

Since \( \|u_0\| + \ldots + \|u_{m0}\| \neq 0 \), then there exists a half-interval \([0, t_1]\), where \( \|u_1(t, \cdot)\| + \ldots + \|u_m(t, \cdot)\| \neq 0 \). Then from (3.10),(3.11) we obtain that

\[ \sum_{j=1}^{m} \|u_j(t, \cdot)\|^2 \geq \left( \frac{p+1}{2 \mu C^2_{s(p+1)}} \right)^\frac{1}{p} = r_0, t \in [0, t_1). \]

It follows that \( \|u_1(t_1, \cdot)\| + \ldots + \|u_m(t_1, \cdot)\| \neq 0 \), therefore (3.7) is also valid on the half-open interval \([t_1, t_2]\), for some \( t_2 > t_1 \). Thus, (3.7) is true on \([0, T_{\text{max}}]\). \( \square \)

**Theorem 3.5.** Suppose that \((u_{10}, \ldots, u_{m0}) \in [H^1]^m \setminus \{0, \ldots, 0\}, (u_{11}, \ldots, u_{1m}) \in [L_2(\mathbb{R}^n)]^m\), and conditions (1.5),(1.6) hold. If \( E(0) < 0 \) or \( E(0) = 0 \) and \((u_{11}, \ldots, u_{1m}) \neq (0, \ldots, 0)\), then \((u_1(t, \cdot), \ldots, u_m(t, \cdot)) \in V_{\delta} \) for \( t \in [0, T_{\text{max}}]\), where \( 0 < \delta < p+1 \).

**Proof.** If \( E(0) < 0 \), then from (3.3) we obtain

\[ J(u_1(t, \cdot), \ldots, u_m(t, \cdot)) \leq E(0) < 0 < d(\delta). \]  

(3.12)

On the other hand,

\[ J(u_1(t, \cdot), \ldots, u_m(t, \cdot)) = \frac{1}{2} \left( 1 - \frac{\delta}{p+1} \right) \sum_{j=1}^{m} \|u_j(t, \cdot)\|^2 \]

\[ + \frac{1}{2(p+1)} I_{\delta}(u_1(t, \cdot), \ldots, u_m(t, \cdot)), \]  

(3.13)

therefore,

\[ I_{\delta}(u_1(t, \cdot), \ldots, u_m(t, \cdot)) < 0, t \in [0, T_{\text{max}}] \quad \text{if} \quad 0 < \delta < p+1. \]  

(3.14)

If \( E(0) = 0 \), then in view of Theorem 3.4, from (3.7),(3.12) we find that inequality (3.14) is also true in this case if \( 0 < \delta < p+1 \).

Thus \((u_1(t, \cdot), \ldots, u_m(t, \cdot)) \in V_{\delta}\), where \( 0 < \delta < p+1 \). \( \square \)

Theorems 3.3-3.5 imply the following result.

**Theorem 3.6.** If \( E(0) < d \), then \( W_1 \) and \( V_1 \) are invariant with respect to the dynamical system generated by problem (1.3),(1.4).
4. Existence of global solutions

Theorem 3.6 implies the following theorem on global solvability:

**Theorem 4.1.** Suppose that \((u_{10}, \ldots, u_{n0}) \in [H^1]^m, (u_{11}, \ldots, u_{1m}) \in [L_2(R^n)]^m, E(0) < d\) and conditions (1.3),(1.4) hold. If \((u_1(t_0, \cdot), \ldots, u_m(t_0, \cdot)) \in W_1\) at some moment of time \(t_0 \in [0, T_{\max})\), then \(T_{\max} = +\infty\) and \((u_1(t, \cdot), \ldots, u_m(t, \cdot))\) satisfies a priori estimate

\[
\sum_{j=1}^{m} [||u_j(t, \cdot)||^2 + ||u_\ell(t, \cdot)||^2] \leq \frac{2d(p+1)}{p}, t \in [0, T_{\max}).
\] (4.1)

**Proof.** By Theorem 3.1 \((u_1(t, \cdot), \ldots, u_m(t, \cdot)) \in W_1, t \in [0, T_{\max})\), therefore \(I_1(u_1(\cdot, \cdot), \ldots, u_m(\cdot, \cdot)) > 0, 0 \leq t < T_{\max}\). Then from (3.8) and (3.13) we obtain that for \(0 \leq t < T_{\max}\) the apriori estimate (4.1) holds, therefore \(T_{\max} = +\infty\), i.e. problem (1.3),(1.4) has a global solution. \(\square\)

Theorem 4.1 implies the following statement:

**Theorem 4.2.** Suppose that \((u_{10}, \ldots, u_{n0}) \in [H^1]^m, (u_{11}, \ldots, u_{1m}) \in [L_2(R^n)]^m, and conditions (1.3),(1.4) hold. If \(0 < E(0) < d\) and \(I_{\delta_2}(u_{10}, \ldots, u_{n0}) > 0\) or \(||u_{10}|| = \ldots = ||u_{n0}|| = 0, where \delta_1 < \delta_2\) are the roots of the equation \(d(\delta) = E(0)\), then problem (1.3),(1.4) has a unique solution \((u_1(\cdot, \cdot), \ldots, u_m(\cdot, \cdot)) \in C([0, \infty); [H^1]^m) \cap C^1([0, \infty); [L_2(R^n)]^m)\) and \((u_1(t, \cdot), \ldots, u_m(t, \cdot)) \in W_\delta, \delta_1 < \delta < \delta_2, 0 \leq t < +\infty.\)

**Proof.** By virtue of Theorem 3.1 and 4.1, it suffices to prove that \(I_1(u_{10}, \ldots, u_{n0}) > 0\). Indeed, otherwise there would exist \(\delta \in [1, \delta_2]\) such that \(I_\delta(u_{10}, \ldots, u_{n0}) = 0\). Since \(I_{\delta_2}(u_{10}, \ldots, u_{n0}) > 0\), therefore \(||u_{10}|| + \ldots + ||u_{n0}|| \neq 0\). Then \(J(u_{10}, \ldots, u_{n0}) \geq d(\delta)\), which contradicts the inequality \(J(u_{10}, \ldots, u_{n0}) \leq E(0) < d(\delta)\), for \(\delta_1 < \delta < \delta_2.\) \(\square\)

5. The absence of global solutions

In this part we consider the case of \(\alpha_j = 1\) and \(\gamma_j = \gamma, j = 1, 2, \ldots, m\)

**Theorem 5.1.** Suppose that \(p \geq 1, (u_{10}, \ldots, u_{n0}) \in [H^s]^m\) and \((u_{11}, \ldots, u_{1m}) \in [H^{s-1}]^m, where s > \frac{n}{2}\). Suppose also that conditions (1.5),(1.6) and one of the following conditions hold:

a) \(E(0) < 0;\)

b) \(0 \leq E(0) < d, I(u_{10}, \ldots, u_{n0}) < 0, \alpha_j = 1, \gamma_j = \gamma, j = 1, 2, \ldots, m\) and \(0 \leq \gamma < \frac{p-1}{2C_\ast}\).

Then \(T_{\max} = +\infty\) and \(\lim_{t \to T_{\max} - 0} \sum_{j=1}^{m} |u_j(t, \cdot)|^2 = +\infty.\)

**Proof.** a) If \(E(0) < 0\), then repeating the proof given in [4], we obtain the assertion of the theorem.

b) Let \(0 \leq E(0) < d, I_1(u_{10}, \ldots, u_{n0}) < 0\) and \(0 \leq \gamma < \frac{p-1}{2C_\ast}\).
Let’s denote

\[ F(t) = \sum_{j=1}^{m} |u_j(t, \cdot)|^2, \quad t \in [0, T_{\text{max}}). \]

Then we obtain

\[ \dot{F}(t) = 2 \sum_{j=1}^{m} \langle u_j(t, \cdot), \dot{u}_j(t, \cdot) \rangle, \quad t \in [0, T_{\text{max}}). \] (5.1)

Assume that the assertion of Theorem 4.1 is not true, i.e. \( T_{\text{max}} = +\infty \). Since \((u_{10}, \ldots, u_{m0}) \in [H^s]^m\) and \((u_{11}, \ldots, u_{1m}) \in [H^{s-1}]^m\), where \( s > \frac{n}{2} \), then \((u_1(t, x), \ldots, u_m(t, x)) \in C([0, \infty); [H^s]^m) \cap C^1([0, \infty); [H^{s-1}]^m)\) and obviously \( \dot{F}(t) \in C[0, \infty) \). Taking into account (1.3), by a simple calculation we obtain

\[ d \frac{d}{dt} [e^{\gamma t} \dot{F}(t)] = 2e^{\gamma t} \sum_{j=1}^{m} |\dot{u}_j(t, \cdot)|^2 + 2(\delta - 1)e^{\gamma t} \sum_{j=1}^{m} \|u_j(t_j)\|^2 
- 2e^{\gamma t} I_\delta(u_1(t, \cdot), \ldots, u_m(t, \cdot)). \] (5.2)

Since \( E(0) < d \), therefore there exists \( \delta_1, \delta_2 \) such that \( \delta_1 < 1 < \delta_2 \) and

\[ d(\delta_i) = E(0), i = 1, 2. \]

In (3.4), we put \( \delta_1 = \delta_2 \). According to Theorem 3.5

\[ I_{\delta_2}(u_1(t, \cdot), \ldots, u_m(t, \cdot)) \leq 0, \] (5.3)

therefore, from (5.2),(5.3) we get

\[ d \frac{d}{dt} [e^{\gamma t} \dot{F}(t)] \geq 2(\delta - 1)e^{\gamma t} \sum_{j=1}^{m} \|u_j(t, \cdot)\|^2. \] (5.4)

On the other hand, applying Lemma 2.4, we have the following estimation

\[ \sum_{j=1}^{m} \|u_j(t, \cdot)\|^2 > r(\delta_2). \] (5.5)

From (5.3) and (5.5) it follows that

\[ d \frac{d}{dt} [e^{\gamma t} \dot{F}(t)] \geq e^{\gamma t} c(\delta_2), \] (5.6)

where \( c(\delta_2) = 2(\delta_2 - 1)r(\delta_2) \). From (5.6), we find that for sufficiently large \( t_0 \)

\[ \dot{F}(t) \geq \frac{c(\delta_2)}{2\lambda}, t \geq t_0. \] (5.7)

Thus,

\[ \lim_{t \to +\infty} F(t) = +\infty. \] (5.8)

On the other hand,

\[ \ddot{F}(t) = 2 \sum_{j=1}^{m} \left[ |\ddot{u}_j(t, \cdot)|^2 - \|u_j(t, \cdot)\|^2 \right] - 2\gamma \sum_{j=1}^{m} \langle u_j(t, \cdot), \dot{u}_j(t, \cdot) \rangle. \]
\[ +2 \cdot \sum_{i,j=1, i<j}^{m} |u_i u_j|^{p+1}_{p+1} = (p + 3) \sum_{j=1}^{m} |\dot{u}_j(t,\cdot)|^2 + (p - 1) \sum_{j=1}^{m} \|u_j(t,\cdot)\|^2 \]

\[-2\gamma \langle u_j(t,\cdot), \dot{u}_j(t,\cdot) \rangle + 2\gamma(p + 1) \sum_{j=1}^{m} \int_{0}^{t} |\dot{u}_j(s,\cdot)|^2 ds - 2(p + 1)E(0). \tag{5.9} \]

Using the Hölder and Young inequalities, we have

\[ \left| 2\gamma \sum_{j=1}^{m} \langle u_j(t,\cdot), \dot{u}_j(t,\cdot) \rangle \right| \leq (p - 1 - \varepsilon) \sum_{j=1}^{m} |\dot{u}_j(t,\cdot)|^2 \]

\[ + \frac{4\gamma^2}{p - 1 - \varepsilon} \sum_{j=1}^{m} |u_j(t,\cdot)|^2. \tag{5.10} \]

When using the imbedding \( H^1 \subset L_2(R^n) \) from (5.8) and (5.10), we get

\[ \ddot{\mathcal{F}}(t) \geq (4 + \varepsilon) \sum_{j=1}^{m} |\dot{u}_j(t,\cdot)|^2 + \psi(t), \tag{5.11} \]

where

\[ \psi(t) = \left( p - 1 - \frac{4\gamma^2C^2_*}{p - 1 - \varepsilon} \right) \sum_{j=1}^{m} \|u_j(t,\cdot)\|^2 - 2(p + 1)E(0). \]

By virtue of (5.8), for sufficiently small \( \varepsilon > 0 \), there exists \( t_1 \) such that for any \( t \geq t_1 \) we have the estimation

\[ \ddot{\mathcal{F}}(t) \geq (4 + \varepsilon) \sum_{j=1}^{m} |\dot{u}_j(t,\cdot)|^2. \tag{5.12} \]

It follows from (5.1) and (5.12)

\[ \ddot{\mathcal{F}}(t)\mathcal{F}(t) - \left( 1 + \frac{\varepsilon}{4} \right) \dot{\mathcal{F}}^2(t) \geq (4 + \varepsilon) \sum_{j=1}^{m} |\dot{u}_j(t,\cdot)|^2 \sum_{j=1}^{m} |u_j(t,\cdot)|^2 \]

\[-4 \left( 1 + \frac{\varepsilon}{4} \right) \left[ \sum_{j=1}^{m} \langle u_j(t,\cdot), \dot{u}_j(t,\cdot) \rangle \right]^2, \quad t \geq t_1. \]

Using Hölder’s inequality, we obtain

\[ \ddot{\mathcal{F}}(t)\mathcal{F}(t) - \left( 1 + \frac{\varepsilon}{4} \right) \dot{\mathcal{F}}^2(t) \geq 0, \quad t \geq t_1. \tag{5.13} \]

From (5.11) and (5.13) we have the following inequality

\[ \left( F^{-\frac{\varepsilon}{4}}(t) \right)'' \leq 0, \quad t \geq t_1, \tag{5.14} \]

From (5.7) it follows that

\[ \left( F^{-\frac{\varepsilon}{4}}(t) \right)' = -\frac{\varepsilon F''(t)}{4 \cdot F^{1+\frac{\varepsilon}{4}}(t)} < 0, \quad t \geq t_1. \tag{5.15} \]
In view of (5.14) and (5.15) there exists $t^* \in (0, t_1)$ such that $\lim_{t \to t^*} F^{-1}(t) = 0$, i.e. $\lim F(t) = +\infty$. This contradiction shows that our assumption is not true. □

**Remark 5.1.** Under the assumptions of Theorem 5.1

$$\lim_{t \to T_{\max} - 0} \sum_{j=1}^{m} \left( \|u_j(t, \cdot)\|^2 + |\dot{u}_j(t, \cdot)|^2 \right) = +\infty.$$  

6. Proofs of Lemmas

**Proof of Lemma 2.1.** Properties of (i) follow directly from

$$J(\lambda \phi_1, \ldots, \lambda \phi_m) = \lambda^2 \sum_{j=1}^{m} (|\nabla \phi_j|^2 + |\phi_j|^2) - \lambda^{2p+2} \frac{1}{p+1} \sum_{i,j=1, i<j}^{m} |\phi_i \phi_j|^{p+1}.$$  

(ii) Elementary computation shows that

$$\frac{d}{d\lambda} J(\lambda \phi_1, \ldots, \lambda \phi_m) = \lambda \sum_{j=1}^{m} \|\phi_j\|^2 - 2\lambda^{2p+1} \sum_{i,j=1, i<j}^{m} \int_{\mathbb{R}^n} |\phi_i \phi_j|^{p+1} dx \quad (6.1)$$

Hence, it is evident that at the point

$$\lambda^* = \left| \frac{\sum_{j=1}^{m} \|\phi_j\|^2}{2 \sum_{i,j=1, i<j}^{m} \int_{\mathbb{R}^n} |\phi_i \phi_j|^{p+1} dx} \right|^{\frac{1}{2p}}$$

the following equality holds

$$\frac{d}{d\lambda} J(\lambda \phi_1, \ldots, \lambda \phi_m)|_{\lambda=\lambda^*} = 0.$$  

(iii) From (6.1) it is clear that

$$\frac{d}{d\lambda} J(\lambda \phi_1, \ldots, \lambda \phi_m) > 0 \quad \text{for} \quad 0 < \lambda < \lambda^*,$$

and

$$\frac{d}{d\lambda} J(\lambda \phi_1, \ldots, \lambda \phi_m) < 0 \quad \text{for} \quad \lambda^* < \lambda < +\infty,$$

i.e. the assertion (iii) is true.

(iv) From definitions of the functionals $J$ and $I_1$ and also from (6.1) it follows that

$$I_1(\lambda \phi_1, \ldots, \lambda \phi_m) = \lambda \frac{d}{d\lambda} J(\lambda \phi_1, \ldots, \lambda \phi_m).$$
Proof of Lemma 2.2. From (2.1) it follows that if \((u_1, \ldots, u_m) \in N_1\), then

\[
J(u_1, \ldots, u_m) = \frac{p}{2(p+1)} \sum_{j=1}^{m} \|u_j\|^2.
\] (6.2)

Let \((u_{1k}, \ldots, u_{mk})\) be a minimizing sequence, i.e.

\[
\lim_{k \to \infty} J(u_{1k}, \ldots, u_{mk}) = \inf_{(u_1, \ldots, u_m) \in N_1} J(u_1, \ldots, u_m) = d.
\]

Let us denote \(u_{j\lambda} = \lambda u_j, j = 1, \ldots, m\) and denote by \(\nu_{jk} = (\mu_k u_{jk})^\ast\) the Schwartz symmetrization \([6, 13, 28]\) of the function \(u_{jk} = \mu_k u_{jk}\) with respect to the variable \(x\), where \(\mu_k\) is chosen so that \((\nu_{1k}, \ldots, \nu_{mk}) \in N_1\). By virtue of (6.2)

\[
J(\nu_{1k}, \ldots, \nu_{mk}) = \frac{p}{2(p+1)} \sum_{j=1}^{m} \|\nu_{jk}\|^2.
\] (6.3)

On the other hand,

\[
\int_{\mathbb{R}^n} \|\nabla \nu_{jk}\|^2 \, dx = \int_{\mathbb{R}^n} \|\nabla (u_{jk})^\ast \mu_k\|^2 \, dx
\]

\[
= \int_{\mathbb{R}^n} \|\nabla (u_{jk})^\ast \|^2 \, dx \leq \int_{\mathbb{R}^n} \|\nabla (u_{jk}) \|^2 \, dx
\] (6.4)

(see \([6, 13, 28]\)).

From (6.3), (6.4) it follows that

\[
J(\nu_{1k}, \ldots, \nu_{mk}) \leq J((u_{1k})^\ast \mu_k, \ldots, (u_{mk})^\ast \mu_k).
\] (6.5)

On the other hand, according to the choice of

\[
J((u_{1k})^\ast \mu_k, \ldots, (u_{mk})^\ast \mu_k) \leq J(u_{1k}, \ldots, u_{mk}).
\] (6.6)

Consequently, \(\lim_{k \to \infty} J(\nu_{1k}, \ldots, \nu_{mk}) = d\). It follows that

\[
\|\nabla \nu_{jk}\| \leq c,
\] (6.7)

where \(c > 0\) is a constant not dependent on \(k = 1, 2, \ldots\). Then we conclude that there exists such a \((\nu_{1\infty}, \ldots, \nu_{m\infty}) \in [H^1]^m\) that, possibly taking along a subsequence,

\[
\nu_{jk} \rightharpoonup \nu_{j\infty} \ \text{weakly in} \ \ H^1 \ \text{as} \ \ k \to +\infty, j = 1, \ldots, m.
\] (6.8)

Then by virtue of the compactness of embedding \(H^1_{\text{radial}} \subset L_q(R^n)\) \([33]\), where \(q \leq \frac{2n}{n-2}\), it follows that

\[
\nu_{jk} \rightharpoonup \nu_{j\infty} \ \text{in} \ \ L_q(R^n) \ \text{as} \ \ k \to +\infty, j = 1, \ldots, m.
\] (6.9)

Let us prove that \((\nu_{1\infty}, \ldots, \nu_{m\infty}) \neq (0, \ldots, 0)\). Assume the opposite, i.e.

\[
(\nu_{1\infty}, \ldots, \nu_{m\infty}) = (0, \ldots, 0).
\] (6.10)

Using the Hölder inequality

\[
\sum_{j=1}^{m} \|\nu_{jk}\|^2 =
\]
\[
= 2 \sum_{i,j=1}^{m} \int_{\mathbb{R}^n} |\nu_{ik}|^{p+1} \cdot |\nu_{jk}|^{p+1} \, dx \leq 2 \sum_{i,j=1, i<j}^{m} \|\nu_{ik}\|_{L_{2(p+1)}} \cdot \|\nu_{jk}\|_{L_{2(p+1)}}, \tag{6.11}
\]
from (1.6), (6.9), (6.10) we obtain
\[G(\nu_{1k}, \ldots, \nu_{mk}) \to 0, \quad \text{as} \quad k \to +\infty.\]
On the other hand, \(I(\nu_{1k}, \ldots, \nu_{mk}) = 0\), so from (6.9) it follows that
\[\nu_{jk} \to 0 \quad \text{star in} \quad H^1 \quad \text{as} \quad k \to +\infty, j = 1, \ldots, m. \tag{6.12}\]
By the multiplicative inequality of Gagliardo-Nirenberg type \([1]\), we have
\[\|\nu_{jk}\|_{L_{2(p+1)}(\mathbb{R}^n)} \leq |\nabla \nu_{jk}|^{(p+1)\theta} \|\nu_{jk}\|^{(p+1)(1-\theta)}, \tag{6.13}\]
where
\[\theta = n \left(\frac{1}{2} - \frac{1}{2(p+1)}\right) = \frac{np}{2(p+1)}, j = 1, \ldots, m.\]
From (6.7) and (6.13) we have
\[\|\nu_{jk}\|_{L_{2(p+1)}(\mathbb{R}^n)} \leq c |\nabla \nu_{jk}|^{(p+1)\theta}, j = 1, \ldots, m.\]
Therefore by virtue of (6.11) and (6.14) we have
\[\sum_{j=1}^{m} \|\nu_{jk}\|^2 \leq C \left(\sum_{j=1}^{m} \|\nu_{jk}\|^2\right)^{\frac{np}{2}},\]
where \(C > 0\) is a constant not dependent on \(k = 1, 2, \ldots\). It follows that
\[\sum_{j=1}^{m} \|\nu_{jk}\|^2 \geq c_1 > 0,\]
where \(c_1 = C^{\frac{n-p-2}{2}}\). Therefore, our assumption isn’t correct. Thus \(d > 0\).

Proof of Lemma 2.3. It follows from (3.11) that

\[G \leq \sum_{i,j=1, i<j}^{m} \|u_i\|_{2(p+1)}^{p+1} \cdot \|u_j\|_{2(p+1)}^{p+1} \leq \mu C_s^{2(p+1)} \left[\sum_{j=1}^{m} \|u_j\|^2\right]^{p+1}.\]

If \(\sum_{j=1}^{m} \|u_j\|^2 < r(\delta)\), then we get \(2G \leq \delta \sum_{j=1}^{m} \|u_j\|^2\). From the definition of \(I_\delta(u_1, \ldots, u_m)\), we have \(I_\delta(u_1, \ldots, u_m) > 0\).

Proof of Lemma 2.4. If \((u_1, \ldots, u_m) \in [H^1]^m, \|u_{10}\| \neq 0, \ldots, \|u_{m0}\| \neq 0\) and \(I_\delta(u_1, \ldots, u_m) < 0\), then we have the following inequality
\[\delta \sum_{j=1}^{m} \|u_j\|^2 < 2 \sum_{i,j=1, i<j}^{m} \|u_iu_j\|_{p+1}^{p+1} \leq 2\mu C_s^{2(p+1)} \left[\sum_{j=1}^{m} \|u_j\|^2\right]^{p+1},\]
from which the required inequality is obtained.
Proof of Lemma 2.5. If \( \|u_1\| + \ldots + \|u_m\| \neq 0 \), then from 
\( I_\delta(u_1, \ldots, u_m) = 0 \) we get 
\[
\delta \sum_{j=1}^{m} \|u_j\|^2 = 2 \sum_{i,j=1, i<j} |u_iu_j|^{p+1} \leq 2\mu C^2_{\ast}(p+1) \left( \sum_{j=1}^{m} \|u_j\|^2 \right)^{p+1} 
\]
Thus, 
\[
\sum_{j=1}^{m} \|u_j\|^2 \geq r(\delta) = \left( \frac{\delta}{2\mu C^2_{\ast}(p+1)} \right)^{\frac{1}{p}}.
\]

Proof of Lemma 2.6. In view of Lemma 2.5, for each \((u_1, \ldots, u_m) \in N_1\) we have 
\[
\sum_{j=1}^{m} \|u_j\|^2 \geq r(\delta).
\]
Therefore, 
\[
J(u_1, \ldots, u_m) = \frac{1}{2} \left( 1 - \frac{\delta}{p+1} \right) \sum_{j=1}^{m} \|u_j\|^2 \geq a(\delta)r(\delta),
\]
where \(0 < \delta < p+1\), \(a(\delta) = \frac{p+1-\delta}{2(p+1)}\). It follows that \(d(\delta) = a(\delta)r(\delta)\). Suppose that \((\bar{u}_1, \ldots, \bar{u}_m) \in N_1\) is a minimizing element, i.e. \(d = J(\bar{u}_1, \ldots, \bar{u}_m)\).

For any \(\delta > 0\), \(\lambda = \lambda(\delta)\) is chosen so that 
\[
\delta \sum_{j=1}^{m} \|\lambda \bar{u}_j\|^2 = 2 \sum_{i,j=1, i<j} |\lambda \bar{u}_i \lambda \bar{u}_j|^{p+1}.
\] (6.14)
Hence we obtain that 
\[
\lambda(\delta) = \left[ \frac{\delta \sum_{j=1}^{m} \|\bar{u}_j\|^2}{2 \sum_{i,j=1, i<j} |\bar{u}_i \bar{u}_j|^{p+1}} \right]^{\frac{1}{2p}} = \delta^{\frac{1}{2p}}.
\]
In view of (6.14), \((\lambda(\delta)\bar{u}_1, \ldots, \lambda(\delta)\bar{u}_m) \in N_\delta\), therefore, by definition of \(d(\delta)\), we have the following inequality
\[
d(\delta) \leq J(\lambda(\delta)\bar{u}_1, \ldots, \lambda(\delta)\bar{u}_m) = \frac{1}{2} \delta^{\frac{1}{p}} \sum_{j=1}^{m} \|\bar{u}_j\|^2 
- \frac{1}{p+1} \delta^{\frac{p+1}{p}} \sum_{i,j=1, i<j} |\bar{u}_i \bar{u}_j|^{p+1}.\] (6.15)
On the other hand, 
\[(\bar{u}_1, \ldots, \bar{u}_m) \in N_1.\] (6.16)
Therefore,
\[
2 \sum_{i,j=1, i<j}^{m} |\bar{u}_i \bar{u}_j|^{p+1} = \sum_{j=1}^{m} \|\bar{u}_j\|^2. \tag{6.17}
\]

It follows from (6.15) and (6.17) that
\[
d(\delta) \leq \frac{1}{2} \delta^\frac{1}{p} \left(1 - \frac{\delta}{p+1}\right) \sum_{j=1}^{m} \|\bar{u}_j\|^2. \tag{6.18}
\]

Since \((\bar{u}_1, \ldots, \bar{u}_m)\) is the minimizing element, we have
\[
d = J(\bar{u}_1, \ldots, \bar{u}_m) = \frac{p}{2(p+1)} \sum_{j=1}^{m} \|\bar{u}_j\|^2
\]
i.e.
\[
\sum_{j=1}^{m} \|\bar{u}_j\|^2 = \frac{2(p+1)}{p} d. \tag{6.19}
\]

It follows from (6.18) and (6.19) that
\[
d(\delta) \leq \frac{1}{2} \delta^\frac{1}{p} \left(1 - \frac{\delta}{p+1}\right) \frac{2(p+1)}{p} d = \frac{p+1 - \delta}{\delta} \frac{1}{2} \frac{1}{p} d. \tag{6.20}
\]

Let \((\tilde{\nu}_1, \ldots, \tilde{\nu}_m) \in N_\delta\) be the minimizing element of the functional \(J(u_1, \ldots, u_m)\), i.e.
\[
J(\tilde{\nu}_1, \ldots, \tilde{\nu}_m) = \min_{(\nu_1, \ldots, \nu_m) \in N_\delta} J(\nu_1, \ldots, \nu_m) = d(\delta).
\]

The parameter \(\mu = \mu(\delta)\) is chosen so that
\[
(\mu \tilde{\nu}_1, \ldots, \mu \tilde{\nu}_m) \in N_1, \text{ i.e. } I_1(\mu \tilde{\nu}_1, \ldots, \mu \tilde{\nu}_m) = 0 \tag{6.21}
\]
Then,
\[
\mu = \mu(\delta) = \left[ \frac{\sum_{j=1}^{m} \|\bar{\nu}_j\|^2}{2 \sum_{i,j=1, i<j}^{m} |\bar{\nu}_i \bar{\nu}_j|^{p+1} (p+1)} \right]^{\frac{1}{p+1}} = \left( \frac{1}{\delta} \right)^{\frac{1}{p+1}}.
\]

By the definition of \(d\), we have
\[
d \leq J(\mu \tilde{\nu}_1, \ldots, \mu \tilde{\nu}_m) = \frac{1}{2} \left( \frac{1}{\delta} \right)^{\frac{1}{p}} \sum_{j=1}^{m} \|\bar{\nu}_j\|^2
\]
\[
- \frac{1}{p+1} \left( \frac{1}{\delta} \right)^{\frac{1}{p}} \sum_{i,j=1, i<j}^{m} |\bar{\nu}_i \bar{\nu}_j|^{p+1} (p+1) = \left( \frac{1}{\delta} \right)^{\frac{1}{p}} \frac{p}{2(p+1)} \sum_{j=1}^{m} \|\bar{\nu}_j\|^2. \tag{6.22}
\]

On the other hand,
\[
J(\tilde{\nu}_1, \ldots, \tilde{\nu}_m) = \left( \frac{1}{2} - \frac{\delta}{2(p+1)} \right) \sum_{j=1}^{m} \|\bar{\nu}_j\|^2.
\]
Hence we have
\[ \sum_{j=1}^{m} \| \bar{\nu}_j \|^2 = \frac{2(p+1)}{p+1-\delta} J(\bar{\nu}_1, \ldots, \bar{\nu}_m) = \frac{2(p+1)}{p+1-\delta} \cdot d(\delta). \] (6.23)

From (6.22) and (6.23) it follows that
\[ d \leq \left( \frac{1}{\delta} \right) \frac{p}{p+1-\delta} \cdot d(\delta), \]
i.e.
\[ d(\delta) \geq \frac{p+1-\delta}{p} \cdot \frac{1}{\delta^{\frac{1}{p}}} d. \] (6.24)

Comparing (6.20) and (6.24), we obtain that
\[ d(\delta) = \frac{p+1-\delta}{p} \cdot \frac{1}{\delta^{\frac{1}{p}}} d. \]

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**References**


Akbar B. Aliev
*Azerbaijan Technical University, Baku, Azerbaijan.*

Institute of Mathematics and Mechanics, NAS of Azerbaijan, Baku, Azerbaijan
E-mail address: alievakbar@gmail.com

Gulshan Kh. Shafiyeva
*Baku State University, Baku, Azerbaijan*

Institute of Mathematics and Mechanics, NAS of Azerbaijan, Baku, Azerbaijan
E-mail address: gulshan.shafiyeva@mail.ru

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