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THE DIRICHLET PROBLEM IN A CLASS OF GENERALIZED WEIGHTED MORREY SPACES

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Abstract. We show continuity in generalized weighted Morrey spaces $M_{p,\varphi}(w)$ of sub-linear integral operators generated by some classical integral operators and commutators. The obtained estimates are used to study global regularity of the solution of the Dirichlet problem for linear uniformly elliptic operators with discontinuous data.

1. Introduction

In the present work we study the global regularity of the solutions of a class of elliptic partial differential equations (PDEs) in *generalized weighted Morrey spaces*. Recall that the classical *Morrey spaces* $L_{p,\lambda}$ were introduced in [24] in order to study the local behavior of the solutions of elliptic systems. In [3] Chiarenza and Frasca show boundedness of the *Hardy-Littlewood maximal operator* $\mathcal{M}$ and the Calderón-Zygmund operator $\mathcal{K}$

$$\mathcal{M}f(x) = \sup_{\mathcal{B}(x)} \frac{1}{|\mathcal{B}(x)|} \int_{\mathcal{B}(x)} |f(y)|\, dy \quad \text{and} \quad \mathcal{K}f(x) = \int_{\mathbb{R}^n} k(x-y)f(y)\, dy$$

in $L_{p,\lambda}(\mathbb{R}^n)$, where the supremum is taken over all balls centered in $x \in \mathbb{R}^n$. Integral operators of that kind appear in the representation formulas of the solutions of various PDEs. Thus the continuity of the Calderón-Zygmund integral in certain functional space permit to study the regularity of the solutions of boundary value problems for linear PDEs in the corresponding space.

Moreover, various Morrey spaces are defined in the process of study. Guliyev, Mizuhara and Nakai [9, 23, 25] introduced generalized Morrey spaces $M_{p,\varphi}$ (see, also [10, 11, 27]). Komori and Shirai [21] defined weighted Morrey spaces $L_{p,\kappa}(w)$; Guliyev [14] gave a concept of the generalized weighted Morrey spaces $M_{p,\varphi}(w)$ which could be viewed as extension of both $M_{p,\varphi}$ and $L_{p,\kappa}(w)$. In [14], the boundedness of the classical operators and their commutators in spaces $M_{p,\varphi}(w)$ was studied. In this paper we apply these estimates to study the regularity of the solution of Dirichlet problem for linear elliptic partial differential equation with discontinuous coefficients. The presented result is a generalization of previous works [6, 17, 29].

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The paper is organized as follows. We begin introducing some function spaces that we are going to use. In Sections 4 and 5 we study continuity in the spaces $M_{p,\varphi}(w)$ of certain sub-linear integrals and their commutators with bounded mean oscillation functions. These results permit to obtain continuity of the Calderón-Zygmund operator and its commutator that is shown in Section 6. In the last section we give an application of these estimates to the study of linear Dirichlet problem for elliptic equations. This problem is firstly studied by Chiarenza, Frasca and Longo. In their pioneer works [4, 5] they prove unique strong solvability of

$$\begin{cases} \mathcal{L}u \equiv a^{ij}(x)D_{ij}u = f(x) \quad \text{a.a. } x \in \Omega, \\ u \in W_p^2(\Omega) \cap \overset{\circ}{W}{}_p^1(\Omega),\ p \in (1,\infty),\ a^{ij} \in VMO \end{cases} \tag{1.1}$$

extending this way the classical theory of operators with continuous coefficients to those with discontinuous coefficients. Later their results have been extended in the Sobolev-Morrey spaces $W^2_{p,\lambda}(\Omega) \cap \overset{\circ}{W}{}^1_p(\Omega)$, $\lambda \in (1,n)$ (see [7]) and the generalized Sobolev-Morrey spaces $W^2_{p,\phi}(\Omega) \cap \overset{\circ}{W}{}^1_p(\Omega)$ (see [29]) with ϕ as in [25]. In [17] we have studied the regularity of the solution of (1.1) in generalized Sobolev-Morrey spaces $W^2_{p,\varphi}(\Omega)$ where the weight function φ satisfies a certain supremal condition as in [11]. We show that $\mathcal{L}u \in M_{p,\varphi}(\Omega)$ implies $D_{ij}u \in M_{p,\varphi}(\Omega)$ satisfying the estimate

$$\|D^2u\|_{p,\varphi;\Omega} \le C\big(\|\mathcal{L}u\|_{p,\varphi;\Omega} + \|u\|_{p,\varphi;\Omega}\big)\,.$$

These studies are extended on divergence form linear elliptic and parabolic equations in [2, 18].

Throughout this paper we use the following notations and conventions. We let Ω be a bounded domain. As usual, D_iu, $D_{ij}u$ and $Du = (D_1u, \dots, D_nu)$ mean the partial derivatives and the gradient of u. The ball in $\mathbb{R}^n$ is denoted by $\mathcal{B}_r(x_0)$ or more generally by $\mathcal{B}_r$ and the unit sphere is $\mathbb{S}^{n-1}$. The complementary of $\mathcal{B}_r$ is $\mathcal{B}_r^c$ and $\mathcal{B}_{2r}$ stands for a ball centered in the same point as $\mathcal{B}_r$ with radius $2r$. For any measurable function f we write $f_{\mathcal{B}} = \frac{1}{|\mathcal{B}|}\int_{\mathcal{B}} f(y)dy$ and

$$\|f\|_{L_p(\Omega)} = \|f\|_{p;\Omega} = \Big(\int_\Omega |f(x)|^p\,dx\Big)^{1/p}, \quad \|\cdot\|_{p;\mathbb{R}^n} \equiv \|\cdot\|_p\,.$$

The letter C is used for various positive constants and may change from one occurrence to another.

2. Weighted spaces

We start with the definitions of some function spaces that we are going to use.

Definition 2.1. (see [19, 26]) Let $a \in L_1^{\mathrm{loc}}(\mathbb{R}^n)$ and $a_{\mathcal{B}_r} = \frac{1}{|\mathcal{B}_r|}\int_{\mathcal{B}_r} a(x)\,dx$. Define

$$\gamma_a(R) = \sup_{r\le R} \frac{1}{|\mathcal{B}_r|}\int_{\mathcal{B}_r} |a(y) - a_{\mathcal{B}_r}|\,dy \qquad \forall\ R > 0.$$

We say that $a \in BMO$ (*bounded mean oscillation*) if

$$\|a\|_* = \sup_{R>0} \gamma_a(R) < +\infty.$$

The quantity $\|a\|_*$ is a norm in BMO modulo constant functions under which BMO is a Banach space. If

$$\lim_{R\to 0}\gamma_a(R)=0,$$

then $a\in VMO$ (*vanishing mean oscillation*) and we call $\gamma_a(R)$ a VMO-modulus of a.

For any bounded domain $\Omega\subset\mathbb{R}^n$ we define $BMO(\Omega)$ and $VMO(\Omega)$ taking $a\in L_1(\Omega)$ and integrating over $\Omega_r=\Omega\cap\mathcal{B}_r$.

According to [1], having a function $a\in BMO(\Omega)$ or $VMO(\Omega)$ it is possible to extend it in the whole $\mathbb{R}^n$ preserving its BMO-norm or VMO-modulus, respectively. In the following we use this extension without explicit references.

Lemma 2.1. (John-Nirenberg lemma, [19]) *Let $a\in BMO$ and $p\in(1,\infty)$. Then for any ball $\mathcal{B}$ there holds*

$$\left(\frac{1}{|\mathcal{B}|}\int_{\mathcal{B}}|a(y)-a_{\mathcal{B}}|^p dy\right)^{\frac{1}{p}}\le C(p)\|a\|_*.$$

As an immediate consequence of Lemma 2.1 we get the following property.

Corollary 2.1. *Let $a\in BMO$, then for all $0<2r<t$ the following inequality holds*

$$\left|a_{\mathcal{B}_r}-a_{\mathcal{B}_t}\right|\le C\|a\|_*\ln\frac{t}{r}, \tag{2.1}$$

where the constant is independent of a,t and r.

We call *weight* a nonnegative locally integrable function w on $\mathbb{R}^n$. Given a weight w and a measurable set $\mathcal{E}$ we denote the w-measure of $\mathcal{E}$ by

$$w(\mathcal{E})=\int_{\mathcal{E}}w(x)\,dx\,.$$

Denote by $L_{p,w}(\mathbb{R}^n)$ or $L_{p,w}$ the weighted L_p spaces. It turns out that the strong type (p,p) inequality

$$\left(\int_{\mathbb{R}^n}(\mathcal{M}f(x))^p w(x)\,dx\right)^{1/p}\le C_p\left(\int_{\mathbb{R}^n}|f(x)|^p w(x)\,dx\right)^{1/p}$$

holds for all $f\in L_{p,w}$ if and only if the weight function satisfies the *Muckenhoupt* A_p-condition

$$[w]_{A_p}:=\sup_{\mathcal{B}}\left(\frac{1}{|\mathcal{B}|}\int_{\mathcal{B}}w(x)\,dx\right)\left(\frac{1}{|\mathcal{B}|}\int_{\mathcal{B}}w(x)^{-\frac{1}{p-1}}\,dx\right)^{p-1}<\infty\,. \tag{2.2}$$

The expression $[w]_{A_p}$ is called *characteristic constant* of w. The function w is A_1 weight if $\mathcal{M}w(x)\le C_1 w(x)$ for almost all $x\in\mathbb{R}^n$. The minimal constant C_1 for which the inequality holds is the A_1 *characteristic constant* of w.

We summarize some basic properties of the A_p weights in the following lemma (see [8, 22] for more details).

Lemma 2.2. *(1) Let $w\in A_p$ for $1\le p<\infty$. Then for each $\mathcal{B}$*

$$1\le[w]^{\frac{1}{p}}_{A_p(\mathcal{B})}=|\mathcal{B}|^{-1}\|w\|^{1/p}_{L_1(\mathcal{B})}\,\|w^{-\frac{1}{p}}\|_{L_{p'}(\mathcal{B})}\le[w]^{1/p}_{A_p}\,. \tag{2.3}$$

(2) The function $w^{-\frac{1}{p-1}}$ *is in* $A_{p'}$*, where* $\frac{1}{p}+\frac{1}{p'}=1$, $1<p<\infty$ *with characteristic constant*

$$[w^{-\frac{1}{p-1}}]_{A_{p'}} = [w]_{A_p}^{\frac{1}{p-1}}.$$

(3) The classes A_p *are increasing as p increases and*

$$[w]_{A_q} \le [w]_{A_p}, \qquad 1\le q<p<\infty.$$

(4) The measure $w(x)dx$ *is doubling, precisely, for all* $\lambda>1$

$$w(\lambda\mathcal{B}) \le \lambda^{np}[w]_{A_p} w(\mathcal{B}).$$

(5) If $w\in A_p$ *for some* $1\le p\le\infty$*, then there exist* $C>0$ *and* $\delta>0$ *such that for any ball* $\mathcal{B}$ *and a measurable set* $\mathcal{E}\subset\mathcal{B}$,

$$\frac{1}{[w]_{A_p}}\left(\frac{|\mathcal{E}|}{|\mathcal{B}|}\right) \le \frac{w(\mathcal{E})}{w(\mathcal{B})} \le C\left(\frac{|\mathcal{E}|}{|\mathcal{B}|}\right)^{\delta}.$$

(6) For each $1\le p<\infty$ *we have*

$$\bigcup_{1\le p<\infty} A_p = A_\infty \quad \text{and} \quad [w]_{A_\infty} \le [w]_{A_p}.$$

(7) For each $a\in BMO$, $1\le p<\infty$ *and* $w\in A_\infty$ *we have*

$$\|a\|_* = C\sup_{\mathcal{B}}\left(\frac{1}{w(\mathcal{B})}\int_{\mathcal{B}} |a(y)-a_{\mathcal{B}}|^p w(y)\,dy\right)^{\frac{1}{p}}. \tag{2.4}$$

The next result follows from [14, Lemma 4.4].

Lemma 2.3. *Let* $w\in A_p$ *with* $1<p<\infty$ *and* $a\in BMO$*. Then*

$$\left(\frac{1}{w^{1-p'}(\mathcal{B}_r)}\int_{\mathcal{B}_r} |a(y)-a_{\mathcal{B}_r}|^{p'} w(y)^{1-p'}\,dy\right)^{\frac{1}{p'}} \le C[w]_{A_p}^{\frac{1}{p}}\|a\|_*, \tag{2.5}$$

where C is independent of a, w and r.

Definition 2.2. Let $\varphi(x,r)$ be a weight in $\Omega\times\mathbb{R}_+\to\mathbb{R}_+$ and $w\in A_p$, $p\in[1,\infty)$. The generalized weighted Morrey space $M_{p,\varphi}(\Omega,w)$ consists of all functions $f\in L_{p,w}(\Omega)$ such that

$$\|f\|_{p,\varphi,w;\Omega} = \sup_{x\in\Omega, r>0}\varphi(x,r)^{-1}\left(w(\mathcal{B}_r(x))^{-1}\int_{\Omega_r(x)}|f(y)|^p w(y)\,dy\right)^{1/p} < \infty,$$

where $\Omega_r(x)=\Omega\cap\mathcal{B}_r(x)$.

Generalized Sobolev-Morrey space $W^2_{p,\varphi}(\Omega,w)$ consists of all functions $u\in W^2_{p.w}(\Omega)$ with distributional derivatives $D^s u\in M_{p,\varphi}(\Omega,w)$, $0\le|s|\le 2$ endowed by the norm

$$\|u\|_{W^2_{p,\varphi}(\Omega,w)} = \sum_{0\le|s|\le 2}\|D^s u\|_{p,\varphi,w;\Omega}.$$

The space $W^2_{p,\varphi}(\Omega,w)\cap \overset{\circ}{W}{}^1_{p,\varphi}(\Omega,w)$ consists of all functions $u\in W^2_{p,w}(\Omega)\cap \overset{\circ}{W}{}^1_{p,w}(\Omega)$ with $D^s u\in M_{p,\varphi}(\Omega,w)$, $0\le|s|\le 2$ and is endowed by the same norm. Recall that $\overset{\circ}{W}{}^1_{p,w}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ with respect to the norm in $W^1_{p,w}(\Omega)$.

Remark 2.1. *The density of the* C_0^∞ *functions in the weighted Lebesgue space* $L_{p,w}$ *is proved in* [28, Chapter 3, Theorem 3.11].

3. Sub-linear operators generated by singular integrals in $M_{p,\varphi}(w)$

Let T be a sub-linear operator such that for any $f \in L_1(\mathbb{R}^n)$ with a compact support. Suppose that for $x \notin \operatorname{supp} f$ the following inequality holds

$$|Tf(x)| \le C \int_{\mathbb{R}^n} \frac{|f(y)|}{|x-y|^n}\, dy, \tag{3.1}$$

where C is independent of f.

The following results generalize some estimates obtained in [6, 9, 11, 15, 16]. The proof follows as in [15] making use of the boundedness of the weighted Hardy operator

$$H_\psi^* g(r) := \int_r^\infty g(t)\psi(t)\, dt, \qquad 0 < r < \infty\,.$$

Theorem 3.1. *(*[12, 13]*) Suppose that* v_1, v_2*, and* ψ *are weights on* $\mathbb{R}_+$*. Then the inequality*

$$\operatorname*{ess\,sup}_{r>0} v_2(r) H_\psi^* g(r) \le C \operatorname*{ess\,sup}_{r>0} v_1(r) g(r) \tag{3.2}$$

holds with some $C > 0$ *for all nonnegative and nondecreasing* g *on* $\mathbb{R}_+$ *if and only if*

$$B := \operatorname*{ess\,sup}_{r>0} v_2(r) \int_r^\infty \frac{\psi(t)}{\operatorname*{ess\,sup}_{t<s<\infty} v_1(s)}\, dt < \infty \tag{3.3}$$

and $C = B$ *is the best constant in* (3.2).

Theorem 3.2. *Let* $1 < p < \infty$*,* $w \in A_p$ *and the pair* (φ_1, φ_2) *satisfy*

$$\int_r^\infty \frac{\operatorname*{ess\,inf}_{t<s<\infty} \varphi_1(x,s) w(\mathcal{B}_s(x))^{\frac{1}{p}}}{w(\mathcal{B}_t(x))^{\frac{1}{p}}} \frac{dt}{t} \le C\, \varphi_2(x,r), \tag{3.4}$$

and T *be a sub-linear operator satisfying* (3.1)*. If* T *is bounded on* $L_{p,w}$ *and* $\|Tf\|_{p,w} \le C[w]_{A_p}^{1/p} \|f\|_{p,w}$*, then* T *is bounded from* $M_{p,\varphi_1}(w)$ *to* $M_{p,\varphi_2}(w)$ *and*

$$\|Tf\|_{p,\varphi_2,w} \le C[w]_{A_p}^{\frac{1}{p}} \|f\|_{p,\varphi_1,w} \tag{3.5}$$

with a constant independent of f*.*

For any $a \in BMO$ consider the commutator $T_a f = aTf - T(af)$ such that for any $f \in L_1(\mathbb{R}^n)$ with a compact support and $x \notin \operatorname{supp} f$ it holds

$$|T_a f(x)| \le C \int_{\mathbb{R}^n} |a(x) - a(y)|\, \frac{|f(y)|}{|x-y|^n}\, dy \tag{3.6}$$

with a constant independent of f, a, and x. Suppose in addition that T_a is bounded in $L_{p,w}$ satisfies the estimate $\|T_a f\|_{p,w} \le C\|a\|_* [w]_{A_p}^{1/p} \|f\|_{p,w}$. Then the following result holds as in [15] by the use of Theorem 3.1.

Theorem 3.3. *Let $p \in (1,\infty)$, $w \in A_p$, $a \in BMO$ and the pair (φ_1, φ_2) satisfy*

$$\int_r^\infty \left(1+\ln\frac{t}{r}\right) \frac{\operatorname*{ess\,inf}_{t<s<\infty} \varphi_1(x,s) w(\mathcal{B}_s(x))^{\frac{1}{p}}}{w(\mathcal{B}_t(x))^{\frac{1}{p}}} \frac{dt}{t} \le C\, \varphi_2(x,r) \tag{3.7}$$

with a constant independent on x and r. Suppose that T_a is bounded in $L_{p,w}$ and satisfies (3.6). *Then T_a is bounded from $M_{p,\varphi_1}(w)$ to $M_{p,\varphi_2}(w)$ and*

$$\|T_a f\|_{p,\varphi_2,w} \le C[w]_{A_p}^{\frac{1}{p}} \|a\|_* \|f\|_{p,\varphi_1,w}\,. \tag{3.8}$$

4. Sub-linear operators generated by nonsingular integrals in $M_{p,\varphi}(w)$

Let $\mathbb{R}^n_+ = \{x = (x_1,\dots,x_n) \in \mathbb{R}^n : x_n > 0\}$. For any $x \in \mathbb{R}^n_+$ define $\tilde{x} = (x_1,\dots,x_{n-1},-x_n)$. Let $\widetilde{T}$ be a sub-linear operator with a nonsingular kernel such that for any $f \in L_1(\mathbb{R}^n_+)$ with a compact support. Suppose that the following inequality holds

$$|\widetilde{T}f(x)| \le C \int_{\mathbb{R}^n_+} \frac{|f(y)|}{|\tilde{x}-y|^n}\, dy, \tag{4.1}$$

where C is independent of f.

Lemma 4.1. *Let $w \in A_p$, $p \in (1,\infty)$, the operator $\widetilde{T}$ satisfy* (4.1) *and be bounded on $L_{p,w}(\mathbb{R}^n_+)$ with $\|\widetilde{T}f\|_{p,w} \le C[w]_{A_p}^{1/p}\|f\|_{p,w}$. Let for any fixed $x_0 \in \mathbb{R}^n_+$ and for any $f \in L^{\mathrm{loc}}_{p,w}(\mathbb{R}^n_+)$*

$$\int_r^\infty w(\mathcal{B}^+_t(x_0))^{-1/p} \|f\|_{p,w;\mathcal{B}^+_t(x_0)} \frac{dt}{t} < \infty\,. \tag{4.2}$$

Then

$$\|\widetilde{T}f\|_{p,w;\mathcal{B}^+_r(x_0)} \le C[w]_{A_p}^{\frac{1}{p}}\, w(\mathcal{B}^+_r(x_0))^{\frac{1}{p}} \int_{2r}^\infty w(\mathcal{B}^+_t(x_0))^{-\frac{1}{p}} \|f\|_{p,w;\mathcal{B}^+_t(x_0)} \frac{dt}{t} \tag{4.3}$$

with a constant independent of x_0, r, and f.

Proof. Consider the decomposition $f = f_1 + f_2$ with $f_1 = f\chi_{2\mathcal{B}^+_r(x_0)}$ and $f_2 = f\chi_{(2\mathcal{B}^+_r(x_0))^c}$. Because of the boundedness of $\widetilde{T}$ in $L_{p,w}(\mathbb{R}^n_+)$ we have as in [17]

$$\|\widetilde{T}f_1\|_{p,w;\mathcal{B}^+_r(x_0)} \le C[w]_{A_p}^{\frac{1}{p}} \|f\|_{p,w;2\mathcal{B}^+_r(x_0)}\,.$$

Since for any $\tilde{x} \in \mathcal{B}^+_r(x_0)$ and $y \in (2\mathcal{B}^+_r(x_0))^c$ it holds

$$\frac{1}{2}|x_0-y| \le |\tilde{x}-y| \le \frac{3}{2}|x_0-y| \tag{4.4}$$

we get as in [17]

$$|\widetilde{T}f_2(x)| \le C\int_{2r}^\infty \left(\int_{\mathcal{B}^+_t(x_0)} |f(y)|dy\right) \frac{dt}{t^{n+1}}.$$

Using the Hölder inequality and (2.3) we get

$$
\begin{aligned}
|\widetilde{T} f_2(x)| &\le C \int_{2r}^{\infty} \|f\|_{p,w;\mathcal{B}_t^+(x_0)} \, \|w^{-1/p}\|_{p';\mathcal{B}_t^+(x_0)} \, \frac{dt}{t^{n+1}} \\
&\le C[w]_{A_p}^{1/p} \int_{2r}^{\infty} w(\mathcal{B}_t^+(x_0))^{-1/p} \|f\|_{p,w;\mathcal{B}_t^+(x_0)} \, \frac{dt}{t}.
\end{aligned} \tag{4.5}
$$

Direct calculations give

$$
\|\widetilde{T} f_2\|_{p,w;\mathcal{B}_r^+(x_0)} \le C[w]_{A_p}^{1/p} \, w(\mathcal{B}_r^+(x_0))^{1/p} \int_{2r}^{\infty} \frac{\|f\|_{p,w;\mathcal{B}_t^+(x_0)}}{w(\mathcal{B}_t^+(x_0))^{1/p}} \, \frac{dt}{t} \tag{4.6}
$$

for all $f \in L_{p,w}(\mathbb{R}_+^n)$ satisfying (4.2). Thus,

$$
\begin{aligned}
\|\widetilde{T} f\|_{p,w;\mathcal{B}_r^+(x_0)} &\le \|\widetilde{T} f_1\|_{p,w;\mathcal{B}_r^+(x_0)} + \|\widetilde{T} f_2\|_{p,w;\mathcal{B}_r^+(x_0)} \\
&\le C[w]_{A_p}^{1/p} \|f\|_{p,w;2\mathcal{B}_r^+(x_0)} \\
&+ C[w]_{A_p}^{1/p} w(\mathcal{B}_r^+(x_0))^{1/p} \int_{2r}^{\infty} \frac{\|f\|_{p,w;\mathcal{B}_t^+(x_0)}}{w(\mathcal{B}_t^+(x_0))^{1/p}} \, \frac{dt}{t}.
\end{aligned} \tag{4.7}
$$

On the other hand, by (2.3)

$$
\begin{aligned}
\|f\|_{p,w;2\mathcal{B}_r^+(x_0)} &\le C|\mathcal{B}_r^+(x_0)| \|f\|_{p,w;2\mathcal{B}_r^+(x_0)} \int_{2r}^{\infty} \frac{dt}{t^{n+1}} \\
&\le C|\mathcal{B}_r^+(x_0)| \int_{2r}^{\infty} \|f\|_{p,w;\mathcal{B}_t^+(x_0)} \, \frac{dt}{t^{n+1}} \\
&\le C[w]_{A_p}^{-\frac{1}{p}} w(\mathcal{B}_r^+(x_0))^{1/p} \int_{2r}^{\infty} \|f\|_{p,w;\mathcal{B}_t^+(x_0)} \, \|w^{-1/p}\|_{p';\mathcal{B}_t^+(x_0)} \, \frac{dt}{t^{n+1}} \\
&\le C[w]_{A_p}^{-\frac{1}{p}} w(\mathcal{B}_r^+(x_0))^{1/p} \int_{2r}^{\infty} [w]_{A_p}^{\frac{1}{p}} w(\mathcal{B}_t^+(x_0))^{-1/p} \|f\|_{p,w;\mathcal{B}_t^+(x_0)} \, \frac{dt}{t} \\
&\le w(\mathcal{B}_r^+(x_0))^{1/p} \int_{2r}^{\infty} w(\mathcal{B}_t^+(x_0))^{-1/p} \|f\|_{p,w;\mathcal{B}_t^+(x_0)} \, \frac{dt}{t}
\end{aligned} \tag{4.8}
$$

which unified with (4.7) gives (4.3).

Theorem 4.1. *Let $w \in A_p$, $p \in (1, \infty)$, the pair (φ_1, φ_2) satisfy (3.4) and $\widetilde{T}$ be bounded in $L_{p,w}(\mathbb{R}_+^n)$. Then it is bounded from $M_{p,\varphi_1}(\mathbb{R}_+^n, w)$ in $M_{p,\varphi_2}(\mathbb{R}_+^n, w)$ and*

$$
\|\widetilde{T} f\|_{p,\varphi_2,w;\mathbb{R}_+^n} \le C[w]_{A_p}^{\frac{1}{p}} \|f\|_{p,\varphi_1,w;\mathbb{R}_+^n} \tag{4.9}
$$

with a constant independent of f.

Proof. By Lemma 4.1 we have

$$
\|\widetilde{T} f\|_{p,\varphi_2,w;\mathbb{R}_+^n} \le C[w]_{A_p}^{\frac{1}{p}} \sup_{x \in \mathbb{R}_+^n, \, r>0} \varphi_2(x,r)^{-1} \int_r^{\infty} w(\mathcal{B}_t^+(x))^{-1/p} \|f\|_{p,w;\mathcal{B}_t^+(x)} \, \frac{dt}{t}.
$$

Applying the Theorem 3.1 with

$$
v_1(r) = \varphi_1(x,r)^{-1} w(\mathcal{B}_r^+(x))^{-1/p}, \qquad v_2(r) = \varphi_2(x,r)^{-1},
$$

$$
\psi(r) = w(\mathcal{B}_r^+(x))^{-1/p} \, r^{-1}, \qquad g(r) = \|f\|_{p,w;\mathcal{B}_r^+(x)}
$$

to the above integral, we get as in [17]

$$\|\widetilde{T}f\|_{p,\varphi_2,w;\mathbb{R}^n_+} \le C[w]^{\frac{1}{p}}_{A_p} \sup_{x\in\mathbb{R}^n_+, r>0} \varphi_1(x,r)^{-1}\, w(\mathcal{B}^+_r(x))^{-1/p}\, \|f\|_{p,w;\mathcal{B}^+_r(x)}$$
$$= C[w]^{\frac{1}{p}}_{A_p} \|f\|_{p,\varphi_1,w;\mathbb{R}^n_+}.$$

5. Commutators of sub-linear operators generated by nonsingular integrals in $M_{p,\varphi}(w)$

For any $a \in BMO$ consider the commutator $\widetilde{T}_a f = a\widetilde{T}f - \widetilde{T}(af)$ where $\widetilde{T}$ is the nonsingular operator satisfying (4.1) and $f \in L_1(\mathbb{R}^n_+)$ with a compact support. Suppose that for $x \notin supp f$

$$|\widetilde{T}_a f(x)| \le C \int_{\mathbb{R}^n_+} |a(x) - a(y)|\, \frac{|f(y)|}{|\tilde{x} - y|^n}\, dy, \tag{5.1}$$

where C is independent of $f, a,$ and x. To estimate the commutator we shall employ the same idea which we used in the proof of Lemma 4.1 (see [17] for details) and the properties of the Muckenhoupt weight.

Lemma 5.1. *Let $w \in A_p$, $p \in (1,\infty)$, $a \in BMO$ and $\widetilde{T}_a$ be a bounded operator such that $\|\widetilde{T}_a f\|_{p,w;\mathbb{R}^n_+} \le C[w]^{1/p}_{A_p}\|a\|_*\, \|f\|_{p,w;\mathbb{R}^n_+}$. Suppose that for all $f \in L^{\rm loc}_{p,w}(\mathbb{R}^n_+)$, $x_0 \in \mathbb{R}^n_+$ and $r > 0$ applies the next condition*

$$\int_r^\infty \Big(1 + \ln\frac{t}{r}\Big) \frac{\|f\|_{p,w;\mathcal{B}^+_t(x_0)}}{w(\mathcal{B}^+_t(x_0))^{1/p}}\, \frac{dt}{t} < \infty\,. \tag{5.2}$$

Then

$$\|\widetilde{T}_a f\|_{p,w;\mathcal{B}^+_r(x_0)} \le C[w]^{\frac{1}{p}}_{A_p}\|a\|_*\, w(\mathcal{B}^+_r(x_0))^{1/p} \int_{2r}^\infty \Big(1 + \ln\frac{t}{r}\Big) \frac{\|f\|_{p,w;\mathcal{B}^+_t(x_0)}}{w(\mathcal{B}^+_t(x_0))^{1/p}}\, \frac{dt}{t}\,. \tag{5.3}$$

Proof. The decomposition $f = f\chi_{2\mathcal{B}^+_r(x_0)} + f\chi_{(2\mathcal{B}^+_r(x_0))^c} = f_1 + f_2$ gives

$$\|\widetilde{T}_a f\|_{p,w;\mathcal{B}^+_r(x_0)} \le \|\widetilde{T}_a f_1\|_{p,w;\mathcal{B}^+_r(x_0)} + \|\widetilde{T}_a f_2\|_{p,w;\mathcal{B}^+_r(x_0)}.$$

From the boundedness of $\widetilde{T}_a$ in $L_{p,w}(\mathbb{R}^n_+)$ it follows

$$\|\widetilde{T}_a f_1\|_{p,w;\mathcal{B}^+_r(x_0)} \le C[w]^{1/p}_{A_p}\, \|a\|_*\, \|f\|_{p,w;2\mathcal{B}^+_r(x_0)}.$$

On the other hand, because of (4.4) we can write

$$\begin{aligned}
&\|\widetilde{T}_a f_2\|_{p,w;\mathcal{B}^+_r(x_0)} \\
&\le C\left(\int_{\mathcal{B}^+_r(x_0)}\left(\int_{(2\mathcal{B}^+_r(x_0))^c} \frac{|a(y) - a_{\mathcal{B}^+_r(x_0)}||f(y)|}{|x_0 - y|^n}\, dy\right)^p w(x)\, dx\right)^{1/p} \\
&+ C\left(\int_{\mathcal{B}^+_r(x_0)}\left(\int_{(2\mathcal{B}^+_r(x_0))^c} \frac{|a(x) - a_{\mathcal{B}^+_r(x_0)}||f(y)|}{|x_0 - y|^n}\, dy\right)^p w(x)\, dx\right)^{1/p} \\
&= I_1 + I_2.
\end{aligned}$$

Where, as in [17], we have

$$I_1 \le C w(\mathcal{B}_r^+(x_0))^{\frac{1}{p}} \int_{2r}^{\infty} \int_{\mathcal{B}_t^+(x_0)} |a(y) - a_{\mathcal{B}_r^+(x_0)}||f(y)|\, dy \,\frac{dt}{t^{n+1}}\,.$$

Applying Hölder's inequality, Lemma 2.1, (2.1) and (2.5), we get

$$\begin{aligned}
I_1 &\le C w(\mathcal{B}_r^+(x_0))^{\frac{1}{p}} \int_{2r}^{\infty} \int_{\mathcal{B}_t^+(x_0)} |a(y) - a_{\mathcal{B}_t^+(x_0)}||f(y)|\, dy \,\frac{dt}{t^{n+1}} \\
&+ C w(\mathcal{B}_r^+(x_0))^{\frac{1}{p}} \int_{2r}^{\infty} \int_{\mathcal{B}_t^+(x_0)} |a_{\mathcal{B}_t^+(x_0)} - a_{\mathcal{B}_r^+(x_0)}||f(y)|\, dy \,\frac{dt}{t^{n+1}} \\
&\le C\, w(\mathcal{B}_r^+(x_0))^{\frac{1}{p}} \int_{2r}^{\infty} \left(\int_{\mathcal{B}_t^+(x_0)} |a(y) - a_{\mathcal{B}_t^+(x_0)}|^{p'} w(y)^{1-p'}\, dy \right)^{\frac{1}{p'}} \\
&\qquad\qquad\qquad \times \|f\|_{p,w;\mathcal{B}_t^+(x_0)} \frac{dt}{t^{n+1}} \\
&+ C[w]_{A_p}^{\frac{1}{p}} w(\mathcal{B}_r^+(x_0))^{\frac{1}{p}} \|a\|_* \int_{2r}^{\infty} \ln\frac{t}{r} \|f\|_{p,w;\mathcal{B}_t(x_0)} w(\mathcal{B}_t(x_0))^{-\frac{1}{p}} \frac{dt}{t} \\
&\le C[w]_{A_p}^{\frac{1}{p}} w(\mathcal{B}_r^+(x_0))^{\frac{1}{p}} \|a\|_* \int_{2r}^{\infty} \|f\|_{p,w;\mathcal{B}_t^+(x_0)} w(\mathcal{B}_t^+(x_0))^{-\frac{1}{p}} \frac{dt}{t^{n+1}} \\
&+ C[w]_{A_p}^{\frac{1}{p}} w(\mathcal{B}_r^+(x_0))^{\frac{1}{p}} \|a\|_* \int_{2r}^{\infty} \ln\frac{t}{r} \|f\|_{p,w;\mathcal{B}_t^+(x_0)} w(\mathcal{B}_t^+(x_0))^{-\frac{1}{p}} \frac{dt}{t} \\
&\le C[w]_{A_p}^{\frac{1}{p}} w(\mathcal{B}_r^+(x_0))^{\frac{1}{p}} \|a\|_* \int_{2r}^{\infty} \left(1 + \ln\frac{t}{r}\right) \|f\|_{p,w;\mathcal{B}_t^+(x_0)} w(\mathcal{B}_t^+(x_0))^{-\frac{1}{p}} \frac{dt}{t}\,.
\end{aligned}$$

By (2.1) and (4.5) we get

$$I_2 \le C[w]_{A_p}^{\frac{1}{p}} \|a\|_*\, w(\mathcal{B}_r^+(x_0))^{\frac{1}{p}} \int_{2r}^{\infty} w(\mathcal{B}_t^+(x_0))^{-\frac{1}{p}} \|f\|_{p,w;\mathcal{B}_t^+(x_0)} \frac{dt}{t}\,.$$

Summing up I_1 and I_2 we get that

$$\|\widetilde{T}_a f_2\|_{p,w;\mathcal{B}_r^+(x_0)} \le C[w]_{A_p}^{\frac{1}{p}} \|a\|_*\, w(\mathcal{B}_r^+(x_0))^{\frac{1}{p}} \int_{2r}^{\infty} \left(1 + \ln\frac{t}{r}\right) \frac{\|f\|_{p,w;\mathcal{B}_t^+(x_0)}}{w(\mathcal{B}_t^+(x_0))^{\frac{1}{p}}} \frac{dt}{t}. \tag{5.4}$$

Finally,

$$\begin{aligned}
\|\widetilde{T}_a f\|_{p,w;\mathcal{B}_r^+(x_0)} &\le C[w]_{A_p}^{\frac{1}{p}} \|a\|_* \Big(\|f\|_{p,w;2\mathcal{B}_r^+(x_0)} \\
&\qquad + w(\mathcal{B}_r^+(x_0))^{\frac{1}{p}} \int_{2r}^{\infty} \left(1 + \ln\frac{t}{r}\right) \frac{\|f\|_{p,w;\mathcal{B}_t^+(x_0)}}{w(\mathcal{B}_t^+(x_0))^{\frac{1}{p}}} \frac{dt}{t} \Big),
\end{aligned}$$

and the statement follows by (4.8).

Theorem 5.1. *Let $w \in A_p$, $p \in (1,\infty)$, $a \in BMO$ and the pair (φ_1, φ_2) satisfy*

$$\int_r^{\infty} \left(1 + \ln\frac{t}{r}\right) \frac{\operatorname*{ess\,inf}_{t<s<\infty} \varphi_1(x,s) w(\mathcal{B}_s^+(x))^{\frac{1}{p}}}{w(\mathcal{B}_t^+(x))^{\frac{1}{p}}} \frac{dt}{t} \le C\, \varphi_2(x,r)\,. \tag{5.5}$$

Suppose $\widetilde{T}_a$ *is a sub-linear operator satisfying* (5.1) *and bounded on* $L_{p,w}(\mathbb{R}^n_+)$. *Then* $\widetilde{T}_a$ *is bounded from* $M_{p,\varphi_1}(\mathbb{R}^n_+, w)$ *to* $M_{p,\varphi_2}(\mathbb{R}^n_+, w)$ *and*

$$\|\widetilde{T}_a f\|_{p,\varphi_2,w;\mathbb{R}^n_+} \le C[w]_{A_p}^{1/p}\|a\|_* \, \|f\|_{p,\varphi_1,w;\mathbb{R}^n_+} \tag{5.6}$$

with a constant independent of f *and* a.

The statement of the theorem follows from Lemma 5.1 and Theorem 3.1 in the same manner as in the proof of Theorem 4.1.

6. Calderón-Zygmund operators in $M_{p,\varphi}(w)$

In the present section we deal with Calderón-Zygmund type integrals and their commutators with BMO functions. We start with the definition of the corresponding kernel.

Definition 6.1. A measurable function $\mathcal{K}(x,\xi) : \mathbb{R}^n \times \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$ is called a variable Calderón-Zygmund kernel if:

i) $\mathcal{K}(x,\cdot)$ is a Calderón-Zygmund kernel for almost all $x \in \mathbb{R}^n$:
   i_a) $\mathcal{K}(x,\cdot) \in C^\infty(\mathbb{R}^n \setminus \{0\})$,
   i_b) $\mathcal{K}(x,\mu\xi) = \mu^{-n}\mathcal{K}(x,\xi) \quad \forall \mu > 0$,
   i_c) $\int_{\mathbb{S}^{n-1}} \mathcal{K}(x,\xi)d\sigma_\xi = 0 \quad \int_{\mathbb{S}^{n-1}} |\mathcal{K}(x,\xi)|d\sigma_\xi < +\infty$,

ii) $\max\limits_{|\beta|\le 2n} \left\| D_\xi^\beta \mathcal{K} \right\|_{\infty;\mathbb{R}^n\times\mathbb{S}^{n-1}} = M < \infty.$

The singular integrals

$$\begin{aligned}\mathfrak{K}f(x) :=& P.V. \int_{\mathbb{R}^n} \mathcal{K}(x, x-y) f(y)\, dy,\\ \mathfrak{C}[a,f](x) :=& P.V. \int_{\mathbb{R}^n} \mathcal{K}(x,x-y)[a(x)-a(y)]f(y)\, dy\\ =& a\mathfrak{K}f(x) - \mathfrak{K}(af)(x)\end{aligned}$$

are bounded in $L_{p,w}$ (see [16] for more references) and satisfy (3.1) and (5.1). Hence the following results hold as a simple application of the estimates from § 3 and § 4 (see [17] for details).

Theorem 6.1. *Let* $w \in A_p$, $p \in (1,\infty)$ *and* φ *be weight such that for all* $x \in \mathbb{R}^n$ *and* $r > 0$

$$\int_r^\infty \left(1 + \ln\frac{t}{r}\right) \frac{\operatorname{ess\,inf}\limits_{t<s<\infty} \varphi(x,s) w(\mathcal{B}_s(x))^{\frac{1}{p}}}{w(\mathcal{B}_t(x))^{\frac{1}{p}}} \frac{dt}{t} \le C\,\varphi(x,r). \tag{6.1}$$

Then for any $f \in M_{p,\varphi}(\mathbb{R}^n, w)$ *and* $a \in BMO$ *there exist constants depending on* n, p, φ, w, *and the kernel such that*

$$\|\mathfrak{K}f\|_{p,\varphi,w} \le C[w]_{A_p}^{\frac{1}{p}}\|f\|_{p,\varphi,w}\,, \qquad \|\mathfrak{C}[a,f]\|_{p,\varphi,w} \le C[w]_{A_p}^{\frac{1}{p}}\|a\|_*\|f\|_{p,\varphi,w}\,. \tag{6.2}$$

The assertion follows by (4.9) and (5.6).

Corollary 6.1. *Let $\Omega \subset \mathbb{R}^n$, $\partial\Omega \in C^{1,1}$, $\mathcal{K} : \Omega \times \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$ be as in Definition 6.1, $a \in BMO(\Omega)$ and $f \in M_{p,\varphi}(\Omega, w)$ with p, φ, and w as in Theorem 6.1. Then*

$$\|\mathfrak{K}f\|_{p,\varphi,w;\Omega} \le C[w]_{A_p}^{\frac{1}{p}} \|f\|_{p,\varphi,w;\Omega}\,, \quad \|\mathfrak{C}[a, f]\|_{p,\varphi,w;\Omega} \le C[w]_{A_p}^{\frac{1}{p}} \|a\|_* \|f\|_{p,\varphi,w;\Omega} \tag{6.3}$$

with $C = C(n, p, \varphi, [w]_{A_p}, |\Omega|, \mathcal{K})$.

Corollary 6.2. *(see* [4, 17]*) Let p, φ, and w be as in Theorem 6.1 and $a \in VMO$ with a VMO-modulus γ_a. Then for any $\varepsilon > 0$ there exists a positive number $\rho_0 = \rho_0(\varepsilon, \gamma_a)$ such that for any ball $\mathcal{B}_r$ with a radius $r \in (0, \rho_0)$ and all $f \in M_{p,\varphi}(\mathcal{B}_r, w)$*

$$\|\mathfrak{C}[a, f]\|_{p,\varphi,w;\mathcal{B}_r} \le C\varepsilon \|f\|_{p,\varphi,w;\mathcal{B}_r} \tag{6.4}$$

with C independent of ε, f, and r.

For any $x, y \in \mathbb{R}^n_+$ define the *generalized reflection* $\mathcal{T}(x; y)$

$$\mathcal{T}(x; y) = x - 2x_n \frac{\mathbf{a}^n(y)}{a^{nn}(y)} \qquad \mathcal{T}(x) = \mathcal{T}(x; x) : \mathbb{R}^n_+ \to \mathbb{R}^n_-\,, \tag{6.5}$$

where $\mathbf{a}^n$ is the last row of the matrix $\mathbf{a} = \{a^{ij}\}_{i,j=1}^n$ and $\mathbb{R}^n_- = \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n : x_n < 0\}$. Then there exist positive constants C_1, C_2 depending on n and Λ, such that

$$C_1|\widetilde{x} - y| \le |\mathcal{T}(x) - y| \le C_2|\widetilde{x} - y| \qquad \forall\, x, y \in \mathbb{R}^n_+. \tag{6.6}$$

Then the nonsingular integrals

$$\widetilde{\mathfrak{K}}f(x) := \int_{\mathbb{R}^n_+} \mathcal{K}(x, \mathcal{T}(x) - y) f(y)\, dy \tag{6.7}$$

$$\widetilde{\mathfrak{C}}[a, f](x) := \int_{\mathbb{R}^n_+} \mathcal{K}(x, \mathcal{T}(x) - y)[a(x) - a(y)] f(y)\, dy$$

are sub-linear and according to the results in Sections 4 and 5 we have.

Theorem 6.2. *Let $a \in BMO(\mathbb{R}^n_+)$, $w \in A_p$, $p \in (1, \infty)$ and φ be a measurable function satisfying* (6.1)*. Then $\widetilde{\mathfrak{K}}f$ and $\widetilde{\mathfrak{C}}[a, f]$ are continuous in $M_{p,\varphi}(\mathbb{R}^n_+, w)$ and for all $f \in M_{p,\varphi}(\mathbb{R}^n_+, w)$ holds*

$$\|\widetilde{\mathfrak{K}}f\|_{p,\varphi,w;\mathbb{R}^n_+} \le C[w]_{A_p}^{\frac{1}{p}} \|f\|_{p,\varphi,w;\mathbb{R}^n_+} \quad \|\widetilde{\mathfrak{C}}[a, f]\|_{p,\varphi,w;\mathbb{R}^n_+} \le C[w]_{A_p}^{\frac{1}{p}} \|a\|_* \, \|f\|_{p,\varphi,w;\mathbb{R}^n_+} \tag{6.8}$$

with a constant dependent on known quantities only.

Corollary 6.3. *(see* [4, 17]*) Let p, φ and w be as in Theorem 6.2 and $a \in VMO$ with a VMO-modulus γ_a. Then for any $\varepsilon > 0$ there exists a positive number $\rho_0 = \rho_0(\varepsilon, \gamma_a)$ such that for any ball $\mathcal{B}^+_r$ with a radius $r \in (0, \rho_0)$ and all $f \in M_{p,\varphi}(\mathcal{B}^+_r, w)$*

$$\|\mathfrak{C}[a, f]\|_{p,\varphi,w;\mathcal{B}^+_r} \le C\varepsilon \|f\|_{p,\varphi,w;\mathcal{B}^+_r}, \tag{6.9}$$

where C is independent of ε, f and r.

7. The Dirichlet problem

Let $\Omega \subset \mathbb{R}^n$, $n \geq 3$ be a bounded $C^{1,1}$-domain. We consider the problem

$$\begin{cases} Lu = a^{ij}(x)D_{ij}u + b^i(x)D_iu + c(x)u = f(x) & \text{a.a. } x \in \Omega, \\ u \in \ W^2_{p,\varphi}(\Omega, w) \cap \overset{\circ}{W}{}^1_p(\Omega, w), \ p \in (1, \infty) \end{cases} \tag{7.1}$$

subject to the following conditions:

H_1) *Strong ellipticity:* there exists a constant $\Lambda > 0$, such that

$$\begin{cases} \Lambda^{-1}|\xi|^2 \leq a^{ij}(x)\xi_i\xi_j \leq \Lambda|\xi|^2 & \text{a.a. } x \in \Omega, \ \forall\, \xi \in \mathbb{R}^n \\ a^{ij}(x) = a^{ji}(x) & 1 \leq i, j \leq n. \end{cases} \tag{7.2}$$

Let $\mathbf{a} = \{a^{ij}\}$, then $\mathbf{a} \in L_\infty(\Omega)$ and $\|\mathbf{a}\| = \sum_{ij=1}^n \|a^{ij}\|_{\infty;\Omega}$ by (7.2).

H_2) *Regularity of the data:* $\mathbf{a} \in VMO(\Omega)$ with VMO-modulus $\gamma_{\mathbf{a}} := \sum \gamma_{a^{ij}}$, $b^i, c \in L_\infty(\Omega)$, and $f \in M_{p,\varphi}(\Omega, w)$ with $w \in A_p$, $1 < p < \infty$ and $\varphi : \Omega \times \mathbb{R}_+ \to \mathbb{R}_+$ measurable.

Let $\mathcal{L} = a^{ij}(x)D_{ij}$, then $\mathcal{L}u = f(x) - b^i(x)D_iu(x) - c(x)u$. As it is well known (see [4, 17] and the references therein) for any $x \in \operatorname{supp} u$, a ball $\mathcal{B}_r \subset \Omega'$ and a function $v \in C_0^\infty(\mathcal{B}_r)$ we have the representation

$$\begin{aligned} D_{ij}v(x) &= P.V. \int_{\mathcal{B}_r} \Gamma_{ij}(x, x - y) \Big[\mathcal{L}v(y) + \big(a^{hk}(x) - a^{hk}(y)\big)D_{hk}v(y)\Big]\, dy \\ &\qquad + \mathcal{L}v(x) \int_{\mathbb{S}^{n-1}} \Gamma_j(x, y)y_i d\sigma_y \\ &= \mathfrak{K}_{ij}\mathcal{L}v(x) + \mathfrak{C}_{ij}[a^{hk}, D_{hk}v](x) + \mathcal{L}v(x) \int_{\mathbb{S}^{n-1}} \Gamma_j(x; y)y_i d\sigma_y. \end{aligned} \tag{7.3}$$

According to Remark 2.1 the formula (7.3) holds for $v \in W^2_{p,w}(\mathcal{B}_r)$. Here $\Gamma_{ij}(x, \xi) = D_{\xi_1}D_{\xi_j}\Gamma(x, \xi)$ and Γ_{ij} are variable Calderón-Zygmund kernels as in Definition 6.1. Then the operators $\mathfrak{K}_{ij}$ and $\mathfrak{C}_{ij}$ are singular integrals as $\mathfrak{K}$ and $\mathfrak{C}$. In view of the results obtained in Section 6 we get for r small enough

$$\|D^2v\|_{p,\varphi,w;\mathcal{B}_r} \leq C\left(\varepsilon\|D^2v\|_{p,\varphi,w;\mathcal{B}_r} + \|\mathcal{L}v\|_{p,\varphi,w;\mathcal{B}_r}\right) .$$

Taking r such that $C\varepsilon < 1$ we can move the norm of D^2v on the left-hand side and write

$$\|D^2v\|_{p,\varphi,w;\mathcal{B}_r} \leq C\|\mathcal{L}v\|_{p,\varphi,w;\mathcal{B}_r} . \tag{7.4}$$

Take a cut-off function $\eta(x) \in C_0^\infty(\mathcal{B}_r)$

$$\eta(x) = \begin{cases} 1 & x \in \mathcal{B}_{\theta r} \\ 0 & x \notin \mathcal{B}_{\theta' r} \end{cases}$$

such that $\theta' = \theta(3 - \theta)/2 > \theta$ for $\theta \in (0, 1)$ and $|D^s\eta| \leq C[\theta(1 - \theta)r]^{-s}$ for $s = 0, 1, 2$. Apply (7.4) to $v(x) = \eta(x)u(x) \in W^2_{p,w}(\mathcal{B}_r)$ we get

$$\begin{aligned} \|D^2u\|_{p,\varphi,w;\mathcal{B}_{\theta r}} &\leq \|D^2v\|_{p,\varphi,w;\mathcal{B}_{\theta' r}} \leq C\|\mathcal{L}v\|_{p,\varphi,w;\mathcal{B}_{\theta' r}} \\ &\leq C\left(\|\mathcal{L}u\|_{p,\varphi,w;\mathcal{B}_{\theta' r}} + \frac{\|Du\|_{p,\varphi,w;\mathcal{B}_{\theta' r}}}{\theta(1 - \theta)r} + \frac{\|u\|_{p,\varphi,w;\mathcal{B}_{\theta' r}}}{[\theta(1 - \theta)r]^2}\right) . \end{aligned}$$

Since $1 < \frac{1}{\theta(1-\theta)r}$ for $r < 4$ and

$$\|\mathcal{L}u\|_{p,\varphi,w;\mathcal{B}_{\theta' r}} \le C\big(\|Lu\|_{p,\varphi,w;\mathcal{B}_{\theta' r}} + \|Du\|_{p,\varphi;w,\mathcal{B}_{\theta' r}} + \|u\|_{p,\varphi;w,\mathcal{B}_{\theta' r}}\big) \tag{7.5}$$

we can write

$$\|D^2u\|_{p,\varphi,w;\mathcal{B}_{\theta r}} \le C\left(\|Lu\|_{p,\varphi,w;\mathcal{B}_{\theta' r}} + \frac{\|Du\|_{p,\varphi,w;\mathcal{B}_{\theta' r}}}{\theta(1-\theta)r} + \frac{\|u\|_{p,\varphi,w;\mathcal{B}_{\theta' r}}}{[\theta(1-\theta)r]^2}\right).$$

Now consider the weighted semi-norm

$$\Theta_s = \sup_{0<\theta<1} \big[\theta(1-\theta)r\big]^s \|D^s u\|_{p,\varphi,w;\mathcal{B}_{\theta r}} \qquad s = 0,1,2.$$

Because of the choice of θ' we have $\theta(1-\theta) \le 2\theta'(1-\theta')$. Thus, after standard transformations and taking the supremum with respect to $\theta \in (0,1)$ we get

$$\Theta_2 \le C\left(r^2\|Lu\|_{p,\varphi,w;\mathcal{B}_{\theta' r}} + \Theta_1 + \Theta_0\right). \tag{7.6}$$

Lemma 7.1 (Interpolation inequality)**.** *There exists a constant C independent of r such that*

$$\Theta_1 \le \varepsilon\Theta_2 + \frac{C}{\varepsilon}\Theta_0 \qquad \textit{for any } \varepsilon \in (0,2).$$

Proof. For functions $u \in W^2_{p,w}(\mathcal{B}_r)$, $p \in (1,\infty)$ and $w \in A_p$ we dispose with the following interpolation inequality proved in [20]

$$\|Du\|_{p,w;\mathcal{B}_r} \le C\left(\|u\|_{p,w;\mathcal{B}_r} + \|u\|^{1/2}_{p,w;\mathcal{B}_r}\|D^2u\|^{1/2}_{p,w;\mathcal{B}_r}\right).$$

Then for any $\epsilon > 0$ we have

$$\|Du\|_{p,w;\mathcal{B}_r} \le C\left(\left(1 + \frac{1}{2\epsilon}\right)\|u\|_{p,w;\mathcal{B}_r} + \frac{\epsilon}{2}\|D^2u\|_{p,w;\mathcal{B}_r}\right).$$

Choosing ϵ small enough, taking $\delta = \frac{C\epsilon}{2}$, dividing all terms of $\varphi(x,r)w(\mathcal{B}_r)^{1/p}$ and taking the supremum over $\mathcal{B}_r$ we get the desired interpolation inequality in $M_{p,\varphi}(w)$

$$\|Du\|_{p,\varphi,w;\mathcal{B}_r} \le \delta\|D^2u\|_{p,\varphi,w;\mathcal{B}_r} + \frac{C}{\delta}\|u\|_{p,\varphi,w;\mathcal{B}_r}. \tag{7.7}$$

We can always find some $\theta_0 \in (0,1)$ such that

$$\begin{aligned}\Theta_1 &\le 2[\theta_0(1-\theta_0)r]\|Du\|_{p,\varphi,w;\mathcal{B}_{\theta_0 r}}\\ &\le 2[\theta_0(1-\theta_0)r]\left(\delta\|D^2u\|_{p,\varphi,w;\mathcal{B}_{\theta_0 r}} + \frac{C}{\delta}\|u\|_{p,\varphi,w;\mathcal{B}_{\theta_0 r}}\right).\end{aligned}$$

The assertion follows choosing $\delta = \frac{\varepsilon}{2}[\theta_0(1-\theta_0)r] < \theta_0 r$ for any $\varepsilon \in (0,2)$.

Interpolating Θ_1 in (7.6) and taking $\theta = 1/2$ as in [17] we get the Caccioppoli-type estimate

$$\|D^2u\|_{p,\varphi,w;\mathcal{B}_{r/2}} \le C\big(\|Lu\|_{p,\varphi,w;\mathcal{B}_r} + \frac{1}{r^2}\|u\|_{p,\varphi,w;\mathcal{B}_r}\big).$$

Further, proceeding as in [17] and using (7.5) and (7.7) we get the following interior a priori estimate.

Theorem 7.1 (Interior estimate). *Let $u \in W^{2,\mathrm{loc}}_{p,w}(\Omega)$ and L be a linear elliptic operator verifying H_1 and H_2 such that $Lu \in M^{\mathrm{loc}}_{p,\varphi}(\Omega, w)$ with $p \in (1,\infty)$, $w \in A_p$ and φ satisfying* (6.1). *Then $D_{ij}u \in L_{p,\varphi}(\Omega', w)$ for any $\Omega' \subset\subset \Omega'' \subset\subset \Omega$ and*

$$\|D^2u\|_{p,\varphi,w;\Omega'} \leq C\big(\|u\|_{p,\varphi,w;\Omega''} + \|Lu\|_{p,\varphi,w;\Omega''}\big), \tag{7.8}$$

where the constant depends on known quantities and $\operatorname{dist}(\Omega', \partial\Omega'')$.

Let $x^0 = (x', 0)$ and denote by C^γ the space of functions $u \in C_0^\infty(\mathcal{B}_r(x^0))$ with $u = 0$ for $x_n \leq 0$. The space $W^{2,\gamma}_{p,w}(\mathcal{B}_r(x_0))$ is the closure of C^γ with respect to the norm of $W^2_{p,w}$. Then for any $v \in W^{2,\gamma}_{p,w}(\mathcal{B}^+_r(x^0))$ the following representation formula holds (see [5])

$$\begin{aligned} D_{ij}v(x) =&\mathfrak{K}_{ij}\mathcal{L}v(x) + \mathfrak{C}_{ij}[a^{hk}D_{hk}v](x) \\ &+\mathcal{L}v(x)\int_{\mathbb{S}^{n-1}} \Gamma_j(x,y)y_i d\sigma_y + I_{ij}(x) \quad \forall\ i,j = 1,\dots,n, \end{aligned}$$

where we have set

$$\begin{aligned} I_{ij}(x) &= \widetilde{\mathfrak{K}}_{ij}\mathcal{L}v(x) + \widetilde{\mathfrak{C}}_{ij}[a^{hk}, D_{hk}v](x), \qquad \forall\ i,j = 1,\dots,n-1, \\ I_{in}(x) &= I_{ni}(x) = \widetilde{\mathfrak{K}}_{il}(D_n\mathcal{T}(x))^l\mathcal{L}v(x) + \mathfrak{C}_{il}[a^{hk}, D_{hk}v](D_n\mathcal{T}(x))^l \\ &\qquad\qquad \forall\ i = 1,\dots,n-1, \\ I_{nn}(x) &= \widetilde{\mathfrak{K}}_{ls}(D_n\mathcal{T}(x))^l(D_n\mathcal{T}(x))^s\mathcal{L}v(x) \\ &\quad + \widetilde{\mathfrak{C}}_{ls}[a^{hk}, D_{hk}v(x)](D_n\mathcal{T}(x))^l(D_n\mathcal{T}(x))^s, \end{aligned}$$

where

$$D_n\mathcal{T}(x) = \big((D_n\mathcal{T}(x))^1, \dots, (D_n\mathcal{T}(x))^n\big) = \mathcal{T}(e_n, x).$$

Applying the estimates (6.8) and (6.9), taking into account the VMO properties of the coefficients a^{ij}'s, it is possible to choose r_0 small and applying the interpolation inequality (7.7)

$$\|D_{ij}v\|_{p,\varphi;w,\mathcal{B}^+_r} \leq C(\|Lv\|_{p,\varphi;w,\mathcal{B}^+_r} + \|u\|_{p,\varphi;w,\mathcal{B}^+_r}) \tag{7.9}$$

for all $r < r_0$ (see [17] for details). By local flattering of the boundary, covering with semi-balls, taking a partition of unity subordinated to that covering and applying of estimate (7.9) we get a boundary a priori estimate that unified with (7.8) gives the following theorem.

Theorem 7.2 (Main result). *Let $u \in W^2_{p,\varphi}(\Omega, w) \cap \overset{\circ}{W}{}^1_p(\Omega, w)$ be a solution of* (7.1) *under the conditions H_1 and H_2. Then for any $w \in A_p$, $p \in (1,\infty)$ and φ satisfying* (6.1) *the following estimate holds*

$$\|D^2u\|_{p,\varphi,w;\Omega} \leq C\big(\|u\|_{p,\varphi,w;\Omega} + \|f\|_{p,\varphi,w;\Omega}\big)$$

and the constant C depends on known quantities only.

Let us note that the solution of (7.1) exists according to Remark 2.1. The proof follows as in [4, 5] using (7.5) and the interpolation inequality in weighted Lebesgue spaces [20].

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