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SOME GENERAL FAMILIES OF THE HURWITZ-LERCH ZETA FUNCTIONS AND THEIR APPLICATIONS: RECENT DEVELOPMENTS AND DIRECTIONS FOR FURTHER RESEARCHES

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Abstract. Our main purpose in this survey-cum-expository article is to systematically investigate several families of the celebrated Hurwitz-Lerch Zeta function including (for example) the so-called λ-generalized Hurwitz-Lerch Zeta functions. We first present here many potentially useful results involving some of these λ-generalized Hurwitz-Lerch Zeta functions including their partial differential equations, new series and Mellin-Barnes type contour integral representations (which are associated with Fox's H-function) and several other summation formulas involving them. We then propose to discuss their potential application in Number Theory by appropriately constructing a presumably new continuous analogue of Lippert's Hurwitz measure and also consider some other statistical applications of these families of the λ-generalized Hurwitz-Lerch Zeta functions in probability distribution theory. A brief description and survey of some recent developments and other applications involving several directions for further researches on the subject-matter of our presentation here will also be given.

1. Introduction, Definitions and Preliminaries

Throughout our present investigation, we use the following standard notations:

$$\mathbb{N} := \{1, 2, 3, \cdots\}, \quad \mathbb{N}_0 := \{0, 1, 2, 3, \cdots\} = \mathbb{N} \cup \{0\}$$

and

$$\mathbb{Z}^- := \{-1, -2, -3, \cdots\} = \mathbb{Z}_0^- \setminus \{0\}.$$

Also, as usual, $\mathbb{Z}$ denotes the set of integers, $\mathbb{R}$ denotes the set of real numbers, $\mathbb{R}^+$ denotes the set of *positive* numbers and $\mathbb{C}$ denotes the set of complex numbers.

One of the fundamentally important higher transcendental functions of *Analytic Number Theory* is the familiar general Hurwitz-Lerch Zeta function $\Phi(z, s, a)$

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defined by (see, for example, [23, p. 27. Eq. 1.11 (1)]; see also [68], [75, p. 121 *et seq.*] and [76, p. 194 *et seq.*])

$$\Phi(z,s,a) := \sum_{n=0}^{\infty} \frac{z^n}{(n+a)^s} \tag{1.1}$$

$$\big(a \in \mathbb{C} \setminus \mathbb{Z}_0^-;\ s \in \mathbb{C} \quad \text{when} \quad |z|<1;\ \Re(s)>1 \quad \text{when} \quad |z|=1\big).$$

It contains, as its *special* cases, not only the Riemann Zeta function $\zeta(s)$, the Hurwitz (or generalized) Zeta function $\zeta(s,a)$ and the Lerch Zeta function $\ell_s(\xi)$ defined by (see, for details, [23, Chapter I] and [75, Chapter 2])

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s} = \Phi(1,s,1) = \zeta(s,1) \qquad \big(\Re(s)>1\big), \tag{1.2}$$

$$\zeta(s,a) := \sum_{n=0}^{\infty} \frac{1}{(n+a)^s} = \Phi(1,s,a) \qquad \big(\Re(s)>1;\ a \in \mathbb{C} \setminus \mathbb{Z}_0^-\big) \tag{1.3}$$

and

$$\ell_s(\xi) := \sum_{n=0}^{\infty} \frac{e^{2n\pi \mathrm{i}\xi}}{(n+1)^s} = \Phi\Big(e^{2\pi \mathrm{i}\xi}, s, 1\Big) \qquad \big(\Re(s)>1;\ \xi \in \mathbb{R}\big), \tag{1.4}$$

respectively, but also such other important functions of *Analytic Number Theory* as the Polylogarithmic function (or *de Jonquière's function*) $\mathrm{Li}_s(z)$:

$$\mathrm{Li}_s(z) := \sum_{n=1}^{\infty} \frac{z^n}{n^s} = z\Phi(z,s,1) \tag{1.5}$$

$$\big(s \in \mathbb{C} \quad \text{when} \quad |z|<1;\ \Re(s)>1 \quad \text{when} \quad |z|=1\big)$$

and the Lipschitz-Lerch Zeta function $\phi(\xi,a,s)$ (see [75, p. 122, Equation 2.5 (11)]):

$$\phi(\xi,s,a) := \sum_{n=0}^{\infty} \frac{e^{2n\pi \mathrm{i}\xi}}{(n+a)^s} = \Phi\Big(e^{2\pi \mathrm{i}\xi}, s, a\Big) \tag{1.6}$$

$$\big(a \in \mathbb{C} \setminus \mathbb{Z}_0^-;\ \Re(s)>0 \quad \text{when} \quad \xi \in \mathbb{R} \setminus \mathbb{Z};\ \Re(s)>1 \quad \text{when} \quad \xi \in \mathbb{Z}\big),$$

which was first studied by Rudolf Lipschitz (1832-1903) and Matyáš Lerch (1860-1922) in connection with Dirichlet's famous theorem on primes in arithmetic progressions (see also [69, Section 5]). Indeed, just as its aforementioned special cases $\zeta(s)$ and $\zeta(s,a)$, the Hurwitz-Lerch Zeta function $\Phi(z,s,a)$ defined by (1.7) can be continued *meromorphically* to the whole complex s-plane, except for a simple pole at $s=1$ with its residue 1. It is also known that [23, p. 27, Equation 1.11 (3)]

$$\Phi(z,s,a) = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{t^{s-1}\, e^{-at}}{1-ze^{-t}}\,\mathrm{d}t = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{t^{s-1}\, e^{-(a-1)t}}{e^t - z}\,\mathrm{d}t \tag{1.7}$$

$$\big(\Re(a)>0;\ \Re(s)>0 \ \text{ when } \ |z| \leqq 1\ (z \neq 1);\ \Re(s)>1 \ \text{ when } \ z=1\big).$$

Recently, Srivastava *et al.* [95] introduced and systematically studied various properties and results involving a natural *multiparameter* extension and generalization of the Hurwitz-Lerch zeta function $\Phi(z,s,a)$ defined by (1.7) (see also [69] and [87]). In order to recall their definition (which was motivated essentially by the earlier works of Goyal and Laddha [31], Lin and Srivastava [40], Garg *et al.* [26], and other authors), each of the following notations will be employed:

$$\nabla^* := \left(\prod_{j=1}^{p} \rho_j^{-\rho_j}\right) \cdot \left(\prod_{j=1}^{q} \sigma_j^{\sigma_j}\right) \tag{1.8}$$

and

$$\Delta := \sum_{j=1}^{q} \sigma_j - \sum_{j=1}^{p} \rho_j \qquad \text{and} \qquad \Xi := s + \sum_{j=1}^{q} \mu_j - \sum_{j=1}^{p} \lambda_j + \frac{p-q}{2} \tag{1.9}$$

The extended Hurwitz-Lerch zeta function

$$\Phi^{(\rho_1,\cdots,\rho_p,\sigma_1,\cdots,\sigma_q)}_{\lambda_1,\cdots,\lambda_p;\mu_1,\cdots,\mu_q}(z,s,a)$$

is then defined by [95, p. 503, Equation (6.2)] (see also [69] and [87])

$$\Phi^{(\rho_1,\cdots,\rho_p,\sigma_1,\cdots,\sigma_q)}_{\lambda_1,\cdots,\lambda_p;\mu_1,\cdots,\mu_q}(z,s,a) := \sum_{n=0}^{\infty} \frac{\prod\limits_{j=1}^{p} (\lambda_j)_{n\rho_j}}{n! \cdot \prod\limits_{j=1}^{q} (\mu_j)_{n\sigma_j}} \frac{z^n}{(n+a)^s} \tag{1.10}$$

$$\Bigg(p, q \in \mathbb{N}_0;\ \lambda_j \in \mathbb{C}\ (j = 1, \cdots, p);\ a, \mu_j \in \mathbb{C} \setminus Z_0^-\ (j = 1, \cdots, q);$$
$$\rho_j, \sigma_k \in \mathbb{R}^+\ (j = 1, \cdots, p;\ k = 1, \cdots, q);$$
$$\Delta > -1 \ \text{ when } \ s, z \in \mathbb{C};$$
$$\Delta = -1 \ \text{ and } \ s \in \mathbb{C} \ \text{ when } \ |z| < \nabla^*;$$
$$\Delta = -1 \ \text{ and } \ \Re(\Xi) > \frac{1}{2} \ \text{ when } \ |z| = \nabla^*\Bigg),$$

where $(\lambda)_\nu$ $(\lambda, \nu \in \mathbb{C})$ denotes the Pochhammer symbol (or the *shifted* factorial) which is defined, in terms of the familiar Gamma function, by

$$(\lambda)_\nu := \frac{\Gamma(\lambda+\nu)}{\Gamma(\lambda)} = \begin{cases} 1 & (\nu = 0;\ \lambda \in \mathbb{C} \setminus \{0\}) \\ \lambda(\lambda+1)\cdots(\lambda+n-1) & (\nu = n \in \mathbb{N};\ \lambda \in \mathbb{C}), \end{cases}$$

it being understood *conventionally* that $(0)_0 := 1$ and assumed *tacitly* that the above Γ-quotient exists. In terms of the extended Hurwitz-Lerch zeta function defined by (1.10), the following unification and generalization of several known integral representations stemming from (1.7) was given by Srivastava *et al.* [95] (see also [71, Theorem 4.1] for a more general sum-integral representation formula):

$$\Phi^{(\rho_1,\cdots,\rho_p,\sigma_1,\cdots,\sigma_q)}_{\lambda_1,\cdots,\lambda_p;\mu_1,\cdots,\mu_q}(z,s,a) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1}\, e^{-at}$$
$$\cdot\, {}_p\Psi_q^* \left[\begin{array}{l} (\lambda_1,\rho_1), \cdots, (\lambda_p,\rho_p); \\ \\ (\mu_1,\sigma_1), \cdots, (\mu_q,\sigma_q); \end{array} ze^{-t} \right] \mathrm{d}t \tag{1.11}$$
$$\big(\min\{\Re(a), \Re(s)\} > 0\big),$$

provided that the integral exists. Here, and in what follows, ${}_p\Psi_q^*$ or ${}_p\Psi_q$ $(p, q \in \mathbb{N}_0)$ denotes the Fox-Wright function, which is a generalization of the familiar generalized hypergeometric function ${}_pF_q$ $(p, q \in \mathbb{N}_0)$, with p numerator parameters $a_1, \cdots, a_p$ and q denominator parameters $b_1, \cdots, b_q$ such that

$$a_j \in \mathbb{C} \;\; (j = 1, \cdots, p) \qquad \text{and} \qquad b_j \in \mathbb{C} \setminus \mathbb{Z}_0^- \;\; (j = 1, \cdots, q),$$

defined by (see, for details, [23, p. 183], [90, p. 21 *et seq.*] and [92, p. 50 *et seq.*]; see also [36, p. 56], [48, p. 30] and [86, p. 19])

$$\begin{aligned}
{}_p\Psi_q^* & \left[\begin{array}{c} (a_1, A_1), \cdots, (a_p, A_p); \\ \\ (b_1, B_1), \cdots, (b_q, B_q); \end{array} z \right] \\
& := \sum_{n=0}^{\infty} \frac{(a_1)_{A_1 n} \cdots (a_p)_{A_p n}}{(b_1)_{B_1 n} \cdots (b_q)_{B_q n}} \frac{z^n}{n!} \\
& = \frac{\Gamma(b_1) \cdots \Gamma(b_q)}{\Gamma(a_1) \cdots \Gamma(a_p)} \, {}_p\Psi_q \left[\begin{array}{c} (a_1, A_1), \cdots, (a_p, A_p); \\ \\ (b_1, B_1), \cdots, (b_q, B_q); \end{array} z \right]
\end{aligned} \tag{1.12}$$

$$\left(A_j > 0 \quad (j = 1, \cdots, p); \; B_j > 0 \quad (j = 1, \cdots, q); \; 1 + \sum_{j=1}^{q} B_j - \sum_{j=1}^{p} A_j \geqq 0 \right),$$

where the equality in the convergence condition holds true for suitably bounded values of $|z|$ given by

$$|z| < \nabla := \left(\prod_{j=1}^{p} A_j^{-A_j} \right) \cdot \left(\prod_{j=1}^{q} B_j^{B_j} \right). \tag{1.13}$$

Definition 1.1. (see Srivastava [70]). By suitably modifying this last integral representation formula (1.11), we now introduce and investigate the various properties of a significantly more general class of Hurwitz-Lerch zeta type functions defined by

$$\begin{aligned}
\Phi_{\lambda_1, \cdots, \lambda_p; \mu_1, \cdots, \mu_q}^{(\rho_1, \cdots, \rho_p, \sigma_1, \cdots, \sigma_q)}(z, s, a; b, \lambda) & := \frac{1}{\Gamma(s)} \int_0^{\infty} t^{s-1} \exp\left(-at - \frac{b}{t^{\lambda}} \right) \\
& \cdot {}_p\Psi_q^* \left[\begin{array}{c} (\lambda_1, \rho_1), \cdots, (\lambda_p, \rho_p); \\ \\ (\mu_1, \sigma_1), \cdots, (\mu_q, \sigma_q); \end{array} z e^{-t} \right] \mathrm{d}t,
\end{aligned} \tag{1.14}$$

$$\big(\min\{\Re(a), \Re(s)\} > 0; \; \Re(b) \geqq 0; \; \lambda \geqq 0 \big),$$

so that, obviously, we have the following relationship:

$$\begin{aligned}
\Phi_{\lambda_1, \cdots, \lambda_p; \mu_1, \cdots, \mu_q}^{(\rho_1, \cdots, \rho_p, \sigma_1, \cdots, \sigma_q)}(z, s, a; 0, \lambda) & = \Phi_{\lambda_1, \cdots, \lambda_p; \mu_1, \cdots, \mu_q}^{(\rho_1, \cdots, \rho_p, \sigma_1, \cdots, \sigma_q)}(z, s, a) \\
& = e^b \, \Phi_{\lambda_1, \cdots, \lambda_p; \mu_1, \cdots, \mu_q}^{(\rho_1, \cdots, \rho_p, \sigma_1, \cdots, \sigma_q)}(z, s, a; b, 0).
\end{aligned} \tag{1.15}$$

In its special case when

$$p - 1 = q = 0 \qquad (\lambda_1 = \mu; \; \rho_1 = 1),$$

the above definition (1.14) would reduce immediately to the following form:

$$\Theta_{\mu}^{\lambda}(z,s,a;b) := \frac{1}{\Gamma(s)} \int_0^{\infty} t^{s-1} \exp\left(-at - \frac{b}{t^{\lambda}}\right) \left(1 - ze^{-t}\right)^{-\mu} \, \mathrm{d}t \tag{1.16}$$

$$\big(\min\{\Re(a), \Re(s)\} > 0;\ \Re(b) \geqq 0;\ \lambda \geqq 0;\ \mu \in \mathbb{C}\big),$$

where we have assumed *further* that

$$\Re(s) > 0 \quad \text{when} \quad b = 0 \quad \text{and} \quad |z| \leqq 1 \quad (z \neq 1)$$

or

$$\Re(s-\mu) > 0 \quad \text{when} \quad b = 0 \quad \text{and} \quad z = 1,$$

provided, of course, that the integral in (1.16) exists. The function $\Theta_{\mu}^{\lambda}(z,s,a;b)$ was introduced and studied by Raina and Chhajed [63, p. 90, Equation (1.6)] and (more recently) by Srivastava *et al.* [91].

Two interesting *further* special cases of the function $\Theta_{\mu}^{\lambda}(z,s,a;b)$ are worthy of note here. First of all, for $b = 0$, we find from the definition (1.16) that

$$\Theta_{\mu}^{\lambda}(z,s,a;0) = \Phi_{\mu}^{*}(z,s,a) = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{t^{s-1}\, e^{-at}}{(1-ze^{-t})^{\mu}} \, \mathrm{d}t \tag{1.17}$$

$$\big(\Re(a) > 0;\ \Re(s) > 0 \quad \text{when} \quad |z| \leqq 1 \ \ (z \neq 1);\ \Re(s-\mu) > 0 \quad \text{when} \quad z = 1\big),$$

where the function $\Phi_{\mu}^{*}(z,s,a)$ defined by

$$\Phi_{\mu}^{*}(z,s,a) := \sum_{n=0}^{\infty} \frac{(\mu)_n}{(a+n)^s} \frac{z^n}{n!} \tag{1.18}$$

was studied by Goyal and Laddha [31, p. 100, Equation (1.5)]. As a matter of fact, in terms of the *Riemann-Liouville fractional derivative operator* $\mathcal{D}_z^{\mu}$ defined by (see, for example, [24, p. 181], [36, p. 70 *et seq.*] and [64])

$$\mathcal{D}_z^{\mu}\{f(z)\} := \begin{cases} \dfrac{1}{\Gamma(-\mu)} \displaystyle\int_0^z (z-t)^{-\mu-1} \ f(t)\, dt & \big(\Re(\mu) < 0\big) \\[4mm] \dfrac{d^m}{dz^m}\Big\{\mathcal{D}_z^{\mu-m}\{f(z)\}\Big\} & \big(m-1 \leqq \Re(\mu) < m \ \ (m \in \mathbb{N})\big), \end{cases}$$

it is easily seen from the series definitions in (1.1) and (1.18) that

$$\Phi_{\mu}^{*}(z,s,a) = \frac{1}{\Gamma(\mu)} \mathcal{D}_z^{\mu-1}\left\{z^{\mu-1}\ \Phi(z,s,a)\right\} \qquad \big(\Re(\mu) > 0\big), \tag{1.19}$$

which (as already remarked by Lin and Srivastava [40, p. 730]) exhibits the interesting (and useful) fact that the function $\Phi_{\mu}^{*}(z,s,a)$ is essentially a Riemann-Liouville fractional derivative of the classical Hurwitz-Lerch zeta function $\Phi(z,s,a)$ (see also the closely-related investigations by Garg *et al.* [27] and Lin *et al.* [41]).

The other interesting special case of the function $\Theta_{\mu}^{\lambda}(z,s,a;b)$ arises when we set $\lambda = \mu = 1$ and $z = 1$ in the definition (1.16). We thus find that

$$\Theta_1^1(1,s,a;b) = \zeta_b(s,a) := \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{t^{s-1}\ \exp\left(-at - \frac{b}{t}\right)}{1 - e^{-t}} \, \mathrm{d}t, \tag{1.20}$$

where $\zeta_b(s,a)$ is the extended Hurwitz zeta function defined in [17, p. 308]. In fact, just as it is already pointed out in [44], the series representation (see [63, p.

91, Equation (2.1)]) given for the function $\Theta_\mu^\lambda(z,s,a;b)$ in (1.16) is incorrect. Obvious *further* specializations in (1.17) and (1.20) would immediately relate these functions with the Riemann zeta function $\zeta(s)$ and the Hurwitz (or generalized) zeta function $\zeta(s,a)$ defined by (1.2) and (1.3), respectively.

Remark 1.1. In a series of recent papers, Bayad *et al.* (see [7], [8] and [25]) introduced and studied the so-called *generalized Hurwitz-Lerch zeta function* $\zeta(s,\mu;a,z)$ *of order* μ, which they defined by (*cf.* [8, p. 608, Equation (6)])

$$\zeta(s,\mu;a,z) := \frac{\Gamma(\mu)}{\Gamma(s)} \int_0^\infty \frac{t^{s-1}\, e^{-at}}{(1-ze^{-t})^\mu}\, \mathrm{d}t \tag{1.21}$$

$$\big(\Re(a)>0;\ \Re(s)>0 \quad \text{when} \quad |z| \leqq 1 \ (z\neq 1);\ \Re(s-\mu)>0 \quad \text{when} \quad z=1\big)$$

or, equivalently, by (*cf.* [8, p. 608, Equation (7)])

$$\zeta(s,\mu;a,z) := \sum_{n=0}^\infty \frac{\Gamma(\mu+n)}{n!}\, \frac{z^n}{(a+n)^s}. \tag{1.22}$$

By comparing the definitions (1.18) and (1.22), it is easily observed that

$$\zeta(s,\mu;a,z) = \Gamma(\mu)\cdot \Phi_\mu^*(z,s,a) \quad \text{and} \quad \Phi_\mu^*(z,s,a) = \frac{1}{\Gamma(\mu)}\, \zeta(s,\mu;a,z). \tag{1.23}$$

Clearly, therefore, Equations (1.23) exhibit the fact that the generalized Hurwitz-Lerch zeta function $\zeta(s,\mu;a,z)$ of order μ, which was considered recently by Bayad *et al.* (see [7], [8] and [25]) is only a constant multiple of the *widely- and extensively-investigated* extended Hurwitz-Lerch Zeta function $\Phi_\mu^*(z,s,a)$ defined by (1.18).

In our present systematic investigation of the λ-generalized Hurwitz-Lerch zeta function

$$\Phi_{\lambda_1,\cdots,\lambda_p;\mu_1,\cdots,\mu_q}^{(\rho_1,\cdots,\rho_p,\sigma_1,\cdots,\sigma_q)}(z,s,a;b,\lambda)$$

defined by (1.14), we make use also of the widely-studied H-function of Charles Fox (1897–1997), which is defined by (see, for details, [48, p. 2, Definition 1.1]; see also [35, p. 1 *et seq.*], [86, p. 10 *et seq.*] and [92, p. 49 *et seq.*])

$$\begin{aligned} H_{p,q}^{m,n}(z) &= H_{p,q}^{m,n}\left[z \,\middle|\, \begin{matrix} (a_p, A_p) \\ \\ (b_q, B_q) \end{matrix} \right] \\ &= H_{p,q}^{m,n}\left[z \,\middle|\, \begin{matrix} (a_1, A_1), \cdots, (a_p, A_p) \\ \\ (b_1, B_1), \cdots, (b_q, B_q) \end{matrix} \right] = \frac{1}{2\pi \mathrm{i}} \int_{\mathcal{L}} \Xi(\mathfrak{s}) z^{-\mathfrak{s}}\, \mathrm{d}\mathfrak{s}, \end{aligned} \tag{1.24}$$

where

$$\Xi(\mathfrak{s}) = \frac{\displaystyle\prod_{j=1}^m \Gamma(b_j + B_j\mathfrak{s}) \prod_{j=1}^n \Gamma(1-a_j-A_j\mathfrak{s})}{\displaystyle\prod_{j=m+1}^q \Gamma(1-b_j-B_j\mathfrak{s}) \prod_{j=n+1}^p \Gamma(a_j+A_j\mathfrak{s})}. \tag{1.25}$$

Here

$$z \in \mathbb{C} \setminus \{0\} \qquad \text{with} \qquad |\arg(z)| < \pi,$$

an empty product is interpreted as 1, m, n, p and q are integers such that $1 \leqq m \leqq q$ and $0 \leqq n \leqq p$,

$$A_j > 0 \quad (j = 1, \cdots, p) \qquad \text{and} \qquad B_j > 0 \quad (j = 1, \cdots, q)\,,$$

$$\alpha_j \in \mathbb{C} \quad (j = 1, \cdots, p) \qquad \text{and} \qquad \beta_j \in \mathbb{C} \quad (j = 1, \cdots, q)\,,$$

and $\mathcal{L}$ is a suitable Mellin-Barnes type contour separating the poles of the gamma functions

$$\{\Gamma(b_j + B_j\mathfrak{s})\}_{j=1}^{m}$$

from the poles of the gamma functions

$$\{\Gamma(1 - a_j - A_j\mathfrak{s})\}_{j=1}^{n}\,.$$

The relatively more familiar G-function $G_{p,q}^{m,n}(z)$ of Cornelis Simon Meijer (1904–1974) is a special case of Fox's H-function defined by (1.24), and we have the following relationship (see, for details, [59, p. 415]; see also [23] and [47]):

$$G_{p,q}^{m,n}(z) = G_{p,q}^{m,n}\left(z \,\middle|\, \begin{matrix} (a_j)_{j=1}^{p} \\ \\ (b_j)_{j=1}^{q} \end{matrix}\right) := H_{p,q}^{m,n}\left[z \,\middle|\, \begin{matrix} (a_1, 1), \cdots, (a_p, 1) \\ \\ (b_1, 1), \cdots, (b_q, 1) \end{matrix}\right], \tag{1.26}$$

where, for convenience,

$$G_{p,q}^{m,n}\left(z \,\middle|\, \begin{matrix} (a_j)_{j=1}^{p} \\ \\ (b_j)_{j=1}^{q} \end{matrix}\right) := G_{p,q}^{m,n}\left(z \,\middle|\, \begin{matrix} a_1, \cdots, a_p \\ \\ b_1, \cdots, b_q \end{matrix}\right). \tag{1.27}$$

2. Explicit Series and Mellin-Barnes Type Contour Integral Representations

Our first set of results are contained in Theorem 2.1 below.

Theorem 2.1. *The following explicit series and Mellin-Barnes type contour integral representation formulas hold true for the extended Hurwitz-Lerch zeta function*

$$\Phi_{\lambda_1,\cdots,\lambda_p;\mu_1,\cdots,\mu_q}^{(\rho_1,\cdots,\rho_p,\sigma_1,\cdots,\sigma_q)}(z, s, a; b, \lambda)$$

defined by (1.14) :

$$\Phi_{\lambda_1,\cdots,\lambda_p;\mu_1,\cdots,\mu_q}^{(\rho_1,\cdots,\rho_p,\sigma_1,\cdots,\sigma_q)}(z, s, a; b, \lambda) = \frac{1}{\lambda\Gamma(s)} \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p} (\lambda_j)_{n\rho_j}}{(a+n)^s \cdot \prod_{j=1}^{q} (\mu_j)_{n\sigma_j}}$$

$$\cdot\, H_{0,2}^{2,0}\left[(a+n)b^{\frac{1}{\lambda}} \,\middle|\, \begin{matrix} \overline{\qquad\qquad} \\ \\ (s, 1), \left(0, \frac{1}{\lambda}\right) \end{matrix}\right] \frac{z^n}{n!} \qquad (\lambda > 0) \tag{2.1}$$

and

$$\Phi_{\lambda_1,\cdots,\lambda_p;\mu_1,\cdots,\mu_q}^{(\rho_1,\cdots,\rho_p,\sigma_1,\cdots,\sigma_q)}(z,s,a;b,\lambda) = \frac{\prod_{j=1}^{q}\Gamma(\mu_j)}{2\pi\mathrm{i}\lambda\Gamma(s)\prod_{j=1}^{p}\Gamma(\lambda_j)}$$

$$\times \int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \frac{\Gamma(\mathfrak{s})\prod_{j=1}^{p}\Gamma(\lambda_j-\mathfrak{s}\rho_j)}{(a-\mathfrak{s})^s\cdot\prod_{j=1}^{q}\Gamma(\mu_j-\mathfrak{s}\sigma_j)}$$

$$\times H_{0,2}^{2,0}\left[(a-\mathfrak{s})b^{\frac{1}{\lambda}}\,\middle|\, \begin{matrix} \overline{\qquad\qquad} \\ (s,1),\left(0,\frac{1}{\lambda}\right)\end{matrix}\right](-z)^{-\mathfrak{s}}\;\mathrm{d}\mathfrak{s} \tag{2.2}$$

$$(\lambda > 0),$$

provided that each member of the assertions (2.1) *and* (2.2) *exists.*

Proof. By making use of the series expansion of the Fox-Wright function

$${}_p\Psi_q^*\left[\begin{matrix}(\lambda_1,\rho_1),\cdots,(\lambda_p,\rho_p); \\ (\mu_1,\sigma_1),\cdots,(\mu_q,\sigma_q);\end{matrix}\; ze^{-t}\right]$$

occurring in the integrand of (1.14) and evaluating the resulting integral, in terms of Fox's H-function defined by (1.24), by means of the following *corrected* version of a known integral formula [48, p. 10, Equation (1.53)]:

$$\int_0^\infty t^{a-1}\,\exp\left(-bt-\frac{c}{t^\rho}\right)\mathrm{d}t = \frac{1}{\rho b^a}\,H_{0,2}^{2,0}\left[bc^{\frac{1}{\rho}}\,\middle|\,\begin{matrix}\overline{\qquad\qquad} \\ (a,1),\left(0,\frac{1}{\rho}\right)\end{matrix}\right] \tag{2.3}$$

$$\Big(\min\{\Re(a),\Re(b),\Re(c)\}>0;\;\rho>0\Big),$$

we obtain the series representation (2.1).

Our demonstration of the Mellin-Barnes type contour integral representation (2.2) is much akin to that of the series representation (2.1). We, therefore, omit the details involved.

In our derivation of each of the representation formulas (2.1) and (2.2), it is assumed that the required inversions of the order of summation and integration are justified by absolute and uniform convergence of the series and integrals involved. The final results (2.1) and (2.2) would thus hold true whenever each member of the assertions (2.1) and (2.2) of Theorem 2.1 exists. □

Remark 2.1. For the function $\Theta_\mu^\lambda(z,s,a;b)$ defined by (1.16), the following special cases of Theorem 2.1 were derived in [44]:

$$\Theta_{\mu}^{\lambda}(z,s,a;b)=\frac{1}{\lambda\Gamma(s)}\sum_{n=0}^{\infty}\frac{(\mu)_n}{(a+n)^s}$$

$$\cdot H_{0,2}^{2,0}\left[(a+n)b^{\frac{1}{\lambda}}\;\middle|\;\begin{array}{c}\overline{\qquad\qquad}\\ (s,1),\left(0,\frac{1}{\lambda}\right)\end{array}\right]\frac{z^n}{n!}\qquad(\lambda>0)\tag{2.4}$$

and

$$\Theta_{\mu}^{\lambda}(z,s,a;b)=\frac{1}{2\pi\mathrm{i}\lambda\Gamma(s)\Gamma(\mu)}\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty}\frac{\Gamma(\mathfrak{s})\Gamma(\mu-\mathfrak{s})}{(a-\mathfrak{s})^s}$$

$$\cdot H_{0,2}^{2,0}\left[(a-\mathfrak{s})b^{\frac{1}{\lambda}}\;\middle|\;\begin{array}{c}\overline{\qquad\qquad}\\ (s,1),\left(0,\frac{1}{\lambda}\right)\end{array}\right](-z)^{-\mathfrak{s}}\,\mathrm{d}\mathfrak{s}\qquad(\lambda>0),\tag{2.5}$$

it being assumed that each member of the assertions (2.4) and (2.5) exists (see, for details, [44]).

We now turn toward some series representations and other related results for the extended Hurwitz-Lerch zeta function

$$\Phi_{\lambda_1,\cdots,\lambda_p;\mu_1,\cdots,\mu_q}^{(\rho_1,\cdots,\rho_p,\sigma_1,\cdots,\sigma_q)}(z,s,a;b,\lambda)$$

defined by (1.14). We first give a pair of new series representations involving the the familiar Laguerre polynomials $L_n^{(\alpha)}(x)$ of order (index) α and degree n in x, defined by

$$L_n^{(\alpha)}(x):=\sum_{k=0}^{n}\binom{n+\alpha}{n-k}\frac{(-x)^k}{k!}=\binom{n+\alpha}{n}\,{}_1F_1\left[\begin{array}{c}-n;\\ \\ \alpha+1;\end{array}\right]\tag{2.6}$$

in terms of the Kummer's confluent hypergeometric function ${}_1F_1$, which are generated by (see, for example, [92, p. 84, Equations 1.11(14)])

$$(1-t)^{-\alpha-1}\exp\left(-\frac{xt}{1-t}\right)=\sum_{n=0}^{\infty}L_n^{(\alpha)}(x)\,t^n\qquad(|t|<1;\;\alpha\in\mathbb{C}).\tag{2.7}$$

Indeed, upon setting

$$t\to 1-t^{\lambda}\qquad\text{and}\qquad x=b$$

in (2.7), we get

$$\exp\left(-\frac{b}{t^{\lambda}}\right)=t^{\lambda(\alpha+1)}\,e^{-b}\sum_{n=0}^{\infty}L_n^{(\alpha)}(b)\left(1-t^{\lambda}\right)^n.\tag{2.8}$$

We now make use of (2.8) and the series expansions of

$$\left(1-t^{\lambda}\right)^n\qquad\text{and}\qquad {}_p\Psi_q^*\left[\begin{array}{c}(\lambda_1,\rho_1),\cdots,(\lambda_p,\rho_p);\\ \\ (\mu_1,\sigma_1),\cdots,(\mu_q,\sigma_q);\end{array}ze^{-t}\right]$$

occurring in the integrand of (1.14). If we evaluate the resulting Eulerian integral, we are led easily to the series representations given by Theorem 2.2 below.

Theorem 2.2. *Each of the following series representations holds true for the generalized Hurwitz-Lerch zeta function*

$$\Phi^{(\rho_1,\cdots,\rho_p,\sigma_1,\cdots,\sigma_q)}_{\lambda_1,\cdots,\lambda_p;\mu_1,\cdots,\mu_q}(z,s,a;b,\lambda)$$

defined by (1.14) :

$$\Phi^{(\rho_1,\cdots,\rho_p,\sigma_1,\cdots,\sigma_q)}_{\lambda_1,\cdots,\lambda_p;\mu_1,\cdots,\mu_q}(z,s,a;b,\lambda) = \frac{e^{-b}}{\Gamma(s)} \sum_{n,\ell=0}^{\infty} \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{\prod_{j=1}^{p} (\lambda_j)_{\ell\rho_j}}{\ell! \cdot \prod_{j=1}^{q} (\mu_j)_{\ell\sigma_j}}$$

$$\cdot\, \Gamma\big(s+\lambda(\alpha+k+1)\big) L_n^{(\alpha)}(b) \frac{z^\ell}{(a+\ell)^{s+\lambda(\alpha+k+1)}} \tag{2.9}$$

$$\big(\Re(a) > 0;\ \Re(s+\lambda\alpha) > -\lambda\big)$$

and

$$\Phi^{(\rho_1,\cdots,\rho_p,\sigma_1,\cdots,\sigma_q)}_{\lambda_1,\cdots,\lambda_p;\mu_1,\cdots,\mu_q}(z,s,a;b,\lambda) = \frac{e^{-b}}{\Gamma(s)} \sum_{n=0}^{\infty} \sum_{k=0}^{n} (-1)^k \binom{n}{k} \Gamma\big(s+\lambda(\alpha+k+1)\big)$$

$$\cdot\, L_n^{(\alpha)}(b)\, \Phi^{(\rho_1,\cdots,\rho_p,\sigma_1,\cdots,\sigma_q)}_{\lambda_1,\cdots,\lambda_p;\mu_1,\cdots,\mu_q}\big(z, s+\lambda(\alpha+j+1), a\big) \tag{2.10}$$

$$\big(\Re(a) > 0;\ \Re(s+\lambda\alpha) > -\lambda\big),$$

provided that each member of the assertions (2.9) *and* (2.10) *exists*,

$$\Phi^{(\rho_1,\cdots,\rho_p,\sigma_1,\cdots,\sigma_q)}_{\lambda_1,\cdots,\lambda_p;\mu_1,\cdots,\mu_q}(z,s,a)$$

being given by (1.10).

Proof. As already outlined above, our demonstration of the first assertion (2.9) of Theorem 2.2 is based essentially upon the representation (2.8) and the following well-known Eulerian integral:

$$\int_0^\infty t^{\rho-1}\, e^{-\sigma t}\, \mathrm{d}t = \frac{\Gamma(\rho)}{\sigma^\rho} \qquad \big(\min\{\Re(\rho), \Re(\sigma)\} > 0\big). \tag{2.11}$$

The second assertion (2.13) follows from the first assertion (2.9) when we interpret the ℓ-series in (2.10) by means of the definition (1.10).

Just as in our demonstration of Theorem 2.1, it is *tacitly* assumed that the required inversions of the order of summation and integration are justified by absolute and uniform convergence of the series and integrals involved. The final results (2.9) and (2.10) would thus hold true whenever each member of the assertions (2.9) and (2.10) of Theorem 2.2 exists. $\square$

Remark 2.2. By suitably specializing Theorem 2.2, we obtain the following known series representations for the generalized Hurwitz-Lerch zeta function $\Theta_\mu^\lambda(z,s,a;b)$ defined by (1.16):

$$\Theta_{\mu}^{\lambda}(z,s,a;b) = \frac{e^{-b}}{\Gamma(s)} \sum_{n,\ell=0}^{\infty} \sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{\mu+\ell-1}{\ell} \cdot \Gamma\big(s+\lambda(\alpha+k+1)\big) L_n^{(\alpha)}(b) \, \frac{z^{\ell}}{(a+\ell)^{s+\lambda(\alpha+k+1)}} \tag{2.12}$$

$$\big(\Re(a) > 0;\ \Re(s+\lambda\alpha) > -\lambda\big)$$

and

$$\Theta_{\mu}^{\lambda}(z,s,a;b) = \frac{e^{-b}}{\Gamma(s)} \sum_{n=0}^{\infty} \sum_{k=0}^{n} (-1)^k \binom{n}{j} \Gamma\big(s+\lambda(\alpha+k+1)\big) \cdot L_n^{(\alpha)}(b) \, \Phi_{\mu}^{*}\big(z, s+\lambda(\alpha+k+1), a\big) \tag{2.13}$$

$$\big(\Re(a) > 0;\ \Re(s+\lambda\alpha) > -\lambda\big),$$

provided that each member of the assertions (2.12) and (2.13) exists, $\Phi_{\mu}^{*}(z,s,a)$ being given by (1.18) (see, for details, [91]).

Remark 2.3. For the extended Hurwitz zeta function $\zeta_b(s,a)$ defined by (1.20), it is easily deduced from the assertion (2.13) of Theorem 2.2 when $\lambda = \mu = 1$ and $z = 1$ that

$$\zeta_b(s,a) = \frac{e^{-b}}{\Gamma(s)} \sum_{n=0}^{\infty} \sum_{j=0}^{n} (-1)^j \binom{n}{j} \Gamma(s+\alpha+j+1) \cdot L_n^{(\alpha)}(b) \, \zeta(s+\alpha+j+1, a) \tag{2.14}$$

$$\big(\Re(a) > 0;\ \Re(s+\alpha) > -1\big),$$

provided that each member of (2.14) exists, $\zeta(s,a)$ being the Hurwitz (or generalized) zeta function given by (1.3). The obvious *further* special case of (2.14) when $a = 1$ *and* $\alpha = 0$ would yield the *corrected* version of a known result (see [17, p. 298, Equation (7.78)]).

Lastly, we choose give several pairs of summation formulas involving the the generalized Hurwitz-Lerch zeta function

$$\Phi_{\lambda_1,\cdots,\lambda_p;\mu_1,\cdots,\mu_q}^{(\rho_1,\cdots,\rho_p,\sigma_1,\cdots,\sigma_q)}(z,s,a;b,\lambda)$$

defined by (1.14). First of all, it is easily seen from the first assertion (2.1) of Theorem 2.1 that

$$\Phi_{\lambda_1,\cdots,\lambda_p;\mu_1,\cdots,\mu_q}^{(\rho_1,\cdots,\rho_p,\sigma_1,\cdots,\sigma_q)}(z,s,a;b,\lambda)+\Phi_{\lambda_1,\cdots,\lambda_p;\mu_1,\cdots,\mu_q}^{(\rho_1,\cdots,\rho_p,\sigma_1,\cdots,\sigma_q)}(-z,s,a;b,\lambda)$$
$$=\frac{2}{\lambda\Gamma(s)}\sum_{n=0}^{\infty}\frac{\prod_{j=1}^{p}(\lambda_j)_{2n\rho_j}}{(a+2n)^s\cdot\prod_{j=1}^{q}(\mu_j)_{2n\sigma_j}}$$
$$\cdot H_{0,2}^{2,0}\left[(a+2n)b^{\frac{1}{\lambda}}\ \middle|\ \begin{matrix}\overline{\qquad\qquad}\\ (s,1),\left(0,\frac{1}{\lambda}\right)\end{matrix}\right]\frac{z^{2n}}{(2n)!}\qquad(\lambda>0)\tag{2.15}$$

and

$$\Phi_{\lambda_1,\cdots,\lambda_p;\mu_1,\cdots,\mu_q}^{(\rho_1,\cdots,\rho_p,\sigma_1,\cdots,\sigma_q)}(z,s,a;b,\lambda)-\Phi_{\lambda_1,\cdots,\lambda_p;\mu_1,\cdots,\mu_q}^{(\rho_1,\cdots,\rho_p,\sigma_1,\cdots,\sigma_q)}(-z,s,a;b,\lambda)$$
$$=\frac{2}{\lambda\Gamma(s)}\sum_{n=0}^{\infty}\frac{\prod_{j=1}^{p}(\lambda_j)_{(2n+1)\rho_j}}{(a+2n+1)^s\cdot\prod_{j=1}^{q}(\mu_j)_{(2n+1)\sigma_j}}$$
$$\cdot H_{0,2}^{2,0}\left[(a+2n+1)b^{\frac{1}{\lambda}}\ \middle|\ \begin{matrix}\overline{\qquad\qquad}\\ (s,1),\left(0,\frac{1}{\lambda}\right)\end{matrix}\right]\frac{z^{2n+1}}{(2n+1)!}\qquad(\lambda>0).\tag{2.16}$$

Alternative expressions for the first members of the summation formulas (2.15) and (2.16) are given by Theorem 2.3 below.

Theorem 2.3. *Each of the following summation formulas holds true for the the generalized Hurwitz-Lerch zeta function*

$$\Phi_{\lambda_1,\cdots,\lambda_p;\mu_1,\cdots,\mu_q}^{(\rho_1,\cdots,\rho_p,\sigma_1,\cdots,\sigma_q)}(z,s,a;b,\lambda)$$

defined by (1.14) :

$$2^{s-1}\left[\Phi_{\lambda_1,\cdots,\lambda_p;\mu_1,\cdots,\mu_q}^{(\rho_1,\cdots,\rho_p,\sigma_1,\cdots,\sigma_q)}(z,s,a;b,\lambda)+\Phi_{\lambda_1,\cdots,\lambda_p;\mu_1,\cdots,\mu_q}^{(\rho_1,\cdots,\rho_p,\sigma_1,\cdots,\sigma_q)}(-z,s,a;b,\lambda)\right]$$
$$=\Phi_{\lambda_1,\cdots,\lambda_p;\frac{1}{2},\mu_1,\cdots,\mu_q}^{(2\rho_1,\cdots,2\rho_p,1,2\sigma_1,\cdots,2\sigma_q)}\left(\frac{z^2}{4},s,\frac{a}{2};2^{\lambda}\,b,\lambda\right)\tag{2.17}$$

and

$$2^{s-1}\left[\Phi_{\lambda_1,\cdots,\lambda_p;\mu_1,\cdots,\mu_q}^{(\rho_1,\cdots,\rho_p,\sigma_1,\cdots,\sigma_q)}(z,s,a;b,\lambda)-\Phi_{\lambda_1,\cdots,\lambda_p;\mu_1,\cdots,\mu_q}^{(\rho_1,\cdots,\rho_p,\sigma_1,\cdots,\sigma_q)}(-z,s,a;b,\lambda)\right]$$
$$=\frac{(\lambda_1)_{\rho_1}\cdots(\lambda_p)_{\rho_p}}{(\mu_1)_{\sigma_1}\cdots(\mu_q)_{\sigma_q}}\,z\,\Phi_{\lambda_1+\rho_1,\cdots,\lambda_p+\rho_p;\frac{3}{2},\mu_1+\sigma_1,\cdots,\mu_q+\sigma_q}^{(2\rho_1,\cdots,2\rho_p,1,2\sigma_1,\cdots,2\sigma_q)}\left(\frac{z^2}{4},s,\frac{a+1}{2};2^{\lambda}\,b,\lambda\right),\tag{2.18}$$

provided that each member of the assertions (2.17) *and* (2.18) *exists.*

Proof. In view of the definition (1.14), we get

$$\begin{aligned}
&\Phi_{\lambda_1,\cdots,\lambda_p;\mu_1,\cdots,\mu_q}^{(\rho_1,\cdots,\rho_p,\sigma_1,\cdots,\sigma_q)}(z,s,a;b,\lambda)+\Phi_{\lambda_1,\cdots,\lambda_p;\mu_1,\cdots,\mu_q}^{(\rho_1,\cdots,\rho_p,\sigma_1,\cdots,\sigma_q)}(z,s,a;b,\lambda)\\
&\quad=\frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}\,\exp\left(-at-\frac{b}{t^\lambda}\right)\\
&\qquad\cdot\left({}_p\Psi_q^*\left[\begin{array}{lr}(\lambda_1,\rho_1),\cdots,(\lambda_p,\rho_p); & \\ & ze^{-t}\\ (\mu_1,\sigma_1),\cdots,(\mu_q,\sigma_q); & \end{array}\right]\right.\\
&\qquad\qquad\left.+{}_p\Psi_q^*\left[\begin{array}{lr}(\lambda_1,\rho_1),\cdots,(\lambda_p,\rho_p); & \\ & -ze^{-t}\\ (\mu_1,\sigma_1),\cdots,(\mu_q,\sigma_q); & \end{array}\right]\right)\mathrm{d}t,
\end{aligned}\tag{2.19}$$

which readily simplifies to the following form:

$$\begin{aligned}
&\Phi_{\lambda_1,\cdots,\lambda_p;\mu_1,\cdots,\mu_q}^{(\rho_1,\cdots,\rho_p,\sigma_1,\cdots,\sigma_q)}(z,s,a;b,\lambda)+\Phi_{\lambda_1,\cdots,\lambda_p;\mu_1,\cdots,\mu_q}^{(\rho_1,\cdots,\rho_p,\sigma_1,\cdots,\sigma_q)}(z,s,a;b,\lambda)\\
&\quad=\frac{2}{\Gamma(s)}\int_0^\infty t^{s-1}\,\exp\left(-at-\frac{b}{t^\lambda}\right)\\
&\qquad\cdot{}_p\Psi_{q+1}^*\left[\begin{array}{rr}(\lambda_1,\rho_1),\cdots,(\lambda_p,\rho_p); & \\ & \dfrac{z^2}{4}\,e^{-2t}\\ \left(\frac{1}{2},1\right),(\mu_1,\sigma_1),\cdots,(\mu_q,\sigma_q); & \end{array}\right]\mathrm{d}t.
\end{aligned}\tag{2.20}$$

Upon setting

$$t\to\frac{t}{2}\qquad\text{and}\qquad\frac{\mathrm{d}t}{2}$$

in (2.20), if we interpret the resulting integral by means of the definition (1.14), we arrive at the first assertion (2.17) of Theorem 2.3. In a similar manner, we can prove the second assertion (2.18) of Theorem 2.3.

Alternatively, we can derive the assertions (2.17) and (2.18) of Theorem 2.3 by applying the series representation in (2.1) in order to interpret the second members of (2.15) and (2.16), respectively. □

Remark 2.4. For the particular case $\Theta_\mu^\lambda(z,s,a;b)$ defined by (1.16), the following interesting analogues of the assertions (2.17) and (2.18) were derived earlier by Srivastava *et al.* [91]:

$$\begin{aligned}
&2^{s-1}\left[\Theta_\mu^\lambda(-z,s,a;b)+\Theta_\mu^\lambda(z,s,a;b)\right]\\
&\qquad=\sum_{n=0}^\infty\frac{(-\mu)_{2n}}{(2n)!}\,\Theta_\mu^\lambda\left(z^2,s,\frac{a}{2}+n;2^\lambda\,b\right)z^{2n}
\end{aligned}\tag{2.21}$$

and

$$\begin{aligned}
&2^{s-1}\left[\Theta_\mu^\lambda(-z,s,a;b)-\Theta_\mu^\lambda(z,s,a;b)\right]\\
&\qquad=\sum_{n=0}^\infty\frac{(-\mu)_{2n+1}}{(2n+1)!}\,\Theta_\mu^\lambda\left(z^2,s,\frac{a+1}{2}+n;2^\lambda\,b\right)z^{2n+1},
\end{aligned}\tag{2.22}$$

provided, of course, that each member of the assertions (2.21) and (2.22) exists. In fact, by putting $\mu = 1$ in (2.21) and (2.22), and upon setting $z \to -z$ and $a \to 2a$, Srivastava *et al.* [91] showed also that

$$2^{s-1}\left[\Theta_1^\lambda(z,s,2a;b)+\Theta_1^\lambda(-z,s,2a;b)\right]=\Theta_1^\lambda\left(z^2,s,a;2^\lambda\, b\right) \tag{2.23}$$

and

$$2^{s-1}\left[\Theta_1^\lambda(z,s,2a;b)-\Theta_1^\lambda(-z,s,2a;b)\right]=z\,\Theta_1^\lambda\left(z^2,s,a+\frac{1}{2};2^\lambda\, b\right). \tag{2.24}$$

In its *further* special case when $z = \lambda = 1$, the summation formula (2.23) can be shown to correspond to known results (see, for example, [17, Theorem 7.9]; see also [17, pp. 308–309]).

3. Derivative Properties and Associated Partial Differential Equations

In this section, we aim at showing that the generalized Hurwitz-Lerch zeta function

$$\Phi_{\lambda_1,\cdots,\lambda_p;\mu_1,\cdots,\mu_q}^{(\rho_1,\cdots,\rho_p,\sigma_1,\cdots,\sigma_q)}(z,s,a;b,\lambda)$$

defined by (1.14) satisfies a partial differential equation when the parameter λ is given by

$$\lambda=\frac{1}{m}\qquad(m\in\mathbb{N})\,.$$

We first derive the following lemma which will be useful in the demonstration of our main result of this section (Theorem 3.1 below).

Lemma 3.1. *(Derivative Property). The following derivative formulas hold true*:

$$\begin{aligned}\frac{(\mu_1)_{\sigma_1}\cdots(\mu_q)_{\sigma_q}}{(\lambda_1)_{\rho_1}\cdots(\lambda_p)_{\rho_p}}\,\frac{\mathrm{d}}{\mathrm{d}z}&\left\{\Phi_{\lambda_1,\cdots,\lambda_p;\mu_1,\cdots,\mu_q}^{(\rho_1,\cdots,\rho_p,\sigma_1,\cdots,\sigma_q)}(z,s,a;b,\lambda)\right\}\\&=\Phi_{\lambda_1+\rho_1,\cdots,\lambda_p+\rho_p;\mu_1+\sigma_1,\cdots,\mu_q+\sigma_q}^{(\rho_1,\cdots,\rho_p,\sigma_1,\cdots,\sigma_q)}(z,s,a+1;b,\lambda)\qquad(\lambda>0)\end{aligned} \tag{3.1}$$

and

$$\begin{aligned}\frac{(\mu_1)_{\sigma_1}\cdots(\mu_q)_{\sigma_q}}{(\lambda_1)_{\rho_1}\cdots(\lambda_p)_{\rho_p}}\,\frac{\mathrm{d}}{\mathrm{d}z}&\left\{\Phi_{\lambda_1,\cdots,\lambda_p;\mu_1,\cdots,\mu_q}^{(\rho_1,\cdots,\rho_p,\sigma_1,\cdots,\sigma_q)}\left(z,s,a;b^{\frac{1}{m}},\frac{1}{m}\right)\right\}\\&=\Phi_{\lambda_1+\rho_1,\cdots,\lambda_p+\rho_p;\mu_1+\sigma_1,\cdots,\mu_q+\sigma_q}^{(\rho_1,\cdots,\rho_p,\sigma_1,\cdots,\sigma_q)}\left(z,s,a+1;b^{\frac{1}{m}},\frac{1}{m}\right)\qquad(m\in\mathbb{N})\end{aligned} \tag{3.2}$$

Proof. Our proofs of the derivative formulas (3.1) and (3.2) are simple and direct. For example, by applying the series representation (2.1), it is easily observed that

$$\begin{aligned}
&\frac{(\mu_1)_{\sigma_1}\cdots(\mu_q)_{\sigma_q}}{(\lambda_1)_{\rho_1}\cdots(\lambda_p)_{\rho_p}}\,\frac{\mathrm{d}}{\mathrm{d}z}\left\{\Phi^{(\rho_1,\cdots,\rho_p,\sigma_1,\cdots,\sigma_q)}_{\lambda_1,\cdots,\lambda_p;\mu_1,\cdots,\mu_q}(z,s,a;b,\lambda)\right\}\\
&\quad=\frac{(\mu_1)_{\sigma_1}\cdots(\mu_q)_{\sigma_q}}{\lambda\Gamma(s)(\lambda_1)_{\rho_1}\cdots(\lambda_p)_{\rho_p}}\sum_{n=1}^{\infty}\frac{\prod\limits_{j=1}^{p}(\lambda_j)_{n\rho_j}}{(a+n)^s\cdot\prod\limits_{j=1}^{q}(\mu_j)_{n\sigma_j}}\\
&\qquad\cdot H^{2,0}_{0,2}\left[(a+n)b^{\frac{1}{\lambda}}\;\middle|\;\begin{matrix}\overline{\qquad\qquad}\\(s,1),\left(0,\frac{1}{\lambda}\right)\end{matrix}\right]\frac{z^{n-1}}{(n-1)!}\\
&\quad=\frac{(\mu_1)_{\sigma_1}\cdots(\mu_q)_{\sigma_q}}{\lambda\Gamma(s)(\lambda_1)_{\rho_1}\cdots(\lambda_p)_{\rho_p}}\sum_{n=0}^{\infty}\frac{\prod\limits_{j=1}^{p}(\lambda_j)_{(n+1)\rho_j}}{(a+n+1)^s\cdot\prod\limits_{j=1}^{q}(\mu_j)_{(n+1)\sigma_j}}\\
&\qquad\cdot H^{2,0}_{0,2}\left[(a+n+1)b^{\frac{1}{\lambda}}\;\middle|\;\begin{matrix}\overline{\qquad\qquad}\\(s,1),\left(0,\frac{1}{\lambda}\right)\end{matrix}\right]\frac{z^{n}}{n!}\\
&\quad=\Phi^{(\rho_1,\cdots,\rho_p,\sigma_1,\cdots,\sigma_q)}_{\lambda_1+\rho_1,\cdots,\lambda_p+\rho_p;\mu_1+\sigma_1,\cdots,\mu_q+\sigma_q}(z,s,a+1;b,\lambda)\qquad(\lambda>0),
\end{aligned}$$

which yields precisely the first assertion (3.1) of the Lemma 3.1. The second assertion (3.2) follows immediately from (3.1) upon setting

$$\lambda=\frac{1}{m}\quad(m\in\mathbb{N})\qquad\text{and}\qquad b\to b^{\frac{1}{m}}\quad(m\in\mathbb{N}).$$

□

Our main result in this section is contained in the following theorem.

Theorem 3.1. *Let $m\in\mathbb{N}$. Then the generalized Hurwitz-Lerch zeta function*

$$\Phi^{(\rho_1,\cdots,\rho_p,\sigma_1,\cdots,\sigma_q)}_{\lambda_1,\cdots,\lambda_p;\mu_1,\cdots,\mu_q}\left(z,s,a;b,\frac{1}{m}\right)$$

satisfies the following partial differential equation:

$$\left[(-1)^{m+1}\,m^m\mathfrak{D}_{\mathfrak{b}}-(a+1)\,b^m\theta_z\right]\left\{\Phi^{(\rho_1,\cdots,\rho_p,\sigma_1,\cdots,\sigma_q)}_{\lambda_1,\cdots,\lambda_p;\mu_1,\cdots,\mu_q}\left(z,s,a;b,\frac{1}{m}\right)\right\}=0,\quad(3.3)$$

where the differential operators $\mathfrak{D}_{\mathfrak{b}}$, θ_z and θ_b are given by

$$\mathfrak{D}_{\mathfrak{b}}:=\theta_b\,(\theta_b-s)\left(\theta_b-\frac{1}{m}\right)\cdots\left(\theta_b-\frac{m-1}{m}\right),\tag{3.4}$$

$$\theta_z:=z\,\frac{\partial}{\partial z}\qquad\textit{and}\qquad\theta_b:=b\,\frac{\partial}{\partial b},\tag{3.5}$$

respectively.

Proof. First of all, let us rewrite the H-function occurring in the Mellin-Barnes type contour integral representation (2.2) as follows:

$$H_{0,2}^{2,0}\left[(a-\mathfrak{s})b^{\frac{1}{\lambda}}\,\middle|\,\begin{matrix}\overline{\qquad\qquad}\\ (s,1),\left(0,\frac{1}{\lambda}\right)\end{matrix}\right]=\frac{1}{2\pi\mathrm{i}}\int_{\mathfrak{L}}\Gamma(s+w)\,\Gamma\left(\frac{w}{\lambda}\right)\left[(a-\mathfrak{s})\,b^{\frac{1}{\lambda}}\right]^{-w}\,\mathrm{d}w, \tag{3.6}$$

where $\mathfrak{L}$ is a suitable Mellin-Barnes type contour integral in the complex w-plane. We now set

$$\frac{1}{\lambda}=m\quad(m\in\mathbb{N})\qquad\text{and}\qquad b\to b^{\frac{1}{m}}\quad(m\in\mathbb{N})$$

in the above equation (3.6) and then apply the following well-known (Gauss-Legendre) multiplication formula (see, for example, [1, p. 256, Entry (6.1.18)]):

$$\Gamma(mz)=(2\pi)^{\frac{1-m}{2}}\,m^{mz-\frac{1}{2}}\prod_{j=1}^{m}\Gamma\left(z+\frac{j-1}{m}\right) \tag{3.7}$$

$$\left(z\neq 0,-\frac{1}{m},-\frac{2}{m},\cdots;\ m\in\mathbb{N}\right).$$

We thus find that

$$\begin{aligned}H_{0,2}^{2,0}\left[(a-\mathfrak{s})b\,\middle|\,\begin{matrix}\overline{\qquad\qquad}\\ (s,1),(0,m)\end{matrix}\right]&=\frac{1}{2\pi\mathrm{i}}\int_{\mathfrak{L}}\Gamma(s+w)\,\Gamma(mw)\,[(a-\mathfrak{s})\,b]^{-w}\,\mathrm{d}w\\ &=\frac{(2\pi)^{\frac{1-m}{2}}}{2\pi\mathrm{i}\sqrt{m}}\int_{\mathfrak{L}}\Gamma(s+w)\prod_{j=1}^{m}\Gamma\left(w+\frac{j-1}{m}\right)\left[(a-\mathfrak{s})\,bm^{-m}\right]^{-w}\,\mathrm{d}w\\ &=\frac{(2\pi)^{\frac{1-m}{2}}}{\sqrt{m}}\,G_{0,m+1}^{m+1,0}\left((a-\mathfrak{s})bm^{-m}\,\middle|\,\begin{matrix}\overline{\qquad\qquad}\\ s,0,\frac{1}{m},\frac{2}{m},\cdots,\frac{m-1}{m}\end{matrix}\right),\end{aligned} \tag{3.8}$$

where

$$G_{0,m+1}^{m+1,0}\left((a-\mathfrak{s})\,bm^{-m}\right)$$

is a very specialized case of Meijer's G-function $G_{p,q}^{m,n}(z)$ defined by (1.26).

We know that the function W defined by

$$W:=G_{p,q}^{m,n}\left(z\,\middle|\,\begin{matrix}a_1,\cdots,a_p\\ b_1,\cdots,b_q\end{matrix}\right) \tag{3.9}$$

satisfies the following differential equation of order $\max(p,q)$ (see, for example, [23, p. 210, Equation 5.4(1)]):

$$\left[(-1)^{p-m-n}\ z\,(\vartheta_z-a_1+1)\cdots(\vartheta_z-a_p+1)-(\vartheta_z-b_1)\cdots(\vartheta_z-b_q)\right]W=0,$$

where

$$\vartheta_z=z\,\frac{\mathrm{d}}{\mathrm{d}z}.$$

Clearly, therefore, the function given by (3.8) satisfies the following differential equation:

$$\begin{aligned}\left[(-1)^{m+1}(a-\mathfrak{s})\,m^{-m}b-\theta_b\,(\theta_b-s)\left(\theta_b-\frac{1}{m}\right)\cdots\left(\theta_b-\frac{m-1}{m}\right)\right]&\\ \cdot\left\{G^{m+1,0}_{0,m+1}\left((a-\mathfrak{s})\,bm^{-m}\,\middle|\,\begin{array}{c}\overline{\qquad\qquad}\\ s,0,\frac{1}{m},\frac{2}{m},\cdots,\frac{m-1}{m}\end{array}\right)\right\}&=0,\end{aligned} \tag{3.10}$$

where, as already stated in (3.5),

$$\theta_b=b\,\frac{\partial}{\partial b}.$$

Now, if we write [see also Equation (3.4)]

$$\mathfrak{D}_{\mathfrak{b}}:=\theta_b\,(\theta_b-s)\left(\theta_b-\frac{1}{m}\right)\cdots\left(\theta_b-\frac{m-1}{m}\right)\qquad\left(\theta_b:=b\,\frac{\partial}{\partial b}\right),$$

then the equation (3.10) becomes

$$\begin{aligned}&\mathfrak{D}_{\mathfrak{b}}\left\{G^{m+1,0}_{0,m+1}\left((a-\mathfrak{s})\,bm^{-m}\,\middle|\,\begin{array}{c}\overline{\qquad\qquad}\\ s,0,\frac{1}{m},\frac{2}{m},\cdots,\frac{m-1}{m}\end{array}\right)\right\}\\ &=(-1)^{m+1}\;m^{-m}\,(a-\mathfrak{s})\,bG^{m+1,0}_{0,m+1}\left((a-\mathfrak{s})\,bm^{-m}\,\middle|\,\begin{array}{c}\overline{\qquad\qquad}\\ s,0,\frac{1}{m},\frac{2}{m},\cdots,\frac{m-1}{m}\end{array}\right).\end{aligned} \tag{3.11}$$

By applying the differential operator $\mathfrak{D}_{\mathfrak{b}}$ to the function

$$\Phi^{(\rho_1,\cdots,\rho_p,\sigma_1,\cdots,\sigma_q)}_{\lambda_1,\cdots,\lambda_p;\mu_1,\cdots,\mu_q}\left(-z,s,a;b^{\frac{1}{m}},\frac{1}{m}\right)$$

given by (2.2) *with*

$$z\to -z,\quad \lambda=\frac{1}{m}\quad (m\in\mathbb{N})\qquad\text{and}\qquad b\to b^{\frac{1}{m}}\quad (m\in\mathbb{N}),$$

we find by making use of (3.11) that

$$
\begin{aligned}
&\mathfrak{D}_{\mathfrak{b}}\left\{\Phi_{\lambda_1,\cdots,\lambda_p;\mu_1,\cdots,\mu_q}^{(\rho_1,\cdots,\rho_p,\sigma_1,\cdots,\sigma_q)}\left(-z,s,a;b^{\frac{1}{m}},\frac{1}{m}\right)\right\} \\
&= \frac{\sqrt{m}\,(2\pi)^{\frac{1-m}{2}}\prod_{j=1}^{q}\Gamma(\mu_j)}{2\pi\mathrm{i}\Gamma(s)\prod_{j=1}^{p}\Gamma(\lambda_j)}\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty}\frac{\Gamma(\mathfrak{s})\,\Gamma\prod_{j=1}^{p}\Gamma(\lambda_j-\mathfrak{s}\rho_j)}{(a-\mathfrak{s})^{s}\prod_{j=1}^{q}\Gamma(\mu_j-\mathfrak{s}\sigma_j)} \\
&\quad\cdot\mathfrak{D}_{\mathfrak{b}}\left\{G_{0,m+1}^{m+1,0}\left((a-\mathfrak{s})\,bm^{-m}\left|\begin{array}{c}\overline{\qquad\qquad}\\ s,0,\frac{1}{m},\frac{2}{m},\cdots,\frac{m-1}{m}\end{array}\right.\right)\right\}z^{-\mathfrak{s}}\,\mathrm{d}\mathfrak{s} \\
&= \frac{(-1)^{m+1}\;m^{\frac{1}{2}-m}\;(2\pi)^{\frac{1-m}{2}}\;b\prod_{j=1}^{q}\Gamma(\mu_j)}{2\pi\mathrm{i}\Gamma(s)\prod_{j=1}^{p}\Gamma(\lambda_j)}\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty}\frac{\Gamma(\mathfrak{s})\prod_{j=1}^{p}\Gamma(\lambda_j-\mathfrak{s}\rho_j)}{(a-\mathfrak{s})^{s}\prod_{j=1}^{q}\Gamma(\mu_j-\mathfrak{s}\sigma_j)} \\
&\quad\cdot(a-\mathfrak{s})G_{0,m+1}^{m+1,0}\left((a-\mathfrak{s})\,bm^{-m}\left|\begin{array}{c}\overline{\qquad\qquad}\\ s,0,\frac{1}{m},\frac{2}{m},\cdots,\frac{m-1}{m}\end{array}\right.\right)z^{-\mathfrak{s}}\,\mathrm{d}\mathfrak{s} \\
&=:(-1)^{m+1}\;m^{-m}\,b\,(a\mathcal{I}_1-\mathcal{I}_2)\,, \qquad (3.12)
\end{aligned}
$$

where the first integral $\mathcal{I}_1$ is actually the generalized Hurwitz-Lerch zeta function given by

$$
\mathcal{I}_1=\Phi_{\lambda_1,\cdots,\lambda_p;\mu_1,\cdots,\mu_q}^{(\rho_1,\cdots,\rho_p,\sigma_1,\cdots,\sigma_q)}\left(-z,s,a;b^{\frac{1}{m}},\frac{1}{m}\right). \qquad (3.13)
$$

The evaluation of the second integral $\mathcal{I}_2$ given by

$$
\begin{aligned}
\mathcal{I}_2 := &\frac{\sqrt{m}\;(2\pi)^{\frac{1-m}{2}}\prod_{j=1}^{q}\Gamma(\mu_j)}{2\pi\mathrm{i}\Gamma(s)\prod_{j=1}^{p}\Gamma(\lambda_j)}\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty}\frac{\Gamma(\mathfrak{s}+1)\prod_{j=1}^{p}\Gamma(\lambda_j-\mathfrak{s}\rho_j)}{(a-\mathfrak{s})^{s}\prod_{j=1}^{q}\Gamma(\mu_j-\mathfrak{s}\sigma_j)} \\
&\cdot G_{0,m+1}^{m+1,0}\left((a-\mathfrak{s})\,bm^{-m}\left|\begin{array}{c}\overline{\qquad\qquad}\\ s,0,\frac{1}{m},\frac{2}{m},\cdots,\frac{m-1}{m}\end{array}\right.\right)z^{-\mathfrak{s}}\,\mathrm{d}\mathfrak{s} \qquad (3.14)
\end{aligned}
$$

is more complicated. Since the residues of $\Gamma(\mathfrak{s}+1)$ at the poles $\mathfrak{s}=-k\;\;(k\in\mathbb{N})$ are computed by

$$
\operatorname*{Res}_{\mathfrak{s}=-k}\{\Gamma(\mathfrak{s}+1)\}=\lim_{\mathfrak{s}\to-k}(\mathfrak{s}+k)\,\Gamma(\mathfrak{s}+1)=\frac{(-1)^{k-1}}{(k-1)!}, \qquad (3.15)
$$

the Residue Theorem 3.1 implies that

$$
\begin{aligned}
\mathcal{I}_2 &= \frac{\sqrt{m}\,(2\pi)^{\frac{1-m}{2}} \prod_{j=1}^{q}\Gamma(\mu_j)}{\Gamma(s)\prod_{j=1}^{p}\Gamma(\lambda_j)} \sum_{k=1}^{\infty} \frac{\prod_{j=1}^{p}\Gamma(\lambda_j+k\rho_j)}{(a+k)^s \prod_{j=1}^{q}\Gamma(\mu_j+k\sigma_j)}\, z^k \operatorname*{Res}_{\mathfrak{s}=-k}\{\Gamma(\mathfrak{s}+1)\} \\
&\qquad \cdot G^{m+1,0}_{0,m+1}\left((a+k)\,bm^{-m} \,\middle|\, \begin{array}{c} \overline{\qquad\qquad} \\ s,0,\frac{1}{m},\frac{2}{m},\cdots,\frac{m-1}{m} \end{array}\right) \\
&= \frac{\sqrt{m}\,(2\pi)^{\frac{1-m}{2}} \prod_{j=1}^{q}\Gamma(\mu_j)}{\Gamma(s)\prod_{j=1}^{p}\Gamma(\lambda_j)} \sum_{k=1}^{\infty} \frac{\prod_{j=1}^{p}\Gamma(\lambda_j+k\rho_j)}{(a+k)^s \prod_{j=1}^{q}\Gamma(\mu_j+k\sigma_j)}\, \frac{(-1)^{k-1}\ z^k}{(k-1)!} \\
&\qquad \cdot G^{m+1,0}_{0,m+1}\left((a+k)\,bm^{-m} \,\middle|\, \begin{array}{c} \overline{\qquad\qquad} \\ s,0,\frac{1}{m},\frac{2}{m},\cdots,\frac{m-1}{m} \end{array}\right) \\
&= \frac{z\,\sqrt{m}\,(2\pi)^{\frac{1-m}{2}} \prod_{j=1}^{q}\Gamma(\mu_j)}{\Gamma(s)\prod_{j=1}^{p}\Gamma(\lambda_j)} \sum_{k=0}^{\infty} \frac{\prod_{j=1}^{p}\Gamma\big(\lambda_j+(k+1)\rho_j\big)}{(a+1+k)^s \prod_{j=1}^{q}\Gamma\big(\mu_j+(k+1)\rho_j\big)}\, \frac{(-z)^k}{k!} \\
&\qquad \cdot G^{m+1,0}_{0,m+1}\left((a+1+k)\,bm^{-m} \,\middle|\, \begin{array}{c} \overline{\qquad\qquad} \\ s,0,\frac{1}{m},\frac{2}{m},\cdots,\frac{m-1}{m} \end{array}\right) \\
&= \frac{mz\lambda_1\cdots\lambda_p}{\mu_1\cdots\mu_q\Gamma(s)} \sum_{k=0}^{\infty} \frac{\prod_{j=1}^{p}\Gamma(\lambda_j+1)_{k\lambda_j}}{(a+1+k)^s \prod_{j=1}^{q}\Gamma(\mu_j+1)_{k\mu_j}}\, \frac{(-z)^k}{k!} \\
&\qquad \cdot H^{2,0}_{0,2}\left[(a+1+k)\,b \,\middle|\, \begin{array}{c} \overline{\qquad\qquad} \\ (s,1),(0,m) \end{array}\right] \\
&= \frac{z\lambda_1\cdots\lambda_p}{\mu_1\cdots\mu_q}\ \Phi^{(\rho_1,\cdots,\rho_p,\sigma_1,\cdots,\sigma_q)}_{\lambda_1+1,\cdots,\lambda_p+1;\mu_1+1,\cdots,\mu_q+1}\left(-z,s,a;b^{\frac{1}{m}},\frac{1}{m}\right)
\end{aligned}
\tag{3.16}
$$

Thus, by applying the derivative formula (3.2) in (3.16), we get

$$
\mathcal{I}_2 = -z\,\frac{\mathrm{d}}{\mathrm{d}z}\left\{\Phi^{(\rho_1,\cdots,\rho_p,\sigma_1,\cdots,\sigma_q)}_{\lambda_1,\cdots,\lambda_p;\mu_1,\cdots,\mu_q}\left(-z,s,a;b^{\frac{1}{m}},\frac{1}{m}\right)\right\}. \tag{3.17}
$$

Now, upon substituting from (3.13) and (3.17) into (3.12), we obtain

$$\begin{aligned}
&\mathfrak{D}_{\mathfrak{b}} \left\{ \Phi^{(\rho_1,\cdots,\rho_p,\sigma_1,\cdots,\sigma_q)}_{\lambda_1,\cdots,\lambda_p;\mu_1,\cdots,\mu_q} \left(-z, s, a; b^{\frac{1}{m}}, \frac{1}{m} \right) \right\} \\
&\quad = (-1)^{m+1} \; m^{-m} \; ab\Phi^{(\rho_1,\cdots,\rho_p,\sigma_1,\cdots,\sigma_q)}_{\lambda_1,\cdots,\lambda_p;\mu_1,\cdots,\mu_q} \left(-z, s, a; b^{\frac{1}{m}}, \frac{1}{m} \right) \\
&\quad + (-1)^{m+1} \; m^{-m} \; bz \, \frac{\partial}{\partial z} \left\{ \Phi^{(\rho_1,\cdots,\rho_p,\sigma_1,\cdots,\sigma_q)}_{\lambda_1,\cdots,\lambda_p;\mu_1,\cdots,\mu_q} \left(-z, s, a; b^{\frac{1}{m}}, \frac{1}{m} \right) \right\}, \qquad (3.18)
\end{aligned}$$

which, after a straightforward simplification, assumes the following form:

$$\left[(-1)^{m+1} \; m^m \; \mathfrak{D}_{\mathfrak{b}} - ab - b\theta_z \right] \left\{ \Phi^{(\rho_1,\cdots,\rho_p,\sigma_1,\cdots,\sigma_q)}_{\lambda_1,\cdots,\lambda_p;\mu_1,\cdots,\mu_q} \left(-z, s, a; b^{\frac{1}{m}}, \frac{1}{m} \right) \right\} = 0 \qquad (3.19)$$

$$\left(\theta_z := z \, \frac{\partial}{\partial z} \right).$$

Finally, by setting

$$b \to b^m \quad (m \in \mathbb{N}) \qquad \text{and} \qquad z \to -z$$

in the last equation (3.19), we readily arrive at the desired result (3.3) asserted by Theorem 3.1. □

Remark 3.1. An interesting special case of Theorem 3.1 occurs when we set $m = 1$. We are thus led immediately to the following results.

Theorem 3.2. *The generalized Hurwitz-Lerch zeta function*

$$\Phi^{(\rho_1,\cdots,\rho_p,\sigma_1,\cdots,\sigma_q)}_{\lambda_1,\cdots,\lambda_p;\mu_1,\cdots,\mu_q} (z, s, a; b, 1)$$

satisfies the following partial differential equation:

$$\begin{aligned}
&\left[b \, \frac{\partial}{\partial b} \left(b \, \frac{\partial}{\partial b} - s \right) \left(b \, \frac{\partial}{\partial b} - 1 \right) - (a+1) \, bz \, \frac{\partial}{\partial z} \right] \\
&\qquad \left\{ \Phi^{(\rho_1,\cdots,\rho_p,\sigma_1,\cdots,\sigma_q)}_{\lambda_1,\cdots,\lambda_p;\mu_1,\cdots,\mu_q} (z, s, a; b, 1) \right\} = 0. \qquad (3.20)
\end{aligned}$$

Furthermore, the generalized Hurwitz-Lerch zeta function

$$\Phi^{(\rho_1,\cdots,\rho_p,\sigma_1,\cdots,\sigma_q)}_{\lambda_1,\cdots,\lambda_p;\mu_1,\cdots,\mu_q} (z, s, a; b, 1),$$

when considered as an analytic function of the variable b, satisfies the following relationship:

$$\begin{aligned}
&\left[b \, \frac{\partial}{\partial b} \left(b \, \frac{\partial}{\partial b} - s \right) \left(b \, \frac{\partial}{\partial b} - 1 \right) \right] \left\{ \Phi^{(\rho_1,\cdots,\rho_p,\sigma_1,\cdots,\sigma_q)}_{\lambda_1,\cdots,\lambda_p;\mu_1,\cdots,\mu_q} (z, s, a; b, 1) \right\} \\
&\qquad = \frac{(a+1) \, b\lambda_1 \cdots \lambda_p}{\mu_1 \cdots \mu_q} \; \Phi^{(\rho_1,\cdots,\rho_p,\sigma_1,\cdots,\sigma_q)}_{\lambda_1,\cdots,\lambda_p;\mu_1,\cdots,\mu_q} (z, s, a+1; b, 1). \qquad (3.21)
\end{aligned}$$

4. Applications Involving the Hurwitz Measure and Probability Distributions

Let $\chi_A(n)$ be the characteristic function of the subset A of the set $\mathbb{N}$ of positive integers (or, in the language of probability theory, the indicator function of the event $A \subseteq \mathbb{N}$). Then it is well known that the following arithmetic density of number theory:

$$\operatorname{dens}(A) = \lim_{k\to\infty} \frac{1}{k} \sum_{n=1}^{k} \chi_A(n) \tag{4.1}$$

does not define a measure on the set $\mathbb{N}$ of positive integers. In order to remedy this deficiency, Golomb [30] defined a probability on the sample space $\mathbb{N}$ and showed that, if the subset A of $\mathbb{N}$ has an arithmetic density, then

$$\lim_{s\to 1} Q_s(A) = \operatorname{dens}(A), \tag{4.2}$$

thereby allowing number-theoretic facts regarding densities of sets of positive integers to be proven by probabilistic means and then showing that such properties are preserved in the limit. Subsequently, in an interesting sequel to Golomb's investigation [30], Lippert [42] gave an analogous definition of the probabilities P_s when the set $\mathbb{N}$ is replaced by the set of all real numbers greater than 1. Thus, for a Borel set $A \subseteq (1,\infty)$, Lippert's Hurwitz measure of the set A is defined by (see, for details, [42, p. 279, Definition 1]

$$P_s(A) = \frac{s}{\zeta(s)} \int_1^\infty \chi_A(a)\, \zeta(s+1,x)\, \mathrm{d}x \tag{4.3}$$

or, equivalently, by

$$P_s(A) = \int_{x\in(1,\infty)} \chi_A(x)\, \mathrm{d}\widetilde{\mu}(x,s), \tag{4.4}$$

where, in terms of the Hurwitz (or generalized) zeta function $\zeta(s,a)$ defined by (1.3), we have

$$\widetilde{\mu}(x,s) := -\frac{\zeta(s,x)}{\zeta(s)} \qquad \text{and} \qquad \mathrm{d}\widetilde{\mu}(x,s) = -\frac{\mathrm{d}\zeta(s,x)}{\zeta(s)} = s\,\frac{\zeta(s+1,x)}{\zeta(s)}\,\mathrm{d}x. \tag{4.5}$$

More recently, Srivastava *et al.* [91] introduced and investigated a new continuous analogue of Lippert's Hurwitz measure in (4.3) by using a special case of the generalized Hurwitz-Lerch zeta function $\Theta_\mu^\lambda(z,s,a;b)$ defined by (1.16), that is,

$$\begin{aligned}\Theta_1^\lambda(1,s,a;b) &= \frac{1}{\Gamma(s)} \int_0^\infty \frac{t^{s-1}}{1-e^{-t}}\, \exp\left(-at - \frac{b}{t^\lambda}\right) \mathrm{d}t \\ &= \Phi_{\lambda_1,\cdots,\lambda_p;\mu_1,\cdots,\mu_q}^{(\rho_1,\cdots,\rho_p,\sigma_1,\cdots,\sigma_q)}(1,s,a;b,\lambda)\Big|_{p-1=q=0 \quad (\lambda_1=\mu=1;\ \rho_1=1)}\end{aligned} \tag{4.6}$$

Definition 4.1. A Borel set [named after Émile Borel (1871–1956)] is any set in a topological space that can be formed from open sets (or, equivalently, from closed sets) through the operations of countable union, countable intersection and relative complement. Thus, for a Borel set $A \subseteq (1,\infty)$, the generalized Hurwitz measure of the set A is defined by

$$P_s(A) = \frac{s}{\Theta_1^\lambda(1,s,1;b)} \int_1^\infty \chi_A(a)\, \Theta_1^\lambda(1,s+1,a;b)\, \mathrm{d}a \tag{4.7}$$

or, equivalently, by

$$P_s(A) = \int_{a\in(1,\infty)} \chi_A(a)\, \mathrm{d}\widetilde{\mu}(a,s;b,\lambda), \tag{4.8}$$

where

$$\widetilde{\mu}(a,s;b,\lambda) := -\frac{\Theta_1^{\lambda}(1,s,a;b)}{\Theta_1^{\lambda}(1,s,1;b)} \tag{4.9}$$

and

$$\mathrm{d}\widetilde{\mu}(a,s;b,\lambda) = -\frac{\mathrm{d}\Theta_1^{\lambda}(1,s,a;b)}{\Theta_1^{\lambda}(1,s,1;b)} = \frac{s}{\Theta_1^{\lambda}(1,s,1;b)}\Theta_1^{\lambda}(1,s+1,a;b)\ \mathrm{d}a, \tag{4.10}$$

since it is easily seen from the definition (1.16) that

$$\frac{\mathrm{d}}{\mathrm{d}a}\left\{\Theta_{\mu}^{\lambda}(z,s,a;b)\right\} = -s\ \Theta_{\mu}^{\lambda}(z,s+1,a;b). \tag{4.11}$$

In view of the following relationship:

$$P_s\big((1,\infty)\big) = \int_1^{\infty} \mathrm{d}\widetilde{\mu}(a,s;b,\lambda) = \lim_{a\to\infty}\widetilde{\mu}(a,s;b,\lambda) - \widetilde{\mu}(1,s;b,\lambda) = 1,$$

the generalized Hurwitz measure $P_s(A)$ in (4.7) or (4.8) also defines a probability measure on $(1,\infty)$.

Remark 4.1. For $\lambda = 1$ and by letting $b \to 0$, we have

$$\lim_{b\to 0} H_{0,2}^{2,0}\left[ab \;\middle|\; \begin{matrix} \overline{\qquad\qquad} \\ (s,1),(0,1) \end{matrix} \right] = \Gamma(s), \tag{4.12}$$

which implies that

$$\lim_{b\to 0}\widetilde{\mu}(a,s;b,1) = -\lim_{b\to 0}\frac{\Theta_1^{1}(1,s,a,b)}{\Theta_1^{1}(1,s,1,b)} = -\frac{\zeta(s,x)}{\zeta(s)} =: \widetilde{\mu}(x,s). \tag{4.13}$$

Thus, clearly, $\widetilde{\mu}(x,s)$ can be continuously approximated by $\widetilde{\mu}(a,s;b,1)$.

Theorem 4.1. *The measure $\widetilde{\mu}(a,s;b,\lambda)$ satisfies the following difference equation:*

$$\widetilde{\mu}(a+1,s;b,\lambda) - \widetilde{\mu}(a,s;b,\lambda) = \frac{H_{0,2}^{2,0}\left[ab^{\frac{1}{\lambda}} \;\middle|\; \begin{matrix} \overline{\qquad\qquad} \\ (s,1),\big(0,\frac{1}{\lambda}\big) \end{matrix} \right]}{\lambda a^s \Gamma(s)\, \Theta_1^{\lambda}(1,s,1;b)} \tag{4.14}$$

$$(s>1;\ a>0;\ b>0;\ \lambda>0).$$

Proof. From the series representation (2.4) of $\Theta_{\mu}^{\lambda}(z,s,a+1;b)$ (with $\mu=1$ and $z=1$), we have

$$\begin{aligned}\Theta_1^{\lambda}(1,s,a+1;b) &= \frac{1}{\lambda\Gamma(s)}\sum_{n=0}^{\infty}\frac{1}{(a+n+1)^s}H_{0,2}^{2,0}\left[(a+n+1)\,b^{\frac{1}{\lambda}}\,\middle|\,\begin{matrix}\overline{\qquad\qquad}\\ (s,1),\left(0,\frac{1}{\lambda}\right)\end{matrix}\right]\\ &= \frac{1}{\lambda\Gamma(s)}\sum_{n=1}^{\infty}\frac{1}{(a+n)^s}H_{0,2}^{2,0}\left[(a+n)\,b^{\frac{1}{\lambda}}\,\middle|\,\begin{matrix}\overline{\qquad\qquad}\\ (s,1),\left(0,\frac{1}{\lambda}\right)\end{matrix}\right]\\ &= \Theta_1^{\lambda}(1,s,a;b)-\frac{1}{\lambda a^s\Gamma(s)}H_{0,2}^{2,0}\left[ab^{\frac{1}{\lambda}}\,\middle|\,\begin{matrix}\overline{\qquad\qquad}\\ (s,1),\left(0,\frac{1}{\lambda}\right)\end{matrix}\right].\end{aligned} \tag{4.15}$$

The difference equation (4.14) now follows on combining (4.9) and (4.15). □

Remark 4.2. For $\lambda=1$ and by letting $b\to 0$, the difference equation (4.14) reduces to the following form:

$$\widetilde{\mu}(a+1,s)-\widetilde{\mu}(a,s)=\frac{1}{a^s\zeta(s)}, \tag{4.16}$$

where $\widetilde{\mu}(x,s)$ is given by (4.5).

For open events, the generalized Hurwitz measure $P_s(A)$ in (4.7) or (4.8) can be evaluated by using (4.8) and the above Proposition. The results are being stated as Theorem 4.2 below.

Theorem 4.2. *If* $A=(a,a+1)$, *then*

$$P_s(A)=P_s\big((a,a+1)\big)=\frac{H_{0,2}^{2,0}\left[ab^{\frac{1}{\lambda}}\,\middle|\,\begin{matrix}\overline{\qquad\qquad}\\ (s,1),\left(0,\frac{1}{\lambda}\right)\end{matrix}\right]}{\lambda a^s\Gamma(s)\,\Theta_1^{\lambda}(1,s,1;b)}. \tag{4.17}$$

More generally, the generalized Hurwitz measure of an open set $A\subseteq(1,\infty)$ *is given by*

$$P_s(A)=\sum_{i\in I}P_s\big((a_i,b_i)\big)=\sum_{i\in I}\left(\frac{\Theta_1^{\lambda}(1,s,a_i;b)-\Theta_1^{\lambda}(1,s,b_i;b)}{\Theta_1^{\lambda}(1,s,1;b)}\right), \tag{4.18}$$

where

$$A=\bigcup_{i\in I}(a_i,b_i)\qquad\big(a_i,b_i\in[1,\infty);\ i\in I\big).$$

The following theorem shows that the generalized Hurwitz measure $P_s(A)$ in (4.7) or (4.8) basically inherits all properties of Lippert's Hurwitz measure given by (4.3) or (4.4).

Theorem 4.3. *Corresponding to the generalized Hurwitz measure given by* (4.18), *let*

$$A(\epsilon)=\bigcup_{i\in\mathbb{N}}(i,i+\epsilon)\qquad(\epsilon\in[0,1]). \tag{4.19}$$

Then

$$\lim_{s\to 1} P_s\left(A\left(\epsilon\right)\right) = \epsilon. \tag{4.20}$$

Proof. From (4.18), we have

$$P_s\left(A\right) = \sum_{i=1}^{\infty}\left(\frac{\Theta_1^{\lambda}\left(1,s,i;b\right) - \Theta_1^{\lambda}\left(1,s,i+\epsilon;b\right)}{\Theta_1^{\lambda}\left(1,s,1;b\right)}\right). \tag{4.21}$$

By expanding the function $\Theta_1^{\lambda}\left(1,s,i+\epsilon,b\right)$ by means of Taylor's series and using the derivative formula (4.11), we get

$$\begin{aligned} P_s\left(A\right) = \frac{1}{\Theta_1^{\lambda}\left(1,s,1;b\right)}\Bigg(&\epsilon s\sum_{i=1}^{\infty}\Theta_1^{\lambda}\left(1,s+1,i;b\right) \\ &- \frac{\epsilon^2}{2}s\left(s+1\right)\sum_{i=1}^{\infty}\Theta_1^{\lambda}\left(1,s+2,i;b\right) + \cdots\Bigg). \end{aligned} \tag{4.22}$$

We now consider each sum in (4.22) separately. We thus find that

$$\begin{aligned} \sum_{i=1}^{\infty}\Theta_1^{\lambda}\left(1,s+m,i;b\right) &= \frac{1}{\lambda\Gamma\left(s+m\right)}\sum_{i=1}^{\infty}\sum_{n=0}^{\infty}\frac{H_{0,2}^{2,0}\left[(i+n)b^{\frac{1}{\lambda}}\,\middle|\,\begin{array}{c}\overline{\qquad\qquad}\\ (s+m,1),\left(0,\frac{1}{\lambda}\right)\end{array}\right]}{\left(i+n\right)^{s+m}} \\ &= \frac{1}{\lambda\Gamma\left(s+m\right)}\sum_{j=0}^{\infty}\sum_{n=0}^{\infty}\frac{H_{0,2}^{2,0}\left[(j+n+1)b^{\frac{1}{\lambda}}\,\middle|\,\begin{array}{c}\overline{\qquad\qquad}\\ (s+m,1),\left(0,\frac{1}{\lambda}\right)\end{array}\right]}{\left(j+n+1\right)^{s+m}}. \end{aligned} \tag{4.23}$$

Since the number of non-negative integer solutions of the Diophantine equation $j+n=N$ is

$$\binom{N+1}{1} = N+1,$$

the double summation in (4.23) can be replaced by a single summation, that is,

$$\begin{aligned} \sum_{i=1}^{\infty}\Theta_1^{\lambda}\left(1,s+m,i;b\right) &= \frac{1}{\lambda\Gamma\left(s+m\right)}\sum_{N=0}^{\infty}\frac{H_{0,2}^{2,0}\left[(N+1)b^{\frac{1}{\lambda}}\,\middle|\,\begin{array}{c}\overline{\qquad\qquad}\\ (s+m,1),\left(0,\frac{1}{\lambda}\right)\end{array}\right]}{\left(N+1\right)^{s+m-1}} \\ &= \Theta_1^{\lambda}\left(1,s+m-1,1;b\right). \end{aligned} \tag{4.24}$$

We thus obtain

$$\begin{aligned} \lim_{s\to 1} P_s\left(A\right) &= \lim_{s\to 1}\left(\epsilon s\frac{\Theta_1^{\lambda}\left(1,s,1;b\right)}{\Theta_1^{\lambda}\left(1,s,1;b\right)} - \frac{\epsilon^2}{2}s\left(s+1\right)\frac{\Theta_1^{\lambda}\left(1,s+1,1;b\right)}{\Theta_1^{\lambda}\left(1,s,1;b\right)} + \cdots\right) \\ &= \epsilon - \frac{\epsilon^2}{2}s\left(s+1\right)\lim_{s\to 1}\frac{\Theta_1^{\lambda}\left(1,s+1,1;b\right)}{\Theta_1^{\lambda}\left(1,s,1;b\right)} + \cdots. \end{aligned} \tag{4.25}$$

We note that, when $s\to 1$, the series for $\Theta_1^{\lambda}\left(1,s,1;b\right)$ is divergent and the series for $\Theta_1^{\lambda}\left(1,s+1,1;b\right)$ is convergent. Therefore, all other terms vanish in (4.25) except the leading term. Consequently, we get

$$\lim_{s\to 1} P_s(A) = \epsilon, \tag{4.26}$$

which completes the proof of Theorem 4.3. □

It does not seem to be difficult to extend the above-detailed investigation of the generalized Hurwitz measure, which was presented earlier by Srivastava *et al.* [91], to analogously cover wider and more general situations involving the Hurwitz-Lerch zeta function

$$\Phi^{(\rho_1,\cdots,\rho_p,\sigma_1,\cdots,\sigma_q)}_{\lambda_1,\cdots,\lambda_p;\mu_1,\cdots,\mu_q}(z,s,a;b,\lambda)$$

defined by (1.14). Nevertheless, we choose to turn instead toward an investigation of the following general probability distribution involving this generalized Hurwitz-Lerch zeta function.

Definition 4.2. A random variable ξ is said to be generalized Hurwitz distributed if its probability density function is given by

$$f_\xi(a) := \begin{cases} \dfrac{s\,\Phi^{(\rho_1,\cdots,\rho_p,\sigma_1,\cdots,\sigma_q)}_{\lambda_1,\cdots,\lambda_p;\mu_1,\cdots,\mu_q}(z,s+1,a;b,\lambda)}{\Phi^{(\rho_1,\cdots,\rho_p,\sigma_1,\cdots,\sigma_q)}_{\lambda_1,\cdots,\lambda_p;\mu_1,\cdots,\mu_q}(z,s,1;b,\lambda)} & (a \geqq 1) \\ \\ 0 & (\text{otherwise}), \end{cases} \tag{4.27}$$

where it is *tacitly* assumed that the arguments z, s, b, λ and μ. and the parameters

$$\lambda_j, \rho_j \quad (j = 1,\cdots,p) \qquad \text{and} \qquad \mu_j, \sigma_j \quad (j = 1,\cdots,q),$$

are fixed and suitably constrained so that the probability density function $f_\xi(a)$ remains nonnegative.

Theorem 4.4. *Suppose that ξ is a continuous random variable ξ with its probability density function defined by* (4.27)*. Then the moment generating function $M(\mathfrak{z})$ of the random variable ξ is given by*

$$M(\mathfrak{z}) := \mathbb{E}_s\left[e^{\mathfrak{z}\xi}\right] = \sum_{n=0}^{\infty} \mathbb{E}_s[\xi^n]\,\frac{\mathfrak{z}^n}{n!} \tag{4.28}$$

with the moment $\mathbb{E}_s[\xi^n]$ of order n given by

$$\mathbb{E}_s[\xi^n] = \sum_{k=0}^{n} \frac{n!}{(n-k)!}\,\frac{\Gamma(s-k)}{\Gamma(s)}\,\frac{\Phi^{(\rho_1,\cdots,\rho_p,\sigma_1,\cdots,\sigma_q)}_{\lambda_1,\cdots,\lambda_p;\mu_1,\cdots,\mu_q}(z,s-k,1;b,\lambda)}{\Phi^{(\rho_1,\cdots,\rho_p,\sigma_1,\cdots,\sigma_q)}_{\lambda_1,\cdots,\lambda_p;\mu_1,\cdots,\mu_q}(z,s,1;b,\lambda)}. \tag{4.29}$$

Proof. The assertion in (4.28) can be derived easily by using the exponential series for $e^{\mathfrak{z}\xi}$. On the other hand, since

$$\frac{\mathrm{d}}{\mathrm{d}a}\left\{\Phi^{(\rho_1,\cdots,\rho_p,\sigma_1,\cdots,\sigma_q)}_{\lambda_1,\cdots,\lambda_p;\mu_1,\cdots,\mu_q}(z,s,a;b,\lambda)\right\}$$

$$= -s\,\Phi^{(\rho_1,\cdots,\rho_p,\sigma_1,\cdots,\sigma_q)}_{\lambda_1,\cdots,\lambda_p;\mu_1,\cdots,\mu_q}(z,s+1,a;b,\lambda), \tag{4.30}$$

which follows readily from the definition (1.14), if we make use of integration by parts, we find from the definition of the moment $\mathbb{E}_s\left[\xi^n\right]$ that

$$\begin{aligned}
\mathbb{E}_s\left[\xi^n\right] &= \int_1^{\infty} a^n\ f_{\xi}\left(a\right)\mathrm{d}a \\
&= \frac{s}{\Phi^{(\rho_1,\cdots,\rho_p,\sigma_1,\cdots,\sigma_q)}_{\lambda_1,\cdots,\lambda_p;\mu_1,\cdots,\mu_q}\left(z,s,1;b,\lambda\right)} \int_1^{\infty} a^n\ \Phi^{(\rho_1,\cdots,\rho_p,\sigma_1,\cdots,\sigma_q)}_{\lambda_1,\cdots,\lambda_p;\mu_1,\cdots,\mu_q}\left(z,s+1,a;b,\lambda\right)\mathrm{d}a \\
&= -\frac{1}{\Phi^{(\rho_1,\cdots,\rho_p,\sigma_1,\cdots,\sigma_q)}_{\lambda_1,\cdots,\lambda_p;\mu_1,\cdots,\mu_q}\left(z,s,1;b,\lambda\right)} \\
&\qquad \cdot \int_1^{\infty} a^n\ \frac{\mathrm{d}}{\mathrm{d}a}\left\{\Phi^{(\rho_1,\cdots,\rho_p,\sigma_1,\cdots,\sigma_q)}_{\lambda_1,\cdots,\lambda_p;\mu_1,\cdots,\mu_q}\left(z,s,a;b,\lambda\right)\right\}\mathrm{d}a \\
&= -\left.\frac{a^n\ \Phi^{(\rho_1,\cdots,\rho_p,\sigma_1,\cdots,\sigma_q)}_{\lambda_1,\cdots,\lambda_p;\mu_1,\cdots,\mu_q}\left(z,s,a;b,\lambda\right)}{\Phi^{(\rho_1,\cdots,\rho_p,\sigma_1,\cdots,\sigma_q)}_{\lambda_1,\cdots,\lambda_p;\mu_1,\cdots,\mu_q}\left(z,s,1;b,\lambda\right)}\right|_{a=1}^{\infty} + \frac{n}{\Phi^{(\rho_1,\cdots,\rho_p,\sigma_1,\cdots,\sigma_q)}_{\lambda_1,\cdots,\lambda_p;\mu_1,\cdots,\mu_q}\left(z,s,1;b,\lambda\right)} \\
&\qquad \cdot \int_1^{\infty} a^{n-1}\ \Phi^{(\rho_1,\cdots,\rho_p,\sigma_1,\cdots,\sigma_q)}_{\lambda_1,\cdots,\lambda_p;\mu_1,\cdots,\mu_q}\left(z,s,a;b,\lambda\right)\mathrm{d}a \\
&= 1 - \lim_{a\to\infty}\left\{\frac{a^n\ \Phi^{(\rho_1,\cdots,\rho_p,\sigma_1,\cdots,\sigma_q)}_{\lambda_1,\cdots,\lambda_p;\mu_1,\cdots,\mu_q}\left(z,s,a;b,\lambda\right)}{\Phi^{(\rho_1,\cdots,\rho_p,\sigma_1,\cdots,\sigma_q)}_{\lambda_1,\cdots,\lambda_p;\mu_1,\cdots,\mu_q}\left(z,s,1;b,\lambda\right)}\right\} + \frac{n}{\Phi^{(\rho_1,\cdots,\rho_p,\sigma_1,\cdots,\sigma_q)}_{\lambda_1,\cdots,\lambda_p;\mu_1,\cdots,\mu_q}\left(z,s,1;b,\lambda\right)} \\
&\qquad \cdot \int_1^{\infty} a^{n-1}\ \Phi^{(\rho_1,\cdots,\rho_p,\sigma_1,\cdots,\sigma_q)}_{\lambda_1,\cdots,\lambda_p;\mu_1,\cdots,\mu_q}\left(z,s,a;b,\lambda\right)\mathrm{d}a \\
&= 1 + \frac{n}{\Phi^{(\rho_1,\cdots,\rho_p,\sigma_1,\cdots,\sigma_q)}_{\lambda_1,\cdots,\lambda_p;\mu_1,\cdots,\mu_q}\left(z,s,1;b,\lambda\right)} \\
&\qquad \cdot \int_1^{\infty} a^{n-1}\ \Phi^{(\rho_1,\cdots,\rho_p,\sigma_1,\cdots,\sigma_q)}_{\lambda_1,\cdots,\lambda_p;\mu_1,\cdots,\mu_q}\left(z,s,a;b,\lambda\right)\mathrm{d}a \qquad (n\in\mathbb{N}),
\end{aligned} \tag{4.31}$$

where, in addition to the derivative property (4.30), we have used the following limit formula:

$$\begin{aligned}
&\lim_{a\to\infty}\left\{a^n\ \Phi^{(\rho_1,\cdots,\rho_p,\sigma_1,\cdots,\sigma_q)}_{\lambda_1,\cdots,\lambda_p;\mu_1,\cdots,\mu_q}\left(z,s,a;b,\lambda\right)\right\} \\
&\quad = \lim_{a\to\infty}\left\{\frac{a^n}{\Gamma\left(s\right)}\int_0^{\infty} t^{s-1}\ \exp\left(-at-\frac{b}{t^{\lambda}}\right)\right\}\mathrm{d}t \\
&\quad = \frac{1}{\Gamma\left(s\right)}\int_0^{\infty} t^{s-1}\ \exp\left(-\frac{b}{t^{\lambda}}\right)\ \lim_{a\to\infty}\left\{a^n\ e^{-at}\right\} \\
&\qquad\qquad \cdot {}_p\Psi^*_q\left[\begin{array}{c}(\lambda_1,\rho_1),\cdots,(\lambda_p,\rho_p);\\ \\ (\mu_1,\sigma_1),\cdots,(\mu_q,\sigma_q);\end{array} ze^{-t}\right]\mathrm{d}t \\
&\quad = 0 \qquad (n\in\mathbb{N}).
\end{aligned} \tag{4.32}$$

Consequently, we have the following reduction formula for $\mathbb{E}_s\left[\xi^n\right]$:

$$\mathbb{E}_s\left[\xi^n\right] = 1 + \frac{\Phi^{(\rho_1,\cdots,\rho_p,\sigma_1,\cdots,\sigma_q)}_{\lambda_1,\cdots,\lambda_p;\mu_1,\cdots,\mu_q}\left(z,s-1,1;b,\lambda\right)}{\Phi^{(\rho_1,\cdots,\rho_p,\sigma_1,\cdots,\sigma_q)}_{\lambda_1,\cdots,\lambda_p;\mu_1,\cdots,\mu_q}\left(z,s,1;b,\lambda\right)}\ \frac{n}{s-1}\ \mathbb{E}_{s-1}\left[\xi^{n-1}\right] \tag{4.33}$$

$$(n\in\mathbb{N}).$$

By iterating the recurrence (4.31), we arrive at the desired result (4.29) asserted by Theorem 4.4. □

Remark 4.3. A special case of Theorem 4.1 when

$$p-1=q=0, \quad \lambda_1=\mu=1 \qquad \text{and} \qquad \rho_1=1$$

was considered by Srivastava *et al.* [91]. Moreover, in an earlier investigation, Gupta *et al.* [32] considered some particularly simple forms of the Hurwitz-Lerch zeta distributions and their applications in reliability theory. On the other hand, in a very recent investigation, Saxena *et al.* [65] made use of some specialized cases of the extended Hurwitz-Lerch zeta function

$$\Phi^{(\rho_1,\cdots,\rho_p,\sigma_1,\cdots,\sigma_q)}_{\lambda_1,\cdots,\lambda_p;\mu_1,\cdots,\mu_q}(z,s,a)$$

of Srivastava *et al.* [95, p. 503, Equation (6.2)] (see also [69] and [87]), which is defined here by (1.10), in statistical inference.

5. Concluding Remarks and Further Applications Including Directions for Further Developments

The main purpose in our presentation here has been to introduce and study the properties and relationships associated with some novel families of the Riemann, the Hurwitz (or generalized) and the Hurwitz-Lerch zeta functions as well as the so-called λ-generalized Hurwitz-Lerch zeta functions. We have successfully presented many potentially useful results involving some of these λ-generalized Hurwitz-Lerch zeta functions including (for example) their partial differential equations, new series and Mellin-Barnes type contour integral representations (which are associated with Fox's H-function) and several other summation formulas involving them. We have discussed their potential application in Number Theory by appropriately constructing a presumably new continuous analogue of Lippert's Hurwitz measure. We have also considered some other statistical applications of these families of the λ-generalized Hurwitz-Lerch zeta functions in probability distribution theory.

As it is widely believed and accepted, *Mathematics* appeals to the intellect. In addition, however, great mathematics possesses a kind of perceptual quality which endows it with a beauty comparable to that of great art or great music. Much of the work of the 18th century Swiss mathematician, Leonhard Euler (1707–1783), belongs in this category. Euler's work on $\zeta(s)$ began around 1730 with approximations to the value of $\zeta(2)$, continued with the evaluation of $\zeta(2n)$ $(n \in \mathbb{N})$, and resulted around 1749 in the discovery of the celebrated functional equation for $\zeta(s)$ almost 110 years before the remarkably influential German mathematician, Georg Friedrich Bernhard Riemann (1826–1866).

Explicit evaluations of such zeta values as $\zeta(3)$, $\zeta(5)$, it et cetera are known to arise naturally in a wide variety of applications such as those in Elastostatics, Quantum Field Theory, *et cetera.* For example, we may refer the reader to the works by Tricomi [99], Witten [103], and Nash and O'Connor (see [55] and

[56]). On the other hand, in the case of *even* integer arguments, the following computationally useful relationship is known:

$$\zeta(2n) = (-1)^{n-1} \frac{(2\pi)^{2n}}{2 \cdot (2n)!} B_{2n} \qquad (n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}) \tag{5.1}$$

together with the *well-tabulated* Bernoulli numbers defined by the following generating function:

$$\frac{z}{e^z - 1} = \sum_{n=0}^{\infty} B_n \frac{z^n}{n!} \qquad (|z| < 2\pi), \tag{5.2}$$

as well as by the following recursion formula:

$$\zeta(2n) = \left(n + \frac{1}{2}\right)^{-1} \sum_{k=1}^{n-1} \zeta(2k)\, \zeta(2n-2k) \qquad (n \in \mathbb{N} \setminus \{1\}). \tag{5.3}$$

For the Hurwitz (or generalized) zeta function $\zeta(s,a)$ defined by (1.3), one can easily derive the following simple consequence of the binomial theorem:

$$\sum_{k=0}^{\infty} \frac{(s)_k}{k!} \zeta(s+k,a)\, t^k = \zeta(s, a-t) \qquad (|t| < |a|). \tag{5.4}$$

Furthermore, for the Hurwitz-Lerch zeta function $\Phi(z,s,a)$ defined by (1.1), we similarly get

$$\sum_{k=0}^{\infty} \frac{(s)_k}{k!} \Phi(z, s+k, a)\, t^k = \Phi(s, a-t) \qquad (|t| < |a|). \tag{5.5}$$

More generally, it follows easily from the definition (1.10) that

$$\sum_{k=0}^{\infty} \frac{(s)_k}{k!} \Phi^{(\rho_1,\cdots,\rho_p,\sigma_1,\cdots,\sigma_q)}_{\lambda_1,\cdots,\lambda_p;\mu_1,\cdots,\mu_q}(z, s+k, a)t^k$$
$$= \Phi^{(\rho_1,\cdots,\rho_p,\sigma_1,\cdots,\sigma_q)}_{\lambda_1,\cdots,\lambda_p;\mu_1,\cdots,\mu_q}(z, s, a-t) \qquad (|t| < |a|). \tag{5.6}$$

On the other hand, for the λ-generalized Hurwitz-Lerch zeta function defined by (1.14), we find that

$$\sum_{k=0}^{\infty} \frac{(s)_k}{k!} \Phi^{(\rho_1,\cdots,\rho_p,\sigma_1,\cdots,\sigma_q)}_{\lambda_1,\cdots,\lambda_p;\mu_1,\cdots,\mu_q}(z, s+k, a; b, \lambda)t^k$$
$$= \Phi^{(\rho_1,\cdots,\rho_p,\sigma_1,\cdots,\sigma_q)}_{\lambda_1,\cdots,\lambda_p;\mu_1,\cdots,\mu_q}(z, s, a-t; b, \lambda) \qquad (|t| < |a|), \tag{5.7}$$

which would immediately yield (5.6) when $b = 0$.

In a series of papers by Srivastava and others (see, for details, a recent survey-cum-expository presentation by Srivastava [72]), the summation formula (5.4) and other results were applied rather extensively for establishing rapidly convergent and computationally useful series and other representations for $\zeta(2n+1)$ $(n \in \mathbb{N})$. Such results for the zeta values at odd positive integers $2n+1$ $(n \in \mathbb{N})$ were revisited and applied recently by Delplace [20] in his study of the elastic membrane model involving Poissons partial differential equation with rectangular boundary. Presumably, the effective usages of one or the other of the summation formulas

(5.5), (5.6) and (5.7) in finding the corresponding rapidly convergent series representations for $s = 2n + 1$ $(n \in \mathbb{N})$ is still an *Open Problem*.

In Geometric Function Theory of Complex Analysis, in the year 2007 Srivastava and Attiya [73] introduced and initiated the study of the linear operator

$$J_{s,a} : \mathcal{A} \to \mathcal{A},$$

which they defined, for a function $f(z)$ in the class $\mathcal{A}$ of normalized analytic functions in the open unit disk

$$\mathbb{U} = \{z : z \in \mathbb{C} \quad \text{and} \quad |z| < 1\},$$

by

$$J_{s,a}(f)(z) := G_{s,a}(z) * f(z) \qquad (a \in \mathbb{C} \setminus \mathbb{Z}_0^-;\ s \in \mathbb{C}), \tag{5.8}$$

where the symbol $*$ denotes the Hadamard product (or convolution) of analytic functions and the function $G_{s,a}$ is defined by

$$G_{s,a}(z) = (a+1)^s[\Phi(z, s, a) - a^{-s}]$$

in terms of the Hurwitz-Lerch zeta function $\Phi(z, s, a)$ given by (1.1).

For a function $f \in \mathbb{A}$ of the form given by

$$f(z) = z + \sum_{n=2}^{\infty} \mathfrak{a}_n\, z^n \qquad (z \in \mathbb{U}),$$

it is easily found from the definitions (5.8) and (1.1) that

$$J_{s,a}(f)(z) = z + \sum_{n=2}^{\infty} \left(\frac{a+1}{a+n}\right)^s \mathfrak{a}_n\, z^n \qquad (z \in \mathbb{U}). \tag{5.9}$$

Ever since the publication of the aforementioned pioneering work by Srivastava and Attiya [73] on the Srivastava-Attiya operator $J_{s,a}$ defined by (5.8), the literature on Geometric Function Theory in Complex Analysis has become flooded by its numerous further studies involving various classes of univalent and multivalent analytic and meromorphic functions as well as by a large variety of its extensions and generalizations. For ready reference of the interested reader, we have chosen to include the citations of most (but, by no means, all) of these recently-published related works dealing, in one way or the other, with the Srivastava-Attiya operator $J_{s,a}$ defined by (5.8).

We choose to conclude our presentation by giving a brief outline and survey of some further recent developments in the subject-matter of our presentation here. Indeed, for several novel properties and results as well as for some other applications of the Hurwitz-Lerch zeta function $\Phi(z, s, a)$ as well as the λ-generalized Hurwitz-Lerch zeta functions, the interested reader may refer to the recent works [45], [65], [69] to [72], [76], [87], [88], [95], [97] and [98]. Moreover, by making use of the λ-generalized Hurwitz-Lerch zeta functions, many recent publications have introduced and studied a number of general Srivastava-Attiya type convolution operators in the context of analytic as well as meromorphic functions in Geometric Function Theory of Complex Analysis (see, for example, [14] and [82] to [85]). Such areas of applications of the Hurwitz-Lerch zeta function $\Phi(z, s, a)$

as well as the λ-generalized Hurwitz-Lerch zeta functions are still growing rather continuously.

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