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DERIVATIVES OF TRIGONOMETRIC POLYNOMIALS AND CONVERSE THEOREM OF THE CONSTRUCTIVE THEORY OF FUNCTIONS IN MORREY SPACES

SADULLA Z. JAFAROV

Abstract. Let $\mathbb{T}$ denote the interval $[-0, 2\pi]$. In this work the relationship between the modulus of smoothness of derivatives of a function and the best approximation in Morrey space $L^{p,\lambda}(\mathbb{T}), 0 < \lambda \leq 2$, $1 < p < \infty$, have been investigated. In addition, the theorems related to the derivatives of trigonometric polynomials in Morrey space $L^{p,\lambda}(\mathbb{T}), 0 < \lambda \leq 2$, $1 < p < \infty$, are proved.

1. Introduction and the main results

Let $\mathbb{T}$ denote the interval $[0, 2\pi]$. Let $L^p(\mathbb{T})$, $1 \leq p < \infty$ be the Lebesgue space of all measurable $2\pi-$periodic functions defined on $\mathbb{T}$ such that

$$\|f\|_p := \left(\int_{\mathbb{T}} |f(x)|^p \, dx \right)^{\frac{1}{p}} < \infty.$$

The *Morrey spaces* $L_0^{p,\lambda}(\mathbb{T})$ for a given $0 \leq \lambda \leq 2$ and $p \geq 1$, we define as the set of functions $f \in L_{loc}^p(\mathbb{T})$ such that

$$\|f\|_{L_0^{p,\lambda}(\mathbb{T})} := \left\{ \sup_I \frac{1}{|I|^{1-\frac{\lambda}{2}}} \int_I |f(t)|^p \, dt \right\}^{\frac{1}{p}} < \infty,$$

where the supremum is taken over all intervals $I \subset [0, 2\pi]$. Note that $L_0^{p,\lambda}(\mathbb{T})$ becomes a Banach spaces, $\lambda = 2$ coincides with $L^p(\mathbb{T})$ and for $\lambda = 0$ with $L^\infty(\mathbb{T})$. If $0 \leq \lambda_1 \leq \lambda_2 \leq 2$, then $L_0^{p,\lambda_1}(\mathbb{T}) \subset L_0^{p,\lambda_2}(\mathbb{T})$. Also, if $f \in L_0^{p,\lambda}(\mathbb{T})$, then $f \in L^p(\mathbb{T})$ and hence $f \in L^1(\mathbb{T})$. The Morrey spaces, were introduced by C. B. Morrey in 1938. The properties of these spaces have been investigated intensively by several authors and together with weighted Lebesgue spaces L_ω^p play an important role in the theory of partial equations, in the study of local behavior of the solutions of elliptic differential equations and describe local regularity more

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precisely than Lebesgue spaces L^p. The detailed information about properties of the Morrey spaces can be found in [13-15], [17], [26], [31], [32], [35], [37], [40], [44] and [45].

In what follows by $L^{p,\lambda}(\mathbb{T})$ we denote the closure of the linear subspace of $L_0^{p,\lambda}(\mathbb{T})$ functions, whose shifts are continuous in $L_0^{p,\lambda}(\mathbb{T})$. Suppose that x, h are real, and let us take into account the following series

$$\Delta_h^\alpha f(x) := \sum_{k=0}^{\infty}(-1)^k \left(\begin{array}{c}\alpha \\ k\end{array}\right) f(x+(\alpha-k)h), \quad \alpha>0, \quad f \in L^{p,\lambda}(\mathbb{T}).$$

Then, by [36, Theorem 11, p.135] the last series converges absolutely almost everywhere (a. e). on $\mathbb{T}$. Hence the operator Δ_h^α by [24] is bounded in the space $L^{p,\lambda}(\mathbb{T})$. Namely,

$$\Delta_h^\alpha f(x)=\sum_{k=0}^{\infty}(-1)^k \left(\begin{array}{c}\alpha \\ k\end{array}\right) f(x+(\alpha-k)h)=\sum_{k=0}^{\alpha}(-1)^{\alpha-k} \left(\begin{array}{c}\alpha \\ k\end{array}\right) f(x+kh).$$

The function

$$\omega_{p,\lambda}^\alpha(f,\delta) := \sup_{|h|\leq\delta} \left\|\Delta_h^\alpha(f,\cdot)\right\|_{L^{p,\lambda}(\mathbb{T})}, \ \alpha\in Z^+$$

is called *α-th modulus of smoothness* of $f \in L^{p,\lambda}(\mathbb{T})$, $0\leq\lambda\leq 2$ and $p\geq 1$.

The modulus of smoothness $\omega_{p,\lambda}^\alpha(f,\delta)_M$ has the following properties [24] :

1) $\omega_{p,\lambda}^\alpha(f,\delta)$ is an increasing function,
2) $\lim\limits_{\delta\to 0}\omega_{p,\lambda}^\alpha(f,\delta)=0$ for every $f\in L^{p,\lambda}(\mathbb{T})$, $0\leq\lambda\leq 2$ and $p\geq 1$,
3) $\omega_{p,\lambda}^\alpha(f+g,\delta)\leq\omega_{p,\lambda}^\alpha(f,\delta)+\omega_{p,\lambda}^\alpha(g,\delta)$ for $f,g\in L^{p,\lambda}(\mathbb{T})$
4) $\omega_{p,\lambda}^\alpha(f,n\delta)\leq n^\alpha\omega_{p,\lambda}^\alpha(f,\delta)$, $\quad n\in\mathbb{N}$,
5) $\omega_{p,\lambda}^\alpha(f,s\delta)\leq(s+1)^\alpha\,\omega_{p,\lambda}^\alpha(f,\delta)$,
6) $\omega_{p,\lambda}^\alpha(f,\delta)\leq[(n+1)\delta+1]^\alpha\,\omega_{p,\lambda}^\alpha(f,\frac{1}{n+1})$, $n\in\mathbb{N}$

For $f\in L^{p,\lambda}(\mathbb{T})$, we define *the derivative of f as a function g* satisfying the condition

$$\lim_{h\to\infty}\left\|\frac{1}{h}\left(f(\cdot+h)-f(\cdot)\right)-g(\cdot)\right\|_{L^{p,\lambda}(\mathbb{T})}=0 \tag{1.1}$$

and we write $g=f'$.

We denote by $E_n(f)_{L^{p,\lambda}(\mathbb{T})}$the best approximation of $f\in L^{p,\lambda}(\mathbb{T})$ by trigonometric polynomials of degree not exceeding n, i.e.,

$$E_n(f)_{L^{p,\lambda}(\mathbb{T})}=\inf\left\{\|f-T_n\|_{L^{p,\lambda}(\mathbb{T})}:T_n\in\Pi_n\right\},$$

where Π_n denotes the class of trigonometric polynomials of degree at most n.

We use the constants $c, c_1, c_2, ...$ (in general, different in different relations) which depend only on the quantities that are not important for the questions of interest.

The problems of approximation theory in the weighted and non-weighted Morrey spaces have been investigated by several authors (see, for example, [6], [7], [9], [18], [24], [25] and [34]).

In this work the relationship between the modulus of smoothness of the derivatives of the function and the best approximation in Morrey space $L^{p,\lambda}(\mathbb{T})$, $0<\lambda\leq 2$ and $1<p<\infty$, have been investigated. In addition, the theorems related to the derivatives of the trigonometric polynomials in Morrey space $L^{p,\lambda}(\mathbb{T}), 0<\lambda\leq 2$ and $1<p<\infty$ are proved.

The similar problems in the different spaces were investigated in [1-5], [10], [11], [16], [19-23], [27-30], [33], [38], [39], [42] and [43].

Our main results are the following.

Theorem 1.1. *If $f\in L^{p,\lambda}(\mathbb{T}),\ 0<\lambda\leq 2$ and $1<p<\infty$, and if the condition*

$$\sum_{m=1}^{\infty} m^{r-1} E_m(f)_{p,\lambda}<\infty,$$

is fulfilled for some $r\in Z^+$. If $T_n\in\Pi_n$ is a near best approximation of f, then has derivative of order r in the sense of (1.1) and the estimate

$$\left\|f^{(r)}-T_n^{(r)}\right\|_{L^{p,\lambda}(\mathbb{T})}\leq c_1\left\{n^r E_n(f)_{L^{p,\lambda}(\mathbb{T})}+\sum_{\mu=n+1}^{\infty}\mu^{r-1}E_\mu(f)_{L^{p,\lambda}(\mathbb{T})}\right\}.$$

holds with a constant $c_1=c_2(p,\lambda,r)>0$ independent of n.

Corollary 1.1. *If $f\in L^{p,\lambda}(\mathbb{T}),\ 0<\lambda\leq 2$, $1<p<\infty$ and the condition*

$$\sum_{m=1}^{\infty} m^{r-1}E_m(f)_{L^{p,\lambda}(\mathbb{T})}<\infty,$$

is fulfilled for some $r\in Z^+$, then f has the $r-th$ derivative $f^{(r)}$ in the sense of $L^{p,\lambda}(\mathbb{T})$ and the estimate

$$\omega_{p,\lambda}^{\alpha}(f^{(r)},\frac{1}{n})\leq c_2\left\{\frac{1}{n^\alpha}\sum_{\nu=0}^{n}(\nu+1)^{(\alpha+r)-1}E_\nu(f)_{L^{p,\lambda}(\mathbb{T})}+\sum_{\nu=n+1}^{\infty}\nu^{r-1}E_\nu(f)_{L^{p,\lambda}(\mathbb{T})}\right\}$$

holds, where $c_2=c_2(p,\lambda,\alpha,r)>0$ is a constant independent of n.

Theorem 1.2. *Let $f\in L^{p,\lambda}(\mathbb{T}),\ 0<\lambda\leq 2$ and $1<p<\infty$ and T_n a sequence of trigonometric polynomials of degree n satisfies the following conditions:*

$$\|f-T_n\|_{L^{p,\lambda}(\mathbb{T})}=o\left(\frac{1}{n}\right)\ \text{and}\ \left\|g-T_n'\right\|_{L^{p,\lambda}(\mathbb{T})}=o(1),\quad n\to\infty.$$

Then we obtain $f'=g$, that is, the function g satisfies the condition (1.1).

Corollary 1.2. *Let $f,g_1,...,g_k\in L^{p,\lambda}(\mathbb{T}),\ 0<\lambda\leq 2$ and $1<p<\infty$ and T_n be a sequence of trigonometric polynomials satisfying, for $i=1,...,k$, the conditions*

$$\begin{aligned}\|f-T_n\|_{L^{p,\lambda}(\mathbb{T})}&=o\left(\frac{1}{n^k}\right),...n\to\infty,\\ \left\|g_i-T_n^{(i)}\right\|_{L^{p,\lambda}(\mathbb{T})}&=o\left(\frac{1}{n^{k-1}}\right),...n\to\infty\end{aligned}$$

Then we obtain $g_i=g_{i-1}'$ $(f=g_0)$ in the sense of (1.1).

Theorem 1.3. *Let* $f \in L^{p,\lambda}(\mathbb{T})$, $0<\lambda\leq 2$ *and* $1<p<\infty$, $\alpha, r \in \mathbb{Z}^{+}$ $(r>\alpha>0)$ *and let* $T_n(f)\in \Pi_n$ *be the polynomial of best approximation to* f *in* $L^{p,\lambda}(\mathbb{T})$. *In order that*

$$\left\| T_n^{(r)}(f) \right\|_{L^{p,\lambda}(\mathbb{T})} = O(n^{r-\alpha})$$

it is necessary and sufficient that

$$E_n(f)_{L^{p,\lambda}(\mathbb{T})} = O(n^{-\alpha}).$$

2. Proofs of main results

The following lemmas for the Morrey spaces $f \in L^{p,\lambda}(\mathbb{T})$, $0<\lambda\leq 2$ and $1<p<\infty$ play an important role in the proofs of the main results.

Lemma 2.1. [24] *Let* $f \in L^{p,\lambda}(\mathbb{T})$, $0<\lambda\leq 2$ *and* $1<p<\infty$. *Then for every* $\alpha \in Z^{+}$ *the inequality*

$$E_n(f)_{L^{p,\lambda}(\mathbb{T})} \leq c_3 \omega_{p,\lambda}^{\alpha}(f, \frac{1}{n+1})$$

holds with a constant $c_3>0$ *independent of* n.

Lemma 2.2. [25] *Let* $f \in L^{p,\lambda}(\mathbb{T})$, $0\leq\lambda\leq 2$ *and* $p\geq 1$. *Then for each trigonometric polynomial* T_n *of degree* n, *the inequality*

$$\left\| T_n^{(k)} \right\|_{L^{p,\lambda}(\mathbb{T})} \leq c_4 n^k \left\| T_n \right\|_{L^{p,\lambda}(\mathbb{T})} \tag{2.1}$$

holds with a constant c_4 *independent of* n.

Lemma 2.3. [25] *Let* $f \in L^{p,\lambda}(\mathbb{T})$,$0<\lambda\leq 2$ *and* $1<p<\infty$. *Then the estimate*

$$\omega_{p,\lambda}^{\alpha}(f, \frac{1}{n}) \leq \frac{c_5}{n^{\alpha}} \sum_{k=1}^{n} k^{\alpha-1} E_k(f)_{L^{p,\lambda}(\mathbb{T})} \ , \ \alpha \in Z^{+}, \ n=1,2,...$$

holds with a const $c_5 = c_5(p,\alpha,\lambda)>0$*independent of* n.

Proof of Theorem 1.1. There exist a sequence of trigonometric polynomils $\{T_n\}_{n=1}^{\infty}$ such that

$$\|f-T_n\|_{L^{p,\lambda}(\mathbb{T})} = E_n(f)_{L^{p,\lambda}(\mathbb{T})}.$$

From the conditions of theorem the following expressions holds:

$$\|T_{2^i} - T_{2^{i-1}}\|_{L^{p,\lambda}(\mathbb{T})} \leq 2E_{2^{i-1}}(f)_{L^{p,\lambda}(\mathbb{T})},$$

$$f = T_1 + \sum_{i=1}^{\infty}(T_{2^i} - T_{2^{i-1}}) = \sum_{i=0}^{\infty} V_{2^i}$$

Now, we show that for $j=1,...,r$ there exist the function $\psi_j(x) \in L^{p,\lambda}(\mathbb{T})$ such that

$$\psi_j(x) = \sum_{i=0}^{\infty} V_{2^i}^{(j)}(x)$$

and

$$\psi_j(x) = f^{(j)}(x).$$

Using (2.1) for $j=1$ we obtain

$$\left\|\frac{f(\cdot+h)-f(\cdot)}{h}-\varphi_1(\cdot)\right\|_{L^{p,\lambda}(\mathbb{T})}$$

$$\leq\left\|\sum_{i=0}^{\infty}\frac{V_{2^i}(\cdot+h)-V_{2^i}(\cdot)}{h}-\sum_{i=0}^{\infty}V'_{2^i}(\cdot)\right\|_{L^{p,\lambda}(\mathbb{T})}$$

$$\leq\sum_{i=0}^{n_0}\left\|\frac{V_{2^i}(\cdot+h)-V_{2^i}(\cdot)}{h}-V'_{2^i}(\cdot)\right\|_{L^{p,\lambda}(\mathbb{T})}$$

$$+\sum_{i=n_0+1}^{\infty}\left(\left\|\frac{V_{2^i}(\cdot+h)-V_{2^i}(\cdot)}{h}\right\|_{L^{p,\lambda}(\mathbb{T})}+\|V_{2^i}\|_{L^{p,\lambda}(\mathbb{T})}\right)$$

$$\leq\sum_{i=0}^{n_0}\left\|\frac{V_{2^i}(\cdot+h)-V_{2^i}(\cdot)}{h}-V'_{2^i}(\cdot)\right\|_{L^{p,\lambda}(\mathbb{T})}+c_6\sum_{i=n_0+1}^{\infty}2^{i\beta}\|V_{2^i}\|_{L^{p,\lambda}(\mathbb{T})}. \tag{2.2}$$

From the inequality (2.2) for $h\to 0$ and $n\geq n_0$ we have

$$f'(x)=\psi_1(x).$$

For $j=2,...,n,$ to prove theorem we use the method of induction.
Taking (2.1) we have

$$\begin{aligned}\left\|T_n^{(r)}-f^{(r)}\right\|_{L^{p,\lambda}(\mathbb{T})}&\leq\left\|T_n^{(r)}-T_{2^m}^{(r)}\right\|_{L^{p,\lambda}(\mathbb{T})}+\sum_{i=m+1}^{\infty}\left\|T_{2^i}^{(r)}-T_{2^{i-1}}^{(r)}\right\|_{L^{p,\lambda}(\mathbb{T})}\\&\leq c_7\left\{n^rE_n(f)_{L^{p,\lambda}(\mathbb{T})}+\sum_{i=m+1}^{\infty}2^{ir}E_{2^{i-1}}(f)_{L^{p,\lambda}(\mathbb{T})}\right\}.\end{aligned} \tag{2.3}$$

For $i=1,2,...,$ the following inequality holds:

$$2^{ir}E_{2^{i-1}}(f)_{L^{p,\lambda}(\mathbb{T})}\leq 2^{2r}\sum_{\mu=2^{i-2}+1}^{2^{i-1}}\mu^{r-1}E_\mu(f)_{L^{p,\lambda}(\mathbb{T})}. \tag{2.4}$$

Choosing m such that $2^{m-1}\leq n<2^m$, using (2.3) and (2.4) we obtain

$$\left\|T_n^{(r)}-f^{(r)}\right\|_{L^{p,\lambda}(\mathbb{T})}\leq c_8\left\{n^rE_n(f)_{L^{p,\lambda}(\mathbb{T})}+\sum_{i=m+1}^{\infty}2^{ri}E_{2^{i-1}}(f)_{L^{p,\lambda}(\mathbb{T})}\right\}$$

$$\leq c_9\left\{n^rE_n(f)_{L^{p,\lambda}(\mathbb{T})}+\sum_{\mu=n+1}^{\infty}\mu^{r-1}E_\mu(f)_{L^{p,\lambda}(\mathbb{T})}\right\}.$$

This completes the proof of Theorem 1.1.
Corollary 1.1 follows immediately from lemma 2.3 and theorem 1.1.

Proof of Theorem 1.2. We take $\varepsilon>0$. We choose natural number $n_0=n_0(\varepsilon)$ such that for $n\geq n_0$

$$\|f-T_n\|_{L^{p,\lambda}(\mathbb{T})}\leq\varepsilon\frac{1}{n},\qquad\left\|g-T'_n\right\|_{L^{p,\lambda}(\mathbb{T})}\leq\varepsilon. \tag{2.5}$$

Taking account of (2.5) for h satisfying the condition $\frac{\sqrt{\varepsilon}}{n} \leq h \leq \frac{1}{n}$ we obtain

$$\left\| \frac{f(\cdot + h) - f(\cdot)}{h} - \frac{T(\cdot + h) - T_n(\cdot)}{h} \right\|^p_{L^{p,\lambda}(\mathbb{T})} \leq 2^{\frac{p}{2}}. \tag{2.6}$$

Considering [12] we have

$$\begin{aligned}
\Delta_h^r T_n(x) &= \sum_{i=0}^{r} \left(\begin{array}{c} r \\ i \end{array} \right) (-1)^i T_n \left(x + \left(\frac{r}{2} - i \right) h \right) \\
&= \sum_{j=r}^{\infty} \sum_{i=0}^{r} \left(\begin{array}{c} r \\ i \end{array} \right) (-1)^i \left(\frac{r}{2} - i \right)^j \frac{h^j}{j!} T_n^{(j)}(x) \\
&= h^r T_n^{(r)}(x) + \sum_{j=r+1}^{\infty} \eta(r,j)^{j-r} T_n^{(j)}(x),
\end{aligned} \tag{2.7}$$

where $-\frac{r}{2} < \eta(r,j) < \frac{r}{2}$ and $\eta(r,j) = 0$ if $j - r$ is odd. Then using (2.7) and Lemma 2.2 for $\frac{\sqrt{\varepsilon}}{n} \leq h < \frac{2\sqrt{\varepsilon}}{n}$ we find that

$$\begin{aligned}
\left\| \frac{T_n(\cdot + h) - T_n(\cdot)}{h} - T_n'(\cdot) \right\|^p_{L^{p,\lambda}(\mathbb{T})} &\leq \sum_{m=2}^{\infty} \left(\frac{h^{m-1}}{m!} \right)^p \left\| T_n^{(m)} \right\|^p_{L^{p,\lambda}(\mathbb{T})} \\
&\leq \sum_{m=2}^{\infty} (hn)^{(m-1)p} \left\| T_n \right\|^p_{L^{p,\lambda}(\mathbb{T})} \\
&\leq 4 \frac{\varepsilon}{1 - 2^p \varepsilon^{p/2}} \left\| T_n \right\|^p_{L^{p,\lambda}(\mathbb{T})} \leq c_{12} \varepsilon^p \left\| T_n \right\|^p_{L^{p,\lambda}(\mathbb{T})}.
\end{aligned} \tag{2.8}$$

Using (2.6), (2.8) and (2.5) for $\frac{\sqrt{\varepsilon}}{n} \leq h < \frac{2\sqrt{\varepsilon}}{n}$ we reach

$$\begin{aligned}
\left\| \frac{f(\cdot + h) - f(\cdot)}{h} - g \right\|^p_{L^{p,\lambda}(\mathbb{T})} &\leq \left\| \frac{f(\cdot + h) - f(\cdot)}{h} - \frac{T_n(\cdot + h) - T_n(\cdot)}{h} \right\|^p_{L^{p,\lambda}(\mathbb{T})} \\
&+ \left\| \frac{T_n(\cdot + h) - T_n(\cdot)}{h} - T_n'(\cdot) \right\|^p_{L^{p,\lambda}(\mathbb{T})} \\
&+ \left\| T_n' - g \right\|^p_{L^{p,\lambda}(\mathbb{T})} \leq c_{10} \left(\varepsilon^{p/2} + \varepsilon^p \left\| f \right\|^p_{L^{p,\lambda}(\mathbb{T})} + \varepsilon^p \right).
\end{aligned}$$

From the last inequality we have $g = f'$ in the sense of (1.1). Then the proof of Theorem 1.2 is completed.

Proof of Theorem 1.3. We suppose that

$$E_n(f)_{L^{p,\lambda}(\mathbb{T})} = \left\| f - T_n(f) \right\|_{L^{p,\lambda}(\mathbb{T})} = O(n^{-\alpha}),\ (\alpha > 0). \tag{2.9}$$

We use the method of the proofs in [12] and [41] we can prove that

$$\left\| T_n^{(r)}(f) \right\|_{L^{p,\lambda}(\mathbb{T})} \leq c_{11} n^r \omega^r_{p,\lambda}(f, \frac{1}{n}) \tag{2.10}$$

$$\omega^r_{p,\lambda}(f, \frac{1}{n}) \leq c_{12} n^{-r} \left\| T_n^{(r)}(f) \right\|_{L^{p,\lambda}(\mathbb{T})} \tag{2.11}$$

Then applying (2.10), lemma 2.3 and relation (2.9) we obtain

$$\left\| T_n^{(r)}(f) \right\|_{L^{p,\lambda}(\mathbb{T})} \leq c_{13} n^{r-\alpha}.$$

Now we suppose that the relation

$$\left\|T_n^{(r)}(f)\right\|_{L^{p,\lambda}(\mathbb{T})} = O(n^{r-\alpha}) \tag{2.12}$$

holds. By virtue of lemma 2.1, (2.11) and (2.12) we get

$$\begin{aligned} &\|T_{2n}(f) - T_n(T_{2n}(f))\|_{L^{p,\lambda}(\mathbb{T})} \le c_{14}\omega^r_{p,\lambda}(f, \frac{1}{n+1}) \\ &\le c_{15}n^{-r}\left\|T_{2n}^{(r)}(f)\right\|_{L^{p,\lambda}(\mathbb{T})} \le c_{16}n^{-r}(n^{r-\alpha}) \le c_{17}n^{-\alpha}. \end{aligned} \tag{2.13}$$

On the other hand, since $T_n(T_{2n}(f))$ is a polynomial of order n the following inequality holds:

$$\begin{aligned} \|T_{2n}(f) - T_n(T_{2n}(f))\|_{L^{p,\lambda}(\mathbb{T})} &= \|f - T_n(T_{2n}(f)) - (f - T_{2n}(f))\|_{L^{p,\lambda}(\mathbb{T})} \\ &\ge \|f - T_n(T_{2n}(f))\|_{L^{p,\lambda}(\mathbb{T})} - \|f - T_{2n}(f))\|_{L^{p,\lambda}(\mathbb{T})} \\ &\ge E_n(f)_{L^{p,\lambda}(\mathbb{T})} - E_{2n}(f)_{L^{p,\lambda}(\mathbb{T})} \ge 0. \end{aligned} \tag{2.14}$$

By (2.13) and (2.14) we obtain

$$0 \le E_n(f)_{L^{p,\lambda}(\mathbb{T})} - E_{2n}(f)_{L^{p,\lambda}(\mathbb{T})} \le c_{18}n^{-\alpha}. \tag{2.15}$$

Since $E_n(f)_{L^{p,\lambda}(\mathbb{T})} \to 0$ the inequality (2.15) yields

$$\sum_{k=n_0}^{\infty}\left\{E_{2^k}(f)_{L^{p,\lambda}(\mathbb{T})} - E_{2^{k+1}}(f)_{L^{p,\lambda}(\mathbb{T})}\right\} \le c_{19}\sum_{k=n_0}^{\infty}2^{-k\alpha}.$$

Then from the last inequality we conclude that

$$E_{2^{n_0}}(f)_{L^{p,\lambda}(\mathbb{T})} \le c_{20}2^{-n_0\alpha}. \tag{2.16}$$

Note that inequality (2.16) is equivalent to $E_n(f)_{L^{p,\lambda}(\mathbb{T})} \le c_{21}(n^{-\alpha})$.The theorem is proved.

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Sadulla Z. Jafarov

Department of Mathematics and Science Education, Faculty of Education, Muş Alparslan University, 49250, Muş, Turkey

Institute of Mathematics and Mechanics, National Academy of Sciences of Azerbaijan, 9 B. Vahabzadeh str., AZ 1141, Baku, Azerbaijan

E-mail address: `s.jafarov@alparslan.edu.tr`

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