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LOCAL SOLVABILITY OF THE $\overline{\partial}$-EQUATIONS ON l^2

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Abstract. In this work, we establish sufficient conditions for the existence of solutions of the Cauchy-Riemann equations on the unit ball B of the Hilbert space l^2 for a particular class of $\overline{\partial}$-closed $\mathcal{C}^\infty$ smooth (0,1)-forms ω. The used method is based on the expansion in Fourier series of the indefinitely differentiable functions on the closed unit ball of $\mathbb{C}^N$.

1. Introduction

This paper addresses a fundamental problem that arises in infinite dimensional complex analysis that concerns the solvability of inhomogeneous Cauchy-Riemann, or $\overline{\partial}$-equations for (0,1)-forms on Banach spaces.

Up to now precious little has been known about the solvability of the infinite equation

$$\overline{\partial} f = \omega, \quad (\overline{\partial}\omega = 0) \tag{1.1}$$

when ω is a (0,1)-form, even on Hilbert space. However, we must mention some important results. First, Coeuré in [3] gave an example of (0,1)-form ω on l^2 of class $\mathcal{C}^1$ for which (1.1) is not solvable on any open set. No other example is known with ω of class $\mathcal{C}^p$ $(1 < p \leq \infty)$. Second, L. Lempert gets local exactness on the space l^1 and on any Banach space when the forms are real analytical [1, 2].

In this work, we study the local solvability of $\overline{\partial}$ in the closed unit ball of l^2 denoted $\overline{B}$ for a particular class of $\mathcal{C}^\infty$ smooth (0,1)-forms of the type

$$\omega(z) = \sum_{k=1}^{\infty} \omega_k(z^k), \quad z = (z_i) \text{ in } l^2 \tag{1.2}$$

where $\mathbb{N} = \cup I_k$ is a partition of $\mathbb{N}$, (card.$I_k = N_k < +\infty$) with $z^k = (z_i)_{i\in I_k}$ standing for the projection of z on $\mathbb{C}^{N_k}$. We assume the following assumptions (H) :

i) Each form ω_k is indefinitely differentiable on the closed unit ball of $\mathbb{C}^{N_k}$ provided with the norm of l^2, and of the form

$$\omega_k(z^k) = \sum_{i\in I_k} z_i \omega_k^i(z^k) d\overline{z}_i.$$

ii) The series (1.2) is supposed to be absolutely convergent.

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The used method is based on the expansion in Fourier series of the indefinitely differentiable functions on the closed unit ball of $\mathbb{C}^N$. According to ([5], Theorem 2.1) each function ω_k^i admits necessarily a Fourier expansion of the form

$$\omega_k^i(z^k) = \sum_{(\alpha,\beta)\in\mathbb{N}^{2N_k}} (z^k)^\alpha (\overline{z^k})^\beta \omega^i_{k,(\alpha,\beta)}(|z^k|^2), \text{ for all } i \text{ and } k,$$

where $|z^k|^2 = (|z_i|^2)_{i\in I_k}$ and $(z^k)^\alpha = \prod_{i\in I_k} z_i^{\alpha_i}$.

In ([5, 6, 7]) the second author studied the local exactness of $\overline{\partial}$ for a restricted class of forms ω which respond moreover to the additional assumption $(\widetilde{H})$:

$$\omega_k^i(z^k) = \sum_{\alpha\in\mathbb{N}^{N_k}} (z^k)^\alpha \omega^i_{k,\alpha}(|z^k|^2), \text{ for all } i \text{ and } k,$$

and gets positive results in some particular cases, especially when the sequence (N_k) is bounded.

In this paper, in order to solve the $\overline{\partial}$-equation for a large class of forms ω, we drop the hypothesis $(\widetilde{H})$, and we prove another positive result more general than the earlier ones. The main result in this work is

Theorem 1.1 (Main Theorem). *Let ω be a closed $(0,1)$-form of class C^∞ on $\overline{B}$ according to the type $\omega = \sum_k \omega_k$ satisfying (H) and one of the following assumptions.*

(1) *There exists a positive integer n such that the coefficients $\omega^i_{k,(\alpha,\beta)}$ are null if $(|\alpha| - |\beta|) > n$, for all k and all i in I_k, and the derivatives $D^m\omega_k^i$ are uniformly bounded in i and k on B for $0 \le m \le n$.*

(2) *There exists a real number $\lambda > 0$ such that the coefficients $\omega^i_{k,(\alpha,\beta)}$ are null if $(|\alpha| - |\beta|) < \lambda N_k$, for all k and all i in I_k, and the derivatives $D^m\omega_k^i$ are uniformly bounded in i and k on B for $0 \le m \le 2$.*

Then there exists a real number $r > 0$, a positive integer M and a function F of class C^∞ on the ball of radius r that satisfies the equation $\overline{\partial}F = \omega$ such that

$$|F(z)| \le C \sup_{\substack{i,k\\0\le m\le M}} \|D^m\omega_k^i\|_\infty \text{ for } \|z\| < r,$$

where C is a constant and D designates the differentiation operator.

2. Preliminaries

2.1. Notations. In this work, unless indicated otherwise, $\| \ \|$ will denote the l^2-norm on l^2 or on $\mathbb{C}^N$: if $z = (z_i) \in l^2$ or $\mathbb{C}^N$, $\|z\| = \sum |z_i|^2$. $B(r)$ resp. $B_N(r)$ will denote the ball $\|z\| < r$ in l^2 resp. $\mathbb{C}^N$. When $r = 1$, we simply write B resp. B_N for $B(1)$ resp. $B_N(1)$.

In the sequel α denote a multiindex. A multiindex $\alpha = (\alpha_i)_{i=1}^\infty$ for us is a sequence of integers $\alpha_i \ge 0$ with $\alpha_i = 0$ for i large enough. The length of α is $|\alpha| = \sum_{i=1}^\infty \alpha_i$. We let $\alpha! = \prod_{i=1}^\infty \alpha_i!$, where the usual convention $0! = 1$ is observed. For a sequence of complex numbers $z = (z_i)_{i=1}^\infty$, we put $z^\alpha = \prod_{i=1}^\infty z_i^{\alpha_i}$, where 0^0 is defined to be 1.

If z and w are in $\mathbb{C}^N$, we denote:

$z_i' = (z_1, \dots, z_i)$; $\quad z_i'' = (z_i, \dots, z_N) \quad (i = 1, \dots, N)$
$|z|^2 = (|z_1|^2, \dots, |z_N|^2), \quad zw = (z_1w_1, \dots, z_Nw_N)$.

If f is in $\mathcal{C}^\infty\left(\overline{B}_N\right)$, then for each $m \in \mathbb{N}$, we put

$$\|D^m f\|_\infty = \sup_{z \in \overline{B}_N} \|D^m f(z)\|,$$

where $\|D^m f(z)\|$ is the operator norm of the mth Fréchet derivative $D^m f$ of f. The norm $\|D^m \omega\|_\infty$ of a $\mathcal{C}^\infty$ smooth (0,1)-form ω on $\overline{B}_N$ is defined by

$$\|D^m \omega\|_\infty = \sup_{(z,h) \in \overline{B}_N^2} \|D^m f(z,h)\|,$$

where $f(z,h) = \omega(z)h$ for $z, h \in \overline{B}_N$.

2.2. Auxiliary results. To prove Theorem 1.1, we require some preliminary results.
If $z = (z_i)_{i=1}^\infty$ is in the unit ball of l^2, we put

$$P_n(z) = \sum_{|\alpha| \geq n} \frac{|\alpha|^{\frac{|\alpha|}{2}}}{\alpha^{\frac{\alpha}{2}}} z^\alpha, \quad n \in \mathbb{N},$$

where $\alpha^{\frac{\alpha}{2}} = \prod_{i=1}^\infty \alpha_i^{\frac{\alpha_i}{2}}$.

Lemma 2.1. *There is a constant $Q > 1$ such that if $z \in B_N(r)$ with $r < \frac{1}{Q}$, then*

$$|P_n(z)| \leq (Qr)^n 2^N.$$

for every $n \in \mathbb{N}$.

Proof. Let us consider in $\mathbb{C}$ the entire function $g(z) = \sum_{\alpha \geq 0} \frac{z^\alpha}{\alpha^{\alpha/2}}$. For every $\epsilon \in \left]0, \frac{1}{2}\right[$, we have

$$|g(z)| \leq \sum_{\alpha \geq 0} \frac{|z|^\alpha}{\epsilon^{\alpha/2}\sqrt{\alpha!}} \epsilon^{\alpha/2}.$$

Using Cauchy-Schwarz inequality, we obtain

$$|g(z)| \leq \left(\sum_{\alpha \geq 0} \frac{|z|^{2\alpha}}{\epsilon^\alpha \alpha!}\right)^{1/2} \left(\sum_{\alpha \geq 0} \epsilon^\alpha\right)^{1/2}.$$

Let $q \in \mathbb{N}$, and $z \in \mathbb{C}^N$, we observe that the series $\sum_{|\alpha|=q} \frac{z^\alpha}{\alpha^{\alpha/2}}$ is the homogeneous component of degree q of the product $g(z_1) \cdots g(z_N)$. It follows, when $z \in B_N(\sqrt{q})$, the majorization

$$\left|\sum_{|\alpha|=q} \frac{z^\alpha}{\alpha^{\alpha/2}}\right| \leq 2^{N/2} \exp\left(\frac{q}{2\epsilon}\right).$$

By homothety on the ball of radius r, we get

$$\left|\sum_{|\alpha|=q} \frac{|\alpha|^{\frac{|\alpha|}{2}}}{\alpha^{\frac{\alpha}{2}}} z^\alpha\right| \leq r^q \exp\left(\frac{q}{2\epsilon}\right) 2^{N/2}.$$

Therefore, if r is sufficiently small, we get

$$|P_n(z)| \leq \left(2e^{1/2\epsilon}r\right)^n 2^N.$$

□

Theorem 2.1. *If f is in $\mathcal{C}^\infty\left(\overline{B}_N\right)$, then it admits the Fourier series expansion*

$$f(z) = \sum_{(\alpha,\beta)\in\mathbb{N}^{2N}} z^\alpha \overline{z}^\beta f_{(\alpha,\beta)}(|z|^2), \tag{2.1}$$

The series (2.1) *is normally convergent with its derivatives on $\overline{B}_N$; the coefficients $f_{(\alpha,\beta)}$ are $\mathcal{C}^\infty$ on the closed unit ball of $\mathbb{R}^N$ provided with the l^1-norm, that satisfy*

$$\forall(\alpha,\beta)\in\mathbb{N}^{2N}; z^\alpha\overline{z}^\beta f_{(\alpha,\beta)}(|z|^2) = \int_{[0,2\pi]^N} f(ze^{i\theta})e^{-i(\alpha\cdot\theta)}e^{i(\beta\cdot\theta)}\frac{d\theta}{(2\pi)^N} \tag{2.2}$$

where $(\alpha\cdot\theta) = \sum_{i=1}^N \alpha_i\theta_i,\ e^{i\theta} = (e^{i\theta_1},\dots,e^{i\theta_N})$, *and* $\frac{d\theta}{(2\pi)^N} = \frac{d\theta_1}{2\pi}\dots\frac{d\theta_N}{2\pi}$.

For the proof see ([5], Theorem 2.1).

3. Closure characterization

Let ω be a $\mathcal{C}^\infty$ smooth $(0,1)$- form on the closed unit ball of $\mathbb{C}^N$ according to the type

$$\omega(z) = \sum_{i=1}^N z_i\omega^i(z)d\overline{z}_i, \quad z = (z_i)_{i=1}^N \text{ in } \overline{B}_N$$

According to Theorem 2.1, for all i the function ω^i admits a Fourier series expansion in the form $\omega^i(z) = \sum_{(\alpha,\beta)\in\mathbb{N}^{2N}} z^\alpha\overline{z}^\beta\omega^i_{(\alpha,\beta)}(|z|^2)$, where the coefficients $\omega^i_{(\alpha,\beta)}$ are functions of class $\mathcal{C}^\infty$ on the closed unit ball of $\mathbb{R}^N$, and the series is normally convergent with its derivatives on $\overline{B}_N$.

Put

$$\omega_{(\alpha,\beta)}(z) = \sum_{i=1}^N z_i z^\alpha\overline{z}^\beta\omega^i_{(\alpha,\beta)}(|z|^2)d\overline{z}_i$$

It is easy to check that ω is closed if and only if $\omega_{(\alpha,\beta)}$ is closed for every $(\alpha,\beta)\in\mathbb{N}^{2N}$.

Proposition 3.1. *Let ω be a $\mathcal{C}^\infty$ smooth $(0,1)$-form on the closed unit ball of $\mathbb{C}^N$ of the type $\omega(z) = \sum_{i=1}^N z_i\omega^i(z)d\overline{z}_i$. Then ω is $\overline{\partial}$-closed on $\overline{B}_N$ if and only if $\Phi_{(\alpha,\beta)}(t) = \sum_{i=1}^N t^\beta\omega^i_{(\alpha,\beta)}(t)dt_i$ is d-closed on the closed unit ball of $\mathbb{R}^N$ for every $(\alpha,\beta)\in\mathbb{N}^{2N}$.*

Proof. ω is closed on $\overline{B}_N$ if and only if

$$z_i z^\alpha\frac{\partial}{\partial\overline{z}_j}\left(\overline{z}^\beta\omega^i_{(\alpha,\beta)}(|z|^2)\right) = z_j z^\alpha\frac{\partial}{\partial\overline{z}_i}\left(\overline{z}^\beta\omega^j_{(\alpha,\beta)}(|z|^2)\right)$$

for every $(\alpha, \beta) \in \mathbb{N}^{2N}$ and $(i, j) \in \mathbb{N}^2$.
By multiplying the above equality by z^β, one obtains

$$z_i z^\alpha \frac{\partial}{\partial \overline{z}_j}\left(|z|^{2\beta}\omega^i_{(\alpha,\beta)}(|z|^2)\right) = z_j z^\alpha \frac{\partial}{\partial \overline{z}_i}\left(|z|^{2\beta}\omega^j_{(\alpha,\beta)}(|z|^2)\right)$$

If we put $t = |z|^2$, this implies

$$z_i z_j z^\alpha \frac{\partial}{\partial t_j}\left(|z|^{2\beta}\omega^i_{(\alpha,\beta)}(|z|^2)\right) = z_j z_i z^\alpha \frac{\partial}{\partial t_i}\left(|z|^{2\beta}\omega^j_{(\alpha,\beta)}(|z|^2)\right)$$

Thus, ω is $\overline{\partial}$-closed on $\overline{B}_N$ if and only if $\Phi_{(\alpha,\beta)}$ is d-closed on the closed unit ball of $\mathbb{R}^N$. □

Now, we consider a closed $(0,1)$-form ω of class C^∞ on the closed unit ball of l^2 according to the type $\omega = \sum_k \omega_k$ and satisfying the assumption (H). For all $z \in \overline{B}$, all t in the closed unit ball of $\mathbb{R}^{N_k}$, and all $(\alpha, \beta) \in \mathbb{N}^{2N_k}$ we put

$$\omega_{k,(\alpha,\beta)}(z^k) = \sum_{i \in I_k} z_i (z^k)^\alpha (\overline{z^k})^\beta \omega^i_{k,(\alpha,\beta)}(|z^k|^2) d\overline{z}_i \tag{3.1}$$

$$\Phi_{k,(\alpha,\beta)}(t) = \sum_{i \in I_k} t^\beta \omega^i_{k,(\alpha,\beta)}(t) dt_i$$

Theorem 3.1. *ω is closed on $\overline{B}$ if and only if for all $k \geq 1$, and $(\alpha, \beta) \in \mathbb{N}^{2N_k}$*

$$(z^k)^\beta \omega_{k,(\alpha,\beta)}(z^k) = \sum_{i \in I_k} z_i (z^k)^\alpha \frac{\partial \Omega_{k,(\alpha,\beta)}}{\partial t_i}(|z^k|^2) d\overline{z}_i, \tag{3.2}$$

where $\Omega_{k,(\alpha,\beta)}$ is an indefinitely differentiable function on the closed unit ball of $\mathbb{R}^{N_k}$.

Proof. ω_k is the restriction to $\mathbb{C}^{N_k}$ of ω, thus ω is closed on $\overline{B}$ if and only if ω_k is closed on $\overline{B}_{N_k}$ for every k. By using Proposition 3.1, and Poincare's lemma, there exists for all k and $(\alpha, \beta) \in \mathbb{N}^{2N_k}$, a function $\Omega_{k,(\alpha,\beta)}$ indefinitely differentiable on the closed unit ball of $\mathbb{R}^{N_k}$ provided with the norm of l^1 such that

$$d\Omega_{k,(\alpha,\beta)} = \Phi_{k,(\alpha,\beta)}. \tag{3.3}$$

Now, (3.2) follows immediately from (3.1) and (3.3). □

4. Solvability

Let ω be a $\overline{\partial}$-closed $\mathcal{C}^\infty$ smooth $(0,1)$-form on the closed unit ball of l^2 according to the type $\omega = \sum_k \omega_k$ such that $\omega_k = \sum_{(\alpha,\beta)\in\mathbb{N}^{2N_k}} \omega_{k,(\alpha,\beta)}$, where

$$\omega_{k,(\alpha,\beta)}(z^k) = \sum_{i \in I_k} z_i (z^k)^\alpha (\overline{z^k})^\beta \omega^i_{k,(\alpha,\beta)}(|z^k|^2) d\overline{z}_i,$$

and $\omega_{k,(\alpha,\beta)}$ is $\overline{\partial}$-closed (0,1)-form of class $\mathcal{C}^\infty$ on the closed unit ball of $\mathbb{C}^{N_k}$.
For each $(\alpha, \beta) \in \mathbb{N}^{2N_k}$ we can solve the equation

$$\overline{\partial} U_{k,(\alpha,\beta)} = (z^k)^\beta \omega_{k,(\alpha,\beta)} \tag{4.1}$$

with $U_{k,(\alpha,\beta)} \in \mathcal{C}^\infty\left(\overline{B}_{N_k}\right)$, by putting

$$U_{k,(\alpha,\beta)}(z^k) = (z^k)^\alpha \Omega_{k,(\alpha,\beta)}(|z^k|^2).$$

This determines $U_{k,(\alpha,\beta)}$ up to a holomorphic term. To determine $U_{k,(\alpha,\beta)}$ unambiguously we assign to each $\alpha \in \mathbb{N}^{N_k}$ a fixed point $M_\alpha = \left(M_\alpha^i\right)_{i\in I_k}$ in the closed unit ball of $\mathbb{R}^{N_k}$ such that $M_\alpha^i = 0$ if $\alpha_i = 0$ and require that $\Omega_{k,(\alpha,\beta)}\left(M_\alpha\right) = 0$.

Lemma 4.1. *Let $\Omega_{k,(\alpha,\beta)}$ be an antiderivative of $\Phi_{k,(\alpha,\beta)}$ that vanishes at the point M_α, then*

$$F_{k,(\alpha,\beta)}(z^k) = \frac{(z^k)^\alpha(\overline{z^k})^\beta}{|z^k|^{2\beta}}\Omega_{k,(\alpha,\beta)}(|z^k|^2)$$

define $\mathcal{C}^\infty$ smooth function on $\overline{B}_{N_k}$ that solves the equation $\overline{\partial}F_{k,(\alpha,\beta)} = \omega_{k,(\alpha,\beta)}$.

Proof. By applying Taylor formula to the function $\Omega_{k,(\alpha,\beta)}$ at the point M_α, and by using (3.3), it follows that

$$\Omega_{k,(\alpha,\beta)}(t) = t^\beta H_{k,(\alpha,\beta)}(t)$$

where $H_{k,(\alpha,\beta)}$ is $\mathcal{C}^\infty$ smooth on the closed unit ball of $\mathbb{R}^{N_k}$. Therefore $F_{k,(\alpha,\beta)}$ is of class $\mathcal{C}^\infty$ on $\overline{B}_{N_k}$.

Since $U_{k,(\alpha,\beta)}(z^k) = (z^k)^\beta F_{k,(\alpha,\beta)}(z^k)$, it follows from (4.1) that

$$\overline{\partial}F_{k,(\alpha,\beta)} = \omega_{k,(\alpha,\beta)}.$$

□

Remark 4.1. Suppose that for some $M \geq 1$, the derivatives $D^m\omega_k^i$ are uniformly bounded in i and k on the unit ball of l^2 for $0 \leq m \leq M$, and put $F_k = \sum_{(\alpha,\beta)\in\mathbb{N}^{2N_k}} F_{k,(\alpha,\beta)}$. Then, by applying ([3],Appendix 3, Lemma 5), for some $0 < r < 1$, the function $F = \sum_{k=1}^\infty F_k$ define a $\mathcal{C}^\infty$-smooth solution of the equation $\overline{\partial}F = \omega$ on $B(r)$ if there exists $C > 0$, such that

$$|F_k(z^k)| \leq C \sup_{\substack{i,k \\ 0\leq m\leq M}} \|D^m\omega_k^i\|_\infty < +\infty$$

for every $z \in B(r)$ and $k \geq 1$.

From this observation the problem is reduced to the existence of an antiderivative $\Omega_{k,(\alpha,\beta)}$ such that the series F_k converge and satisfies an estimate independent of the dimension N_k. We give a positive response for two particular cases.

The polynomial case. Let us choose for $\Omega_{k,(\alpha,\beta)}$ the integral of $\Phi_{k,(\alpha,\beta)}$ along a path connecting the origin to $|z^k|^2$ inside the unit ball of $\mathbb{R}^{N_k}$ provided with the l^1-norm, then we get the following result

Proposition 4.1. *If there exists a positive integer M such that the coefficients $\omega_{k,(\alpha,\beta)}^i$ are null if $(|\alpha| - |\beta|) > M$, for all k and all i in I_k, and if the derivatives $D^m\omega_k^i$ are uniformly bounded in i and k on B for $0 \leq m \leq M$. then the series F_k converge and define indefinitely differentiable functions on $\overline{B}_{N_k}$ that satisfies*

$$|F_k(z^k)| \leq C\|z^k\|^2 \sup_{m\leq M} \|D^m\omega\|_\infty < +\infty,$$

where C is a constant independent of k.

Proof. By applying Lemma 4.1, we have

$$F_k(z^k) = \sum_{q=-\infty}^{M} \sum_{|\alpha|-|\beta|=q} \int_0^1 (z^k)^\alpha (\overline{z^k})^\beta u^{|\beta|} \sum_{i\in I_k} \omega^i_{k,(\alpha,\beta)}(u|z^k|^2)\cdot |z_i|^2 du$$
$$= \sum_{q=-\infty}^{M} \int_0^1 \int_0^{2\pi} \frac{1}{(\sqrt{u})^{q+1}} < \omega_k(\sqrt{u}z^k e^{i\theta}), z^k > e^{-iq\theta}\frac{d\theta}{2\pi} du,$$

where $< \omega_k(\sqrt{u}z^k e^{i\theta}), z^k >= \omega_k(\sqrt{u}z^k e^{i\theta})z^k = \sum_{i\in I_k} |z_i|^2 \omega^i_k(\sqrt{u}z^k e^{i\theta})$.
By making q integrations by parts relatively to θ in each term of the above sum when $1 \le q \le M$, we obtain

$$|F_k(z^k)| \le C\|z^k\|^2 \sup_{m\le M} \|D^m\omega\|_\infty,$$

where C is a constant independent of k, and hence

$$|F(z)| \le C\|z\|^2 \sup_{m\le M} \|D^m\omega\|_\infty.$$

□

Thus we have proved the first part of Theorem 1.1.

Non-polynomial case. Let ω be a $\overline{\partial}$-closed $\mathcal{C}^\infty$ smooth $(0,1)$- form on the closed unit ball of $\mathbb{C}^N$ according to the type

$$\omega(z) = \sum_{i=1}^{N} z_i \omega^i(z) d\overline{z}_i, \quad z = (z_i)_{i=1}^N \text{ in } \overline{B}_N$$

We recall that ω^i admits a Fourier series expansion of the type

$$\omega^i(z) = \sum_{(\alpha,\beta)\in\mathbb{N}^{2N}} z^\alpha \overline{z}^\beta \omega^i_{(\alpha,\beta)}(|z|^2).$$

We also recall that $\Phi_{(\alpha,\beta)}$ designates the closed form in $\mathbb{R}^N$ defined by

$$\Phi_{(\alpha,\beta)}(t) = \sum_{i=1}^{N} t^\beta \omega^i_{(\alpha,\beta)}(t) dt_i.$$

The anti-derivates of $\Phi_{(\alpha,\beta)}$ is given by $\Omega_{(\alpha,\beta)}(|z|^2) = \int_\gamma \Phi_{(\alpha,\beta)}$, where the path γ defined below joins the point $|z|^2$ to a fixed point of the closed unit ball of $\mathbb{R}^N_+$. We can take the function

$$F(z) = \sum_{(\alpha,\beta)\in\mathbb{N}^{2N}} \frac{z^\alpha \overline{z}^\beta}{|z|^{2\beta}} \Omega_{(\alpha,\beta)}(|z|^2)$$

for a $\overline{\partial}$- anti-derivate of ω conditioned by its series convergence.

If we suppose that there exists a real number $\lambda > 0$ such that the coefficients $\omega^i_{(\alpha,\beta)}$ are null if $(|\alpha|-|\beta|) < \lambda N$, then we shall prove that for some $0 < r < 1$ and for a suitable choice of the path γ, the series F converge and define $\mathcal{C}^\infty$-functions on $B_N(r)$ with estimates independent of N.

Put for $\alpha \in \mathbb{N}^N$

$$M_\alpha = \begin{cases} 0 & \text{if } \alpha = 0 \\ \alpha/|\alpha| & \text{otherwise,} \end{cases}$$

Given $\|z\| < r < 1$, and $(\alpha, \beta) \in \mathbb{N}^{2N}$, we denote by γ the union of the adjacent segments $[M^m, M^{m+1}]$ $(m = 0, 1, 2)$, connecting $M^0 = |z|^2$ to the point $M^3 = M_\alpha$ with $M^1 = \frac{1}{r^2}|z|^2$ and

$$M_i^2 = \begin{cases} \frac{1}{r^2}|z_i|^2 & \text{if } \alpha_i \neq 0 \\ r^2|z_i|^2 & \text{if } \alpha_i = 0 \end{cases}$$

In the sequel, we shall use the majorization of the following lemma.

Lemma 4.2. *Let n be a natural number, then for any $z \in B_N(r)$, we have*

$$\sum_{|\alpha|=n} \int_{M^2}^{M^3} \frac{|z|^\alpha}{t^{\frac{\alpha}{2}}} dt \leq (Qr)^n \, 2^N. \tag{4.2}$$

where Q is the constant of Lemma 2.1 .

Proof. Let $[0,1] \ni u \mapsto t(u) = (1-u)M^2 + uM^3$ be the parametrization of the segment $[M^2, M^3]$, we can write

$$\int_{M^2}^{M^3} \frac{|z|^\alpha}{t^{\frac{\alpha}{2}}} dt = \int_0^{\frac{1}{2}} \frac{|z|^\alpha}{(t(u))^{\frac{\alpha}{2}}} du + \int_{\frac{1}{2}}^1 \frac{|z|^\alpha}{(t(u))^{\frac{\alpha}{2}}} du.$$

When u describes $[0, \frac{1}{2}]$, we observe that $(t(u))^{\frac{\alpha}{2}} \geq \frac{|z|^\alpha}{(\sqrt{2}r)^{|\alpha|}}$. Therefore

$$\int_0^{\frac{1}{2}} \frac{|z|^\alpha}{(t(u))^{\frac{\alpha}{2}}} du \leq \frac{1}{2}\left(\sqrt{2}r\right)^{|\alpha|} \leq \left(\sqrt{2}r\right)^{|\alpha|}. \tag{4.3}$$

On the other hand, we have the inequality $t(u) \geq \frac{\alpha}{2|\alpha|}$ for every $u \in [\frac{1}{2}, 1]$. Hence

$$\int_{\frac{1}{2}}^1 \frac{|z|^\alpha}{(t(u))^{\frac{\alpha}{2}}} du \leq \frac{|\alpha|^{\frac{|\alpha|}{2}}}{\alpha^{\frac{\alpha}{2}}} |2z|^\alpha. \tag{4.4}$$

From (4.3) and (4.4) we get the majorization

$$\sum_{|\alpha|=n} \int_{M^2}^{M^3} \frac{|z|^\alpha}{t^{\frac{\alpha}{2}}} dt \leq \sum_{|\alpha|=n} \frac{|\alpha|^{\frac{|\alpha|}{2}}}{\alpha^{\frac{\alpha}{2}}} \left(|2z|^\alpha + \left(2r^2 \frac{\alpha}{|\alpha|}\right)^{\frac{\alpha}{2}} \right).$$

If we choose r sufficiently small then an application of Lemma 2.1 implies the required estimate (4.2). □

Remark 4.2. Following Proposition 4.1, we may suppose $|\alpha| - |\beta| \geq 2$.

The proof of the second part of Theorem 1.1 is a direct consequence of the forthcoming proposition.

Proposition 4.2. *Let ω be a $\overline{\partial}$-closed $\mathcal{C}^\infty$ smooth $(0,1)$- form on $\overline{B}_N$ according to the type $\omega(z) = \sum_{i=1}^N z_i\omega^i(z)d\overline{z}_i$ such that there exists a real number $\lambda > 0$ such that the coefficients $\omega^i_{(\alpha,\beta)}$ are null if $(|\alpha| - |\beta|) < \lambda N$. We assume furthermore that the derivatives $D^m\omega^i$ are uniformly bounded in i and k on B for $0 \leq m \leq 2$. Then there exists $0 < r < R < 1$ such that the series F converge and defines a C^∞-smooth $\overline{\partial}$-antiderivative of ω on $\overline{B}_N(r)$ for which*

$$|F(z)| \leq C\left(\|z\|^2 + R^N\right) \sup_{\substack{i \\ 0\leq m\leq 2}} \|D^m\omega^i\|_\infty \tag{4.5}$$

where C is a constant independent of N.

Proof. For every $z \in B_N$, we put

$$F^m(z) = \sum_{(|\alpha|-|\beta|)\geq\lambda N} \frac{z^\alpha\overline{z}^\beta}{|z|^{2\beta}} \int_{M^m}^{M^{m+1}} \Phi_{(\alpha,\beta)}(t)dt, \quad m = 0,1,2.$$

According to Lemma 4.1, we have $F = F^0 + F^1 + F^2$, so it will be enough to prove that, for $m = 0,1,2$, the series F^m converges and satisfies an estimate independent of N.

Let us start with the case $m = 0$.

$$\begin{aligned} F^0(z) &= \sum_{(|\alpha|-|\beta|)\geq\lambda N} \int_1^{\frac{1}{r^2}} z^\alpha\overline{z}^\beta u^{|\beta|} \sum_{i=1}^N \omega^i_{(\alpha,\beta)}(u|z|^2)\cdot|z_i|^2du \\ &= \sum_{q=\lambda N}^{+\infty} \int_1^{\frac{1}{r^2}} \int_0^{2\pi} \frac{1}{(\sqrt{u})^{q+1}} < \omega(\sqrt{u}ze^{i\theta}), z > e^{-iq\theta}\frac{d\theta}{2\pi}du. \end{aligned}$$

By making two integrations by parts relatively to θ in each term of the above sum, we obtain

$$|F^0(z)| \leq C\|z\|^2 \sup_{m\leq 2} \|D^m\omega\|_\infty. \tag{4.6}$$

where C is a constant independent of N.

Now, let us consider the series F^1.
Let J be a (possibly empty) subset of $\{1,2,\dots,N\}$, we denote by A_J the set of multi-indices $(\alpha,\beta) \in \mathbb{N}^{2N}$ such that $\alpha_i = 0$ if and only if $i \in J$.
Note that A_J constitute a partition of $\mathbb{N}^{2N}$. Let

$$\omega_J = \sum_{i\in J} z_i \left[\sum_{(\alpha,\beta)\in A_J} z^\alpha\overline{z}^\beta\omega^i_{(\alpha,\beta)}(|z^k|^2)\right] d\overline{z}_i.$$

We have $F^1 = \sum_J F^1_J$, with

$$F^1_J(z) = \sum_{(\alpha,\beta)\in A_J} \int_{\frac{1}{r^2}}^{r^2} z^\alpha\overline{z}^\beta u^{|\beta|} \sum_{i\in J} \omega^i_{(\alpha,\beta)}(t(u))\cdot|z_i|^2du.$$

Note that if $J = \emptyset$ we have $F^1_J = 0$, hence it is enough to consider $J \neq \emptyset$.
By a change of numerotation we may suppose that $J = \{\nu,\dots,N\}$ $(\nu \leq N)$, thus

we can write

$$F_J^1(z) = \sum_{q-p\geq\lambda N}\sum_{|\alpha|=q}\sum_{|\beta|=p}\int_{\frac{1}{r^2}}^{r^2} z^\alpha\overline{z}^\beta u^p \sum_{i=\nu}^{N}\omega^i_{(\alpha,\beta)}(\frac{1}{r^2}|z'_{\nu-1}|^2, u|z''_\nu|^2)\cdot|z_i|^2du$$

$$= \sum_{p=0}^{+\infty}\sum_{q=p+\lambda N}^{+\infty}\int_{\frac{1}{r^2}}^{r^2}\int_0^{2\pi}\int_0^{2\pi} r^q\sqrt{u}^p < \omega_J(\frac{1}{r}z'_{\nu-1}e^{i\theta}, \sqrt{u}z''_\nu e^{i\varphi}), z''_{\nu+1} >$$

$$\times\, e^{-i(q\theta-p\varphi)}\frac{dud\theta d\varphi}{4\pi^2}.$$

An easy computation shows that

$$r^p\int_{r^2}^{1/r^2}\sqrt{u}^p du \leq \frac{1}{r^2}.$$

By making two integrations by parts relatively to φ in each term of the above sum we are led to the majorization

$$|F_J^1(z)| \leq Cr^{\lambda N}\|z\|^2\sup_{m\leq 2}\|D^m\omega\|_\infty$$

Hence

$$|F^1(z)| \leq C\|z\|^2\sup_{m\leq 2}\|D^m\omega\|_\infty \tag{4.7}$$

where C is a constant independent of N.

It remains to consider the series F^2, we have

$$F^2(z) = \sum_{q-p\geq\lambda N}\sum_{|\alpha|=q}\sum_{|\beta|=p}\int_0^1 z^\alpha\overline{z}^\beta(1-u)^p r^{2p}\sum_{i=1}^{N}\omega^i_{(\alpha,\beta)}(t(u))t'_i(u)du$$

$$= \sum_{p=0}^{+\infty}\sum_{q=p+\lambda N}^{+\infty}\int_0^1\int_0^{2\pi}\frac{z^\alpha(1-u)^{\frac{p}{2}}r^p}{(t(u))^{\frac{\alpha}{2}}}\sum_{i=1}^{N}\omega^i(\sqrt{(t(u)}e^{i\theta})e^{-i(q-p)\theta}t'_i(u)\frac{dud\theta}{2\pi}$$

By applying Lemma 4.2 and by choosing r sufficiently small, we get

$$|F^2(z)| \leq CR^N\sup_i\|\omega^i\|_\infty \tag{4.8}$$

where $0 < R < 1$ and C are independent of N.

Now, the required estimate (4.5) follows immediately from (4.6),(4.7) and (4.8). □

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