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# MAXIMAL AND POTENTIAL OPERATORS ASSOCIATED WITH GEGENBAUER DIFFERENTIAL OPERATOR ON GENERALIZED MORREY SPACES

ELMAN J. IBRAHIMOV, SAADAT A. JAFAROVA, AND S. ELIFNUR EKINCIOGLU

**Abstract**. In this paper we study the boundedness of the maximal ($G$-maximal) and potential ($G$-potential) operators associated with Gegenbauer differential operator on generalized $G$-Morrey spaces. The results of this paper are generalizations of the corresponding results to generalized $G$-Morrey spaces and modified Morrey spaces.We obtain also analogs of E.Nakai's results for the Hardy-Littlewood maximal operator and the Riesz potential in generalized Morrey spaces.

## 1. Introduction

In 2011, in the paper [11] new integral transformations that formed the basis of theory of Harmonic analysis of the Gegenbauer differential operator were constructed. Later, this theory was intensively developed in various directions: approximation theory, imbedding theory, transformation theory, theory of maximal functions and potential theory (see [4-8, 9-11]). The basis of this theory was the Gegenbauer differential operator $G$ (see [1]). In [1], various representations (through integral and hypergeometrical functions) of eigen-functions of this operator, relations between them, formulas of addition and product for these functions, asymptotic formulas, etc are given. The reader can find detailed information in the mentioned paper [1].

One of the important directions of the Gegenbauer harmonic analysis is the boundedness of maximal operator and potential generated by the Gegenbauer differential operator $G$.

The boundedness of the maximal ($G$-maximal) and potential ($G$-potential) operators associated with Gegenbauer differential operator $G$.

$$G \equiv G_\lambda = (x^2-1)^{\frac{1}{2}-\lambda}\frac{d}{dx}(x^2-1)^{\lambda+\frac{1}{2}}\frac{d}{dx},\ x\in(1,\infty),\ \lambda\in(0,\frac{1}{2})$$

on the Lebesgue, Morrey and modified Morrey spaces is considered in [3, 4, 5].

In the present paper, we introduce a generalized Gegenbauer-Morrey ($G$-Morrey) space $\mathcal{M}_{p,\lambda,\omega}(\mathbb{R}_+,G)$, and estimate $G$-maximal and $G-$ potential operators generated by Gegenbauer differential operator $G$. The obtained result is an analog of

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the corresponding theorems obtained for the Hardy-Littlewood maximal operator and the Riesz potential in [16].

## 2. Definition and notation

Let $H(x,r) = (x-r, x+r) \cap (0,\infty),\ r \in (0,\infty),\ x \in (0,\infty) = \mathbb{R}_+$. For all measurable sets $E \subset (0,\infty)$, put $\mu E = |E|_\lambda = \int_E sh^{2\lambda} t dt$.

For $1 \le p \le \infty$ let $L_{p,\lambda}(\mathbb{R}_+, G)$ be the space of functions measurable on $\mathbb{R}_+$ with the finite norm

$$\|f\|_{L_{p,\lambda}} = \Big( \int_0^\infty |f(cht)|^p sh^{2\lambda} t dt \Big)^{\frac{1}{p}},\ 1 \le p < \infty,$$

$$\|f\|_{\infty,\lambda} \equiv \|f\|_\infty = \operatorname*{ess\,sup}_{t\in(0,\infty)} |f(cht)|,\ p = \infty.$$

In [5], the following notation is introduced.

Let $1 \le p < \infty,\ 0 < \lambda < \frac{1}{2},\ 0 \le \gamma \le 2\lambda + 1,\ [r]_1 = \min\{1, r\}$. We denote by $L_{p,\lambda,\gamma}(\mathbb{R}_+, G)$, $\mathbb{R}_+ = (0,\infty)$, the $G$-Morrey space, and by $\widetilde{L}_{p,\lambda,\gamma}(\mathbb{R}_+, G)$ the modified $G$-Morrey space, as the set of locally integrable functions $f(chx)$, $x \in \mathbb{R}_+$, with the finite norms

$$\|f\|_{L_{p,\lambda,\gamma}} = \sup_{x,r>0} \left( r^{-\gamma} \int_{H(x,r)} |f(cht)|^p sh^{2\lambda} t dt \right)^{\frac{1}{p}},$$

$$\|f\|_{\widetilde{L}_{p,\lambda,\gamma}} = \sup_{x,r>0} \left( [r]_1^{-\gamma} \int_{H(x,r)} |f(cht)|^p sh^{2\lambda} t dt \right)^{\frac{1}{p}},$$

respectively.

Note that $\widetilde{L}_{p,\lambda,0}(\mathbb{R}_+, G) = L_{p,\lambda,0}(\mathbb{R}_+, G) = L_{p,\lambda}(\mathbb{R}_+, G)$.

If $1 \le p < \infty$, $0 < \lambda < \frac{1}{2}$, $0 \le \gamma \le 2\lambda + 1$, then

$$\widetilde{L}_{p,\lambda,\gamma}(\mathbb{R}_+, G) = L_{p,\lambda,\gamma}(\mathbb{R}_+, G) \cap L_{p,\lambda}(\mathbb{R}_+, G)$$

and

$$\|f\|_{\widetilde{L}_{p,\lambda,\gamma}} = \max\{\|f\|_{L_{p,\lambda,\gamma}, \|f\|_{L_{p,\lambda}}}\}$$

(see [5], Lemma 2.2).

If $\gamma < 0$ or $\gamma > 2\lambda + 1$, then $L_{p,\lambda,\gamma}(\mathbb{R}_+, G) = \widetilde{L}_{p,\lambda,\gamma}(\mathbb{R}_+, G) = \Theta$, where $\theta$ is the set of all functions equivalent to 0 on $\mathbb{R}_+$.

Let $1 \le p < \infty$, $0 < \lambda < \frac{1}{2}$, $0 \le \gamma \le 2\lambda + 1$. We denote by $WL_{p,\lambda,\gamma}(\mathbb{R}_+, G)$ the weak $G$-Morrey space, and $W\widetilde{L}_{p,\lambda,\gamma}(\mathbb{R}_+, G)$ the modified weak $G$-Morrey space as the set of locally integrable functions $f(chx)$, $x \in \mathbb{R}_+$, with the finite norms

$$\|f\|_{WL_{p,\lambda,\gamma}} = \sup_{r>0} r \sup_{t,x>0} \Big( t^{-\gamma} |\{y \in H(x,t) : |f(chy)| > r\}|_\gamma \Big)^{\frac{1}{p}},$$

$$\|f\|_{W\widetilde{L}_{p,\lambda,\gamma}} = \sup_{r>0} r \sup_{t,x>0} \Big( [t]_1^{-\gamma} |\{y \in H(x,t) : |f(chy)| > r\}|_\gamma \Big)^{\frac{1}{p}},$$

respectively.

Note that $WL_{p,\lambda}(\mathbb{R}_+,G) = WL_{p,\lambda,0}(\mathbb{R}_+,G) = W\widetilde{L}_{p,\lambda,0}(\mathbb{R}_+,G)$, $L_{p,\lambda,\gamma}(\mathbb{R}_+,G) \subset WL_{p,\lambda,\gamma}(\mathbb{R}_+,G)$ and $\|f\|_{WL_{p,\lambda,\gamma}} \le \|f\|_{L_{p,\lambda,\gamma}}$, $\widetilde{L}_{p,\lambda,\gamma}(\mathbb{R}_+,G) \subset WL_{p,\lambda,\gamma}(\mathbb{R}_+,G)$ and $\|f\|_{W\widetilde{L}_{p,\lambda,\gamma}} \le \|f\|_{\widetilde{L}_{p,\lambda,\gamma}}$.

The generalized shift operator associated with the operator $G_\lambda$ is of the form (see [6, 9])

$$A^\lambda_{cht} f(chx) = \frac{\Gamma(\lambda+\frac{1}{2})}{\Gamma(\lambda)\Gamma(\frac{1}{2})} \int_0^\pi f(chxcht - shxsht\cos\varphi)(sin\varphi)^{2\lambda-1} d\varphi.$$

This operator possesses properties similar to those of the generalized shift operator in Levitan's works [13] and [14].

By analogy with [16], we introduce the following notation.

**Definition 2.1.** Let $1 \le p < \infty$ and let $w : \mathbb{R}_+ \to \mathbb{R}_+$ be a Lebesgue measurable function. The generalized Gegenbauer-Morrey ($G$-Morrey) space $M_{p,\lambda,w}(\mathbb{R}_+,G)$ associated with the Gegenbauer differential operator $G_\lambda$ are the set of locally integrable functions $f(chx)$, $x \in \mathbb{R}_+$ with the finite norm

$$\|f\|_{M_{p,\lambda,w}(\mathbb{R}_+,G)} \equiv \|f\|_{M_{p,\lambda,w}} := \sup_{x\in\mathbb{R}_+, r>0} \left(\frac{1}{w(r)} \int_{H(0,r)} A^\lambda_{cht}|f(chx)|^p sh^{2\lambda} t dt\right)^{\frac{1}{p}},$$

and the weak Morrey space $WM_{p,\lambda,w}(\mathbb{R}_+,G)$ are the set of locally integrable functions $f(chx)$, $x \in \mathbb{R}_+$, with the finite norm

$$\begin{aligned}&\|f\|_{WM_{p,\lambda,w}}(\mathbb{R}_+,G) \equiv \|f\|_{WM_{p,\lambda,w}} \\ &= \sup_{r>0} r \sup_{x\in\mathbb{R}_+,t>0} \left(\frac{1}{w(t)} \Big|\Big\{y \in H(0,t) : |A^\lambda_{chy} f(chx)| > r\Big\}\Big|_\lambda\right)^{\frac{1}{p}} \\ &= \sup_{r>0} r \sup_{x\in\mathbb{R}_+,t>0} \left(\frac{1}{w(t)} \int_{\{y\in H(0,t): A^\lambda_{chy}|f(chx)|>r\}} sh^{2\lambda} y dy\right)^{\frac{1}{p}}.\end{aligned}$$

Under the choice $w(r) = r^\gamma$, $0 \le \gamma \le 2\lambda+1$, or $w(r) = [r]_1^\gamma$, we can write that $L_{p,\lambda,\gamma}(\mathbb{R}_+,G) \equiv M_{p,\lambda,w}(\mathbb{R}_+,G)|_{w(r)=r^\gamma}$, and $\widetilde{L}_{p,\lambda,\gamma}(\mathbb{R}_+,G) \equiv M_{p,\lambda,w}(\mathbb{R}_+,G)\Big|_{w(r)=[r]_1^\gamma}$, respectively (see [5]).

Let $M_G$ be the Gegenbauer maximal operator(see [9]) for $f \in L^{loc}_{1,\lambda}(\mathbb{R}_+)$

$$M_G(chx) = \sup_{r>0} \frac{1}{|H(0,r)|_\lambda} \int_{H(0,r)} A^\lambda_{cht}|f(chx)| sh^{2\lambda} t dt,$$

where $|H(0,r)|_\lambda = \int_0^r sh^{2\lambda} t dt$.

For $q \ge 1$ let

$$M^q_G f(chx) = (M_G|f|^q(chx))^{\frac{1}{q}}.$$

The Riesz-Gegenbauer ((R-G)-potential) $I^\alpha_G$ is defined as follows (see [3, 4, 5])

$$I^\alpha_G f(chx) = \frac{1}{\Gamma(\frac{\alpha}{2})} \int_0^\infty \left(\int_0^\infty r^{\frac{\alpha}{2}-1} h_r(cht) dr\right) A^\lambda_{cht} f(chx) sh^{2\lambda} t dt,$$

where

$$h_r(cht) = \int_1^\infty e^{-u(u+2\lambda)r} P^\lambda_u(cht) sh^{2\lambda} u du$$

and $P^\lambda_u(cht)$ is an eigen function of the operator $G$.

Throughout in the paper, we will denote by $shx, chx$ the hyperbolic functions and by $A \lesssim B$ we mean that $A \leq CB$ with some positive constant $C$ which can depend on some parameters. If $A \lesssim B$ and $B \lesssim A$, we write $A \approx B$ and say that they are equivalent.

## 3. Main results

Let $0 < \delta \leq 1$. Assume that $w(r)$ satisfies the conditions: for any $r > 0$

$$r \leq t \leq 2r \Rightarrow w(t) \approx w(r), \tag{3.1}$$

$$\int_r^\infty \frac{w(t)}{t^{\gamma\delta+1}} dt \lesssim \begin{cases} r^{-(2\lambda+1)\delta} w(r), \ \gamma = 2\lambda+1; \ 0 < r < 2. \\ r^{-4\lambda\delta} w(r), \ \gamma = 4\lambda; \ 2 \leq r < \infty. \end{cases} \tag{3.2}$$

**Theorem 3.1.** *Let conditions* (3.1) *and* (3.2) *be valid. Then*

($i$) *For* $f \in M_{p,\lambda,w}(\mathbb{R}_+, G)$ *and* $1 \leq q < p < \infty$

$$\|M_G^q f\|_{M_{p,\lambda,w}} \lesssim \|f\|_{M_{p,\lambda,w}}. \tag{3.3}$$

($ii$) *For* $f \in WM_{p,\lambda,w}(\mathbb{R}_+, G)$ , $1 \leq p < \infty$ *and for any* $t > 0$

$$\|M_G^p f\|_{WM_{p,\lambda,w}} \lesssim \|f\|_{M_{p,\lambda,w}}. \tag{3.4}$$

Now, we consider the Riesz-Gegenbauer potential ((R-G)-potential) $I_G^\alpha$.

**Theorem 3.2.** *Let* $0 < \lambda < \frac{1}{2}$ , $0 < \alpha < 2\lambda+1$, $1 \leq p < \frac{\alpha}{2\lambda+1}$ *and* $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{2\lambda+1}$. *Assume that* $w$ *satisfies the conditions* (3.1) *and* (3.2). *Then*

($i$) *if* $p > 1$ *then for* $f \in M_{p,\lambda,w}(\mathbb{R}_+, G)$

$$\|I_G^\alpha f\|_{M_{q,\lambda,w^{\frac{q}{p}}}} \lesssim \|f\|_{M_{p,\lambda,w}}, \tag{3.5}$$

($ii$) *if* $p = 1$ *and* $f \in M_{1,\lambda,w}(\mathbb{R}_+, G)$. *Then*

$$\|I_G^\alpha f\|_{WM_{q,\lambda,w}} \lesssim \|f\|_{M_{1,\lambda,w}}. \tag{3.6}$$

**Corollary 3.1.** [3] *Let* $0 < \alpha < 2\lambda+1$, $0 < \gamma < 2\lambda+1-\alpha$ *and* $1 \leq p < \frac{2\lambda+1-\gamma}{\alpha}$.

($i$) *If* $1 < p < \frac{2\lambda+1-\gamma}{\alpha}$, *then condition* $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{2\lambda+1-\gamma}$ *is necessary and sufficient for the boundedness of* $I_G^\alpha$ *from* $L_{p,\lambda,\gamma}(\mathbb{R}_+, G)$ *to* $L_{q,\lambda,\gamma}(\mathbb{R}_+, G)$.

($ii$) *If* $p = 1 < \frac{2\lambda+1-\gamma}{\alpha}$, *then the condition* $1 - \frac{1}{q} = \frac{\alpha}{2\lambda+1-\gamma}$ *is necessary and sufficient for the boundedness of* $I_G^\alpha$ *from* $L_{1,\lambda,\gamma}(\mathbb{R}_+, G)$ *to* $WL_{q,\lambda,\gamma}(\mathbb{R}_+, G)$.

**Corollary 3.2.** [5] *Let* $0 \leq \alpha < 2\lambda+1$, $0 \leq \gamma < 2\lambda+1-\alpha$ *and* $1 \leq p < \frac{2\lambda+1-\gamma}{\alpha}$.

1) *If* $1 < p < \frac{2\lambda+1-\gamma}{\alpha}$, *then the condition* $\frac{\alpha}{2\lambda+1} \leq \frac{1}{p} - \frac{1}{q} \leq \frac{\alpha}{2\lambda+1-\gamma}$ *is necessary and sufficient for the boundedness of* $I_G^\alpha$ *from* $\widetilde{L}_{p,\lambda,\gamma}(\mathbb{R}_+, G)$ *to* $\widetilde{L}_{q,\lambda,\gamma}(\mathbb{R}_+, G)$.

2) *If* $p = 1 < \frac{2\lambda+1-\gamma}{\alpha}$, *then the condition* $\frac{\alpha}{2\lambda+1} \leq 1 - \frac{1}{q} \leq \frac{\alpha}{2\lambda+1-\gamma}$ *is necessary and sufficient for the boundedness of* $I_G^\alpha$ *from* $\widetilde{L}_{1,\lambda,\gamma}(\mathbb{R}_+, G)$ *to* $W\widetilde{L}_{q,\lambda,\gamma}(\mathbb{R}_+, G)$.

## 4. Auxiliary results

Further we need the following results.

**Lemma 4.1.** [9] *For $0 < \lambda < \frac{1}{2}$ the following relations are true:*

$$|H(0,r)|_\lambda \approx \begin{cases} \left(sh\frac{r}{2}\right)^{2\lambda+1}, & 0 < r < 2, \\ (ch\frac{r}{2})^{4\lambda}, & 2 \le r < \infty. \end{cases}$$

*Let $\chi_H$ be the characteristic function of $H = H(0,r)$.*

**Lemma 4.2.** *[10]. For $x \in \mathbb{R}_+$, $r > 0$, and $0 < \lambda < \frac{1}{2}$ the following relation*

$$M_G \chi_H(chx) \approx \begin{cases} \left(\frac{sh\frac{r}{2}}{sh\frac{x+r}{2}}\right)^{2\lambda+1}, & 0 < x+r < 2, \\ \left(\frac{sh\frac{r}{2}}{sh\frac{x+r}{2}}\right)^{4\lambda}, & 2 \le x+r < \infty \end{cases}$$

*is valid.*

**Lemma 4.3.** *For every nonnegative function $f(ch\,x)$, $x \in \mathbb{R}_+$ the following relation*

$$\int_0^r A^\lambda_{cht} f(ch\,x) sh^{2\lambda} t dt \approx \int_{H(x,r)} f(ch\,u) sh^{2\lambda} u\, du.$$

*is valid.*

*Proof.* In the work [9] it is proved that (see [9], proof of Theorem 2.1)

$$J(x,r) = \int_0^r A^\lambda_{cht} f(chx) sh^{2\lambda} t\, dt$$

$$= C_\lambda \int_{ch(x-r)}^{ch(x+r)} f(z)(z^2-1)^{\lambda-\frac{1}{2}} \int_{\varphi(z,x,r)} (1-u^2)^{\lambda-1} du\, dz,$$

where $\varphi(z,x,r) = \frac{z\ ch\ x - ch\ r}{\sqrt{z^2-1} sh\ x}$ and $-1 \le \varphi(z,x,r) \le 1, C_\lambda = \frac{\Gamma\left(\lambda+\frac{1}{2}\right)}{\Gamma(\lambda)\Gamma\left(\frac{1}{2}\right)}$.

Then

$$A(z,x,r) = C_\lambda \int_{\varphi(z,x,r)}^1 (1-u^2)^{\lambda-1} du \le C_\lambda \int_{-1}^1 (1-u^2)^{\lambda-1} du = 1.$$

Now estimate the integral $A(z,x,r)$. Let $-1 \le \varphi(z,x,r) \le 0$. Then

$$A(z,x,r) = C_\lambda \int_{\varphi(z,x,r)}^1 (1-u^2)^{\lambda-1} du \ge C_\lambda \int_0^1 (1-u^2)^{\lambda-1} du$$

$$\ge 2^{\lambda-1} C_\lambda \int_0^1 (1-u)^{\lambda-1} du = \frac{2^{\lambda-1}}{\lambda} C_\lambda.$$

Now, let $0 \le \varphi(z,x,r) \le 1$, then

$$A(z,x,r) = C_\lambda \int\limits_{\varphi(z,x,r)}^{1} (1-u)^{\lambda-1}(1+u)^{\lambda-1}du = C_\lambda \int\limits_{0}^{1-\varphi(z,x,r)} u^{\lambda-1}(2-u)^{\lambda-1}du$$

$$= C_\lambda \int\limits_{\frac{1}{1-\varphi(z,x,r)}}^{\infty} u^{-\lambda-1}\left(2-\frac{1}{u}\right)^{\lambda-1} du = C_\lambda \int\limits_{\frac{1}{1-\varphi(z,x,r)}}^{\infty} u^{-2\lambda}\left(2u-1\right)^{\lambda-1} du$$

$$= 2^{2\lambda-1}C_\lambda \int\limits_{\frac{2}{1-\varphi(z,x,r)}}^{\infty} u^{-2\lambda}\left(u-1\right)^{\lambda-1} du = 2^{2\lambda-1}C_\lambda \int\limits_{\frac{1-\varphi(z,x,r)}{1+\varphi(z,x,r)}}^{\infty} (u+1)^{-2\lambda}u^{\lambda-1}du$$

$$= 2^{2\lambda-1}\cdot C_\lambda \int\limits_{0}^{\frac{1+\varphi(z,x,r)}{1-\varphi(z,x,r)}} (1+u)^{-2\lambda}u^{\lambda-1}du \ge 2^{2\lambda-1}C_\lambda \int\limits_{0}^{1} (1+u)^{-2\lambda}u^{\lambda-1}du$$

$$\ge 2^{2\lambda-1}C_\lambda \int\limits_{0}^{1} \frac{u^{\lambda-1}}{(1+u)^{2\lambda}}du \ge \frac{C_\lambda}{2}\int\limits_{0}^{1} u^{\lambda-1}du = \frac{C_\lambda}{2\lambda}\ .$$

Consequently,

$$A(z,x,r) = \int\limits_{\varphi(z,x,r)}^{1} (1-u^2)^{\lambda-1}du \approx 1,$$

and

$$J(x,r) \approx \int\limits_{ch(x-r)}^{ch(x+r)} f(z)(z^2-1)^{\lambda-\frac{1}{2}}dz = \int\limits_{H(x,r)} f\left(chu\right)sh^{2\lambda}udu.$$

□

**Theorem 4.1.** *(Calderon-Zygmund decomposition of $\mathbb{R}^n$). Suppose that $f$ is nonnegative integrable on $\mathbb{R}^+$. Then for any fixed $\alpha > 0$, there exists a sequence $\{H_j(x_j,r_j)\} = \{H_j\}$ of disjoint interval such that*

(1) *$f(chx) \le \alpha$ for a.e. $x \notin \bigcup\limits_j H_j$;*

(2) *$|\bigcup\limits_j H_j|_\lambda \le \frac{1}{\alpha}\|f\|_{L_{1,\lambda}}$;*

(3) *$\alpha < \frac{1}{|H_j|_\lambda}\int_{H_j} f(chy)sh^{2\lambda}ydy \lesssim 2^{(2\lambda+1)n}\alpha$, $n = 1,2,\ldots$.*

The proof of this theorem is similar to Theorem 1.2.1 from [15].

**Theorem 4.2.** *(Fefferman-Stein type inequality)* (*i*) *For every nonnegative measurable functions* $f$ *and* $g$ *on* $\mathbb{R}_+$ *every* $1 \leq p < \infty$ *and every* $0 < t < \infty$,

$$\int_{\mathbb{R}_+} A^\lambda_{cht}(M_G f(chx))^p g(chx) sh^{2\lambda} x dx \lesssim \int_{\mathbb{R}_+} A^\lambda_{cht} f(chx)^p M_G g(chx) sh^{2\lambda} x dx, \tag{4.1}$$

(*ii*) *For any measurable function on* $\mathbb{R}_+$ $f \geq 0$ *and* $g \geq 0$

$$\int_{\{x\in\mathbb{R}_+ : A^\lambda_{cht} M_G f(chx) > \alpha\}} g(chx) sh^{2\lambda} x dx \lesssim \frac{1}{\alpha} \int_{\mathbb{R}_+} A^\lambda_{cht} f(chx) M_G g(chx) sh^{2\lambda} x dx, \tag{4.2}$$

*Proof.* First assertion follows from the inequality (see[3], Theorem 1.4)

$$\int_0^r A^\lambda_{cht}(M_G f(chx))^p g(chx) sh^{2\lambda} x dx \lesssim \int_0^r A^\lambda_{cht} f(chx)^p M_G g(chx) sh^{2\lambda} x dx$$

as $r \to \infty$.

We prove (4.2). Using the relation from Lemma 4.3

$$\int_{H(0,r)} A^\lambda_{cht} f(chx) sh^{2\lambda} t dt \approx \int_{H(x,r)} f(chu) sh^{2\lambda} u du \approx \alpha,$$

we obtain

$$\begin{aligned} &\int_{H_i(0,r)} A^\lambda_{cht} f(chx) M_G g(ch\, x) sh^{2\lambda} x dx \\ &\geq \int_{H_i(0,r)} A^\lambda_{cht} f(chx) \Big( \frac{1}{|H_i(0,r)|_\lambda} \int_{H_i(0,r)} A^\lambda_{cht} g(chx) sh^{2\lambda} y dy \Big) sh^{2\lambda} x dx \\ &\geq \alpha \int_{\{u\in R_+ : M_G f(chu) > \alpha\}} g(chu) sh^{2\lambda} u du. \end{aligned}$$

Summing over $i$, we get

$$\begin{aligned} \int_{\mathbb{R}_+} A^\lambda_{cht} f(chx) M_G g(chx) sh^{2\lambda} x dx &\gtrsim \alpha \int_{\mathbb{R}_+} g(chu) sh^{2\lambda} u du \\ &\gtrsim \alpha \int_{\{u\in\mathbb{R}_+ : M_G f(chu) > \alpha\}} g(chu) sh^{2\lambda} u du. \end{aligned}$$

From this it follows (4.2). □

**Lemma 4.4.** *Let the conditions (3.1) and (3.2) hold.*

*Then for* $1 \leq p < \infty$ *and* $f \in M_{p,\lambda,w}(\mathbb{R}_+, G)$ *we have*

$$\int_{\mathbb{R}_+} A^\lambda_{cht} |f(chx)|^p (M_G \chi_H(chx))^\delta sh^{2\lambda} x dx \lesssim w(r) \|f\|^p_{M_{p,\lambda,w}}.$$

*Proof.* Let $\chi_H$ be the characteristic function of $H(0,r)$. Then $M_G\chi_H \leq 1$. On the other hand, by Lemma 4.2 for $0 < x + r < 2$ we have

$$\begin{aligned}
&\int_{\mathbb{R}_+} A^\lambda_{cht}|f(chx)|^p (M_G\chi_H(cht))^\delta sh^{2\lambda}tdt \\
&\approx \int_0^r A^\lambda_{cht}|f(chx)|^p sh^{2\lambda}tdt \\
&+ \sum_{k=0}^\infty \int_{2^k r}^{2^{k+1}r} \Big(\frac{sh\frac{r}{2}}{sh\frac{x+r}{2}}\Big)^{(2\lambda+1)\delta} A^\lambda_{cht}|f(chx)|^p sh^{2\lambda}tdt \\
&\approx w(r)\Big(\frac{1}{w(r)}\int_0^r A^\lambda_{cht}|f(chx)|^p sh^{2\lambda}tdt\Big) \\
&+ \sum_{k=0}^\infty \int_{2^k r}^{2^{k+1}r} \Big(\frac{sh\frac{r}{2}}{sh(2^{k+1}+1)\frac{r}{2}}\Big)^{(2\lambda+1)\delta} A^\lambda_{cht}|f(chx)|^p sh^{2\lambda}tdt
\end{aligned}$$

(since $shax \geq ashx$ for $a \geq 1$)

$$\begin{aligned}
&\lesssim \big(w(r) + \sum_{k=0}^\infty 2^{-(2\lambda+1)\delta} w(2^{k+1}r)\big)\, \|f\|^p_{M_{p,\lambda,w}} \\
&\lesssim \big(r^{(2\lambda+1)\delta} \sum_{k=0}^\infty \frac{w(2^k r)}{(2\lambda+1)^{(2\lambda+1)\delta}}\big)\, \|f\|^p_{M_{p,\lambda,w}}.
\end{aligned}$$

By (3.1)

$$\frac{w(2^k r)}{(2^k r)^{(2\lambda+1)\delta}} \lesssim \int_{2^k r}^{2^{k+1}r} \frac{w(t)}{t^{(2\lambda+1)\delta+1}}dt,$$

we have

$$\begin{aligned}
&\int_{\mathbb{R}_+} A^\lambda_{cht}|f(chx)|^p (M_G\chi_H(cht))^\delta sh^{2\lambda}tdt \\
&\lesssim \Big(r^{(2\lambda+1)\delta}\int_r^\infty \frac{w(t)}{t^{(2\lambda+1)\delta+1}}dt\Big)\|f\|_{M^p_{p,\lambda,w}} \lesssim w(r)\|f\|_{M^p_{p,\lambda,w}}. \qquad (4.3)
\end{aligned}$$

If $2 \leq x + r < \infty$, then by Lemma 4.2 and previous case we obtain

$$\begin{aligned}
&\int_{\mathbb{R}_+} A^\lambda_{cht}|f(chx)|^p \Big(M_G\chi_H(cht)\Big)^\delta sh^{2\lambda}tdt \\
&\lesssim \Big(r^{4\lambda}\int_r^\infty \frac{w(t)}{t^{4\lambda\delta+1}}dt\Big)\|f\|_{M_{p,\lambda,w}} \lesssim w(r)\|f\|_{M_{p,\lambda,w}}. \qquad (4.4)
\end{aligned}$$

Now the assertion of Lemma 3.4 follows from (4.3) and (4.4). □

## 5. Proofs of the main results

*Proof of Theorem 3.1.* $(i)$ We use (4.1) for $|f|^q$ and $\chi_H \geq 0$, the characteristic function of $H(0,r)$.

Then

$$\int_{H(0,r)} A^\lambda_{cht}(M^q_G f(chx))^p sh^{2\lambda}xdx \lesssim \int_{\mathbb{R}_+} A^\lambda_{cht}|f(chx)|^p M_G\chi_H(chx) sh^{2\lambda}xdx$$

It follows from Lemma 4.4 with $\delta = 1$ that

$$\int_{H(0,r)} A^{\lambda}_{cht}(M^{q}_{G}f(chx))^{p} sh^{2\lambda}x dx \lesssim w(r)\|f\|_{M_{p,\lambda,w}}.$$

Therefore we obtain (3.3).

$(ii)$ We use (4.2). By Lemma 4.4 with $\delta = 1$ we have

$$\Big|\Big\{x \in H(0,r) : A^{\lambda}_{cht} M_G f(chx) > \alpha\Big\}\Big|_{\lambda}$$
$$= \int_{\{x\in\mathbb{R}_+ : A^{\lambda}_{cht} M_G|f|^p(chx)>\alpha^p\}} \chi_H(chx) sh^{2\lambda}x dx$$
$$\lesssim \alpha^{-p} \int_{\mathbb{R}_+} A^{\lambda}_{cht}|f(chx)|^p M\chi_H(chx) sh^{2\lambda}x dx$$
$$\lesssim \alpha^{-p} w(r)\|f\|^p_{M_{p,\lambda,w}}.$$

From this it follows (3.4).

To prove Theorem 3.2 we need the following result (see [4], Theorem 3)

**Theorem 5.1.** [4] *Let* $0 < \lambda < \frac{1}{2}$, $0 < \alpha < 2\lambda + 1$ *and* $1 \le p < \frac{2\lambda+1}{\alpha}$.

*(a) If* $1 < p < \frac{2\lambda+1}{\alpha}$, *then the condition* $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{2\lambda+1}$ *is necessary and sufficient for the boundedness of the operator* $I^{\alpha}_{G}$ *from* $L_{p,\lambda}(\mathbb{R}_+, G)$ *to* $L_{q,\lambda}(\mathbb{R}_+, G)$.

*(b) If* $p = 1$, *the condition is necessary and sufficient for the boundedness of the operator* $I^{\alpha}_{G}$ *from* $L_{1,\lambda}(\mathbb{R}_+, G)$ *to* $L_{q,\lambda}(\mathbb{R}_+, G)$.

Proof of Theorem 3.2. $(i)$ For $f \in M_{p,\lambda,w}(\mathbb{R}_+, G)$ and for $H(0,r)$, let $f = f_1 + f_2$, $f_1 = f\chi_H$. Since $I^{\alpha}_{G}$ is bounded from $L_{p,\lambda}(\mathbb{R}_+, G)$ to $L_{q,\lambda}(\mathbb{R}_+, G)$,

$$\int_{H(0,r)} A^{\lambda}_{cht}|I^{\alpha}_{G} f_1(chx)|^q sh^{2\lambda}x dx \lesssim \|I^{\alpha}_{G} f_1\|^q_{L_{q,\lambda}(H(0,r))}$$
$$\lesssim \|f_1\|^q_{L_{q,\lambda}(H(0,r))} \lesssim \Big(\int_{H(0,r)} A^{\lambda}_{cht}|f(chx)|^p sh^{2\lambda}x dx\Big)^{\frac{q}{p}}.$$

Therefore,

$$\Big(w(r)^{-\frac{q}{p}} \int_{H(0,r)} A^{\lambda}_{cht}|I^{\alpha}_{G} f_1(chx)|^q sh^{2\lambda}x dx\Big)^{\frac{1}{q}}$$
$$\lesssim \Big(\frac{1}{w(r)} \int_{H(0,r)} A^{\lambda}_{cht}|f(chx)|^p sh^{2\lambda}t dt\Big)^{\frac{1}{p}} \lesssim \|f\|_{M_{p,\lambda,w}}. \qquad (5.1)$$

For $x \in H(0,r)$ and for $t \in (r,\infty)$ we have

$$|I^{\alpha}_{G} f_2(chx)| \lesssim \begin{cases} \int_r^\infty \frac{A^{\lambda}_{cht}|f_2(chx)| sh^{2\lambda}t}{(cht)^{2\lambda+1}} dt, & 0 < r < 2, \\ \int_r^\infty \frac{A^{\lambda}_{cht}|f_2(chx)| sh^{2\lambda}t}{(cht)^{4\lambda}} dt, & 2 < r < \infty, \end{cases}$$

$$\lesssim \begin{cases} \int_r^\infty \frac{A^{\lambda}_{cht}|f_2(chx)| sh^{2\lambda}t}{(cht)^{2\lambda+1-\alpha}} dt, & 0 < r < 2, \\ \int_r^\infty \frac{A^{\lambda}_{cht}|f_2(chx)| sh^{2\lambda}t}{(cht)^{4\lambda-\alpha}} dt, & 2 < r < \infty, \end{cases}$$

$$\lesssim \begin{cases} \left(sh\frac{r}{2}\right)^{\alpha-2\lambda+1} \int\limits_r^\infty A^\lambda_{cht}|f_2(chx)|\left(\frac{sh\frac{r}{2}}{sh\frac{t+r}{2}}\right)^{2\lambda+1-\alpha} sh^{2\lambda}tdt, \; 0<r<2, \\ \left(sh\frac{r}{2}\right)^{\alpha-4\lambda} \int\limits_r^\infty A^\lambda_{cht}|f_2(chx)|\left(\frac{sh\frac{r}{2}}{sh\frac{t+r}{2}}\right)^{4\lambda-\alpha} sh^{2\lambda}tdt, \; 2<r<\infty, \end{cases}$$

$$\lesssim \begin{cases} \left(sh\frac{r}{2}\right)^{-(2\lambda+1)(1-\frac{\alpha}{2\lambda+1})} \int\limits_r^\infty A^\lambda_{cht}|f_2(chx)|\left(\frac{sh\frac{r}{2}}{sh\frac{t+r}{2}}\right)^{(2\lambda+1)(1-\frac{\alpha}{2\lambda+1})} sh^{2\lambda}tdt, \; 0<r<2, \\ \left(sh\frac{r}{2}\right)^{-4\lambda(1-\frac{\alpha}{4\lambda})} \int\limits_r^\infty A^\lambda_{cht}|f_2(chx)|\left(\frac{sh\frac{r}{2}}{sh\frac{t+r}{2}}\right)^{4\lambda(1-\frac{\alpha}{4\lambda})} sh^{2\lambda}tdt, \; 2<r<\infty, \end{cases}$$

$$\approx \begin{cases} |H(0,r)|_\lambda^{\frac{\alpha}{2\lambda+1}-1} \int\limits_r^\infty A^\lambda_{cht}|f_2(chx)|(M_G\chi_H(cht))^{1-\frac{\alpha}{2\lambda+1}} sh^{2\lambda}tdt, \; 0<r<2 \\ |H(0,r)|_\lambda^{\frac{\alpha}{4\lambda}-1} \int\limits_r^\infty A^\lambda_{cht}|f_2(chx)|(M_G\chi_H(cht))^{1-\frac{\alpha}{4\lambda}} sh^{2\lambda}tdt, \; 2<r<\infty. \end{cases} \tag{5.2}$$

First we consider the case $0<r<2$ and $0<\alpha<2\lambda+1$. Let $0<\delta<1-\frac{\alpha p}{2\lambda+1}$. By Hölder's inequality, we have

$$\begin{aligned} &|I^\alpha_G f_2(chx)| \\ &\lesssim \frac{1}{|H(0,r)|_\lambda^{1-\frac{\alpha}{2\lambda+1}}} \int_r^\infty A^\lambda_{cht}|f_2(chx)|(M_G\chi_H(chx))^{\frac{\delta}{p}}(M_G\chi_H(cht))^{1-\frac{\alpha}{2\lambda+1}-\frac{\delta}{p}} sh^{2\lambda}tdt \\ &\lesssim \frac{1}{|H(0,r)|_\lambda^{1-\frac{\alpha}{2\lambda+1}}} \left(\int_r^\infty A^\lambda_{cht}|f_2(chx)|^p(M_G\chi_H(cht))^\delta sh^{2\lambda}tdt\right)^{\frac{1}{p}} \\ &\qquad \times \left(\int_r^\infty (M_G\chi_H(cht))^{p-\frac{\alpha p}{2\lambda+1}-\delta} sh^{2\lambda}tdt\right)^{\frac{p-1}{p}} \\ &\qquad = \frac{1}{|H(0,r)|_\lambda^{1-\frac{1}{p}+\frac{1}{q}}} \left(\int_r^\infty A^\lambda_{cht}|f_2(chx)|^p(M_G\chi_H(cht))^\delta sh^{2\lambda}tdt\right)^{\frac{1}{p}} \\ &\qquad \times \left(\int_r^\infty (M_G\chi_H(cht))^{\frac{p-\frac{\alpha p}{2\lambda+1}-\delta}{p-1}} sh^{2\lambda}tdt\right)^{\frac{p-1}{p}} \\ &\qquad = \frac{1}{|H(0,r)|_\lambda^{\frac{1}{q}}} \left(\int_r^\infty A^\lambda_{cht}|f_2(chx)|^p(M_G\chi_H cht)^\delta sh^{2\lambda}tdt\right)^{\frac{1}{p}} \\ &\qquad \times \left(\frac{1}{|H(0,r)|_\lambda}\int_r^\infty (M_G\chi_H(cht))^{\frac{p-\frac{\alpha p}{2\lambda+1}-\delta}{p-1}} sh^{2\lambda}tdt\right)^{\frac{p-1}{p}} \end{aligned} \tag{5.3}$$

Further

$$\begin{aligned} &\frac{1}{|H(0,r)|_\lambda}\int_r^\infty (M_G\chi_H(cht))^{\frac{p-\frac{\alpha p}{2\lambda+1}-\delta}{p-1}} sh^{2\lambda}tdt \\ &\approx \frac{1}{|H(0,r)|_\lambda}\int_r^\infty \left(\frac{sh\frac{r}{2}}{sh\frac{t+r}{2}}\right)^{\frac{(2\lambda+1)(p-\delta)-\alpha p}{p-1}} sh^{2\lambda}tdt \end{aligned}$$

$$\lesssim \frac{\left(sh\frac{r}{2}\right)^{\frac{(2\lambda+1)(p-\delta)-\alpha p}{p-1}}}{\left(sh\frac{r}{2}\right)^{2\lambda+1}}\int_r^\infty \frac{sh^{2\lambda}\frac{t}{2}ch^{2\lambda}\frac{t}{2}}{(sh\frac{t}{2})^{\frac{(2\lambda+1)(p-\delta)-\alpha p}{p-1}}}dt$$

$$\lesssim \left(sh\frac{r}{2}\right)^{\frac{(2\lambda+1)(p-\delta)-\alpha p}{p-1}-(2\lambda+1)}\int_r^\infty \frac{sh^{2\lambda}\frac{t}{2}ch\frac{t}{2}}{(sh\frac{t}{2})^{\frac{(2\lambda+1)(p-\delta)-\alpha p}{p-1}}}dt$$

$$\lesssim \left(sh\frac{r}{2}\right)^{\frac{(2\lambda+1)(1-\delta)-\alpha p}{p-1}}\int_r^\infty \frac{d(sh\frac{t}{2})}{(sh\frac{t}{2})^{\frac{(2\lambda+1)(p-\delta)-\alpha p}{p-1}-(2\lambda+1)+1}}$$

$$\lesssim \left(sh\frac{r}{2}\right)^{\frac{(2\lambda+1)(1-\delta)-\alpha p}{p-1}}\int_r^\infty \frac{d(sh\frac{t}{2})}{(sh\frac{t}{2})^{\frac{(2\lambda+1)(1-\delta)-\alpha p}{p-1}+1}} \lesssim 1. \tag{5.4}$$

From (5.3) and (5.4) we obtain

$$|I_G^\alpha f_2(chx)| \lesssim |H(0,r)|_\lambda^{-\frac{1}{q}}\Big\{\int_r^\infty A_{cht}^\lambda |f_2(chx)|^p (M_G\chi_H(chx))^\delta sh^{2\lambda}tdt\Big\}^{\frac{1}{p}} \tag{5.5}$$

for $0 < x + r < 2$ and $0 < \alpha < 2\lambda + 1$.

Now we consider the case $2 \le x + r < \infty$ and $0 < \alpha \le 4\lambda$.

Let $0 < \delta < 1 - \frac{\alpha p}{4\lambda}$. By Hölder's inequality we have

$$|I_G^\alpha f_2(chx)|$$

$$\lesssim \frac{1}{|H(0,r)|_\lambda^{1-\frac{\alpha}{4\lambda}}}\int_r^\infty A_{cht}^\lambda |f_2(chx)|(M_G\chi_H(cht))^{\frac{\delta}{p}}(M_G\chi_H(cht))^{1-\frac{\alpha}{4\lambda}-\frac{\delta}{p}}sh^{2\lambda}tdt$$

$$\lesssim \frac{1}{|H(0,r)|_\lambda^{1-\frac{\alpha}{4\lambda}}}\left(\int_r^\infty A_{cht}^\lambda |f_2(chx)|^p (M_G\chi_H(cht))^\delta sh^{2\lambda}tdt\right)^{\frac{1}{p}}$$

$$\times\frac{\left(sh\frac{r}{2}\right)^{1-2\lambda}}{\left(sh\frac{r}{2}\right)^{2\lambda+1-\alpha}}\left(\int_r^\infty A_{cht}^\lambda |f_2(chx)|^p (M_G\chi_H(cht))^{\frac{p-\frac{\alpha p}{4\lambda}-\delta}{p-1}}sh^{2\lambda}tdt\right)^{\frac{p-1}{p}}$$

$$\lesssim \frac{\left(sh\frac{r}{2}\right)^{1-2\lambda}}{\left(sh\frac{r}{2}\right)^{2\lambda+1-\alpha}}\left(\int_r^\infty A_{cht}^\lambda |f_2(chx)|^p (M_G\chi_H(cht))^\delta sh^{2\lambda}tdt\right)^{\frac{1}{p}}$$

$$\times\left(\int_r^\infty \Big(\frac{sh\frac{r}{2}}{sh\frac{t+r}{2}}\Big)^{\frac{4\lambda(p-\delta)-\alpha p}{p-1}}sh^{2\lambda}tdt\right)^{\frac{p-1}{p}}$$

$$\lesssim \frac{|H(0,r)|_\lambda^{\frac{1-2\lambda}{1+2\lambda}}}{|H(0,r)|_\lambda^{\frac{1}{q}+1-\frac{1}{q}}}\left(\int_r^\infty A_{cht}^\lambda |f_2(chx)|^p (M_G\chi_H(cht))^\delta sh^{2\lambda}tdt\right)^{\frac{1}{p}}$$

$$\times\frac{\left(sh\frac{r}{2}\right)^{1-2\lambda}}{|H(0,r)|_\lambda^{\frac{1}{q}}}\left(\frac{1}{\left(sh\frac{r}{2}\right)^{2\lambda+1}}\int_r^\infty \frac{sh^{2\lambda}\frac{t}{2}ch^{2\lambda}\frac{t}{2}}{(sh\frac{r}{2})^{\frac{4\lambda(p-\delta)-\alpha p}{p-1}}}dt\right)^{\frac{p-1}{p}}$$

$$\lesssim \frac{1}{|H(0,r)|_\lambda^{\frac{1}{q}}} \left( \int_r^\infty A_{cht}^\lambda |f_2(chx)|^p (M_G \chi_H(cht))^\delta sh^{2\lambda} t dt \right)^{\frac{1}{p}}$$

$$\times \left(sh\frac{r}{2}\right)^{1-2\lambda} \left( \frac{1}{\left(sh\frac{r}{2}\right)^{2\lambda+1}} \int_r^\infty \frac{d(sh\frac{t}{2})}{\left(sh\frac{t}{2}\right)^{\frac{4\lambda(1-\delta)-\alpha p}{p-1}+1}} \right)^{\frac{p-1}{p}} \tag{5.6}$$

We estimate the expression

$$\frac{1}{\left(sh\frac{r}{2}\right)^{2\lambda+1}} \int_r^\infty \frac{d(sh\frac{t}{2})}{\left(sh\frac{t}{2}\right)^{\frac{4\lambda(1-\delta)-\alpha p}{p-1}+1}} \approx \frac{1}{\left(sh\frac{r}{2}\right)^{2\lambda+1+\frac{4\lambda(1-\delta)-\alpha p}{p-1}}}.$$

From (5.6) we get

$$\left(sh\frac{r}{2}\right)^{1-2\lambda} \left( \frac{1}{\left(sh\frac{r}{2}\right)^{2\lambda+1}} \int_r^\infty \frac{d(sh\frac{t}{2})}{\left(sh\frac{t}{2}\right)^{\frac{4\lambda(1-\delta)-\alpha p}{p-1}+1}} \right)^{\frac{p-1}{p}}$$
$$\lesssim \frac{1}{\left(sh\frac{r}{2}\right)^{\frac{1}{p}(4\lambda(1-\delta)+(4\lambda-\alpha)p-2\lambda-1)}} \lesssim 1.$$

From this and (5.6) we obtain

$$|I_G^\alpha f_2(chx)| \lesssim \frac{1}{|H(0,r)|_\lambda^{\frac{1}{q}}} \left( \int_r^\infty A_{cht}^\lambda |f_2(chx)|^p (M_G X_H(cht))^\delta sh^{2\lambda} t dt \right)^{\frac{1}{p}} \tag{5.7}$$

for $2 \le x + r < \infty$ and $0 < \alpha \le 4\lambda$.

It remains to consider the case $2 \le x + r < \infty$ and $4\lambda < \alpha < 2\lambda + 1$.

Let $\delta < 1 - \frac{(8\lambda-\alpha)p}{4\lambda}$,

$$|I_G^\alpha f_2(chx)| \lesssim \int_r^\infty A_{cht}^\lambda |f_2(chx)| \frac{sh^{2\lambda} t}{(cht)^{2\lambda+1}} dt$$
$$\lesssim \int_r^\infty A_{cht}^\lambda |f_2(chx)| \frac{sh^{2\lambda} t}{ch^\alpha t} dt$$
$$\lesssim \frac{1}{\left(sh\frac{r}{2}\right)^{\alpha-4\lambda}} \int_r^\infty A_{cht}^\lambda |f_2(chx)| \left( \frac{sh\frac{r}{2}}{sh\frac{t+r}{2}} \right)^{\alpha-4\lambda} sh^{2\lambda} t dt$$
$$\lesssim \frac{1}{\left(sh\frac{r}{2}\right)^{\alpha-4\lambda}} \int_r^\infty A_{cht}^\lambda |f_2(chx)| (M_G \chi_H(cht))^{\frac{\alpha-4\lambda}{4\lambda}} sh^{2\lambda} t dt$$
$$\lesssim \frac{1}{\left(sh\frac{r}{2}\right)^{\alpha-4\lambda}} \int_r^\infty A_{cht}^\lambda |f_2(chx)| (M_G \chi_H(cht))^{\frac{\delta}{p}} (M_G \chi_H(cht))^{\frac{\alpha-4\lambda}{4\lambda}-\frac{\delta}{p}} sh^{2\lambda} t dt.$$

By Hölder's inequality we have

$$|I_G^\alpha f_2(chx)| \lesssim \frac{1}{\left(sh\frac{r}{2}\right)^{\alpha-4\lambda}} \left( \int_r^\infty A_{cht}^\lambda |f_2(chx)|^p (M_G \chi_H(cht))^\delta sh^{2\lambda} t dt \right)^{\frac{1}{p}}$$
$$\times \left( \int_r^\infty (M_G \chi_H(cht))^{(\frac{\alpha-4\lambda}{4\lambda}-\frac{\delta}{p})\frac{p-1}{p}} sh^{2\lambda} t dt \right)^{\frac{p-1}{p}}$$

$$= \frac{\left(sh\frac{r}{2}\right)^{2\lambda+1-\alpha}}{\left(sh\frac{r}{2}\right)^{2\lambda+1-\alpha}\left(sh\frac{r}{2}\right)^{\alpha-4\lambda}} \left(\int_r^\infty A_{cht}^\lambda |f_2(chx)|(M_G\chi_H(cht))^\delta sh^{2\lambda}tdt\right)^{\frac{1}{p}}$$

$$\times\left(\int_r^\infty \left(\frac{sh\frac{r}{2}}{sh\frac{t+r}{2}}\right)^{\frac{(\alpha-4\lambda)p-4\lambda\delta}{p-1}} sh^{2\lambda}tdt\right)^{\frac{p-1}{p}}$$

$$\lesssim \frac{\left(sh\frac{r}{2}\right)^{6\lambda+1-2\alpha}}{|H(0,r)|_\lambda^{\frac{1}{q}+1-\frac{1}{p}}} \left(\int_r^\infty A_{cht}^\lambda |f_2(chx)|^p(M_G\chi_H(cht))^\delta sh^{2\lambda}tdt\right)^{\frac{1}{p}}$$

$$\times\left(\int_r^\infty \left(\frac{sh\frac{r}{2}}{sh\frac{t+r}{2}}\right)^{\frac{(\alpha-4\lambda)p-4\lambda\delta}{p-1}} sh^{2\lambda}tdt\right)^{\frac{p-1}{p}}$$

$$\lesssim \frac{1}{|H(0,r)|_\lambda^{\frac{1}{q}}} \left(\int_r^\infty A_{cht}^\lambda |f_2(chx)|(M_G\chi_H(cht))^\delta sh^{2\lambda}tdt\right)^{\frac{1}{p}}$$

$$\times\left(sh\frac{r}{2}\right)^{2\lambda+1-\alpha}\left(\frac{1}{\left(sh\frac{r}{2}\right)^{2\lambda+1}}\int_r^\infty \frac{sh^{2\lambda}\frac{t}{2}ch^{2\lambda}\frac{t}{2}}{\left(sh\frac{t}{2}\right)^{\frac{(\alpha-4\lambda)p-4\lambda\delta}{p-1}}}dt\right)^{\frac{p-1}{p}}. \tag{5.8}$$

Further

$$\frac{1}{\left(sh\frac{r}{2}\right)^{2\lambda+1}}\int_r^\infty \frac{sh^{2\lambda}\frac{t}{2}ch^{2\lambda}\frac{t}{2}}{\left(sh\frac{t}{2}\right)^{\frac{(\alpha-4\lambda)p-4\lambda\delta}{p-1}}}dt$$

$$\le \frac{1}{\left(sh\frac{r}{2}\right)^{2\lambda+1}}\int_r^\infty \frac{d(sh\frac{t}{2})}{\left(sh\frac{t}{2}\right)^{\frac{(\alpha-4\lambda)p-4\lambda\delta}{p-1}-4\lambda+1}}$$

$$\lesssim \frac{1}{\left(sh\frac{r}{2}\right)^{2\lambda+1}}\cdot\frac{1}{\left(sh\frac{r}{2}\right)^{\frac{(\alpha-4\lambda)p-4\lambda\delta-4\lambda(p-1)}{p-1}}}$$

$$\lesssim \frac{1}{\left(sh\frac{r}{2}\right)^{\frac{(\alpha-8\lambda)p+4\lambda-4\lambda\delta+(2\lambda+1)(p-1)}{p-1}}} = \frac{1}{\left(sh\frac{r}{2}\right)^{\frac{(\alpha-6\lambda+1)p-4\lambda\delta+2\lambda-1}{p-1}}}.$$

From this and (5.8) we obtain

$$\left(sh\frac{r}{2}\right)^{2\lambda+1-\alpha}\left(\frac{1}{\left(sh\frac{r}{2}\right)^{2\lambda+1}}\int_r^\infty \frac{sh^{2\lambda}\frac{t}{2}ch^{2\lambda}\frac{t}{2}}{\left(sh\frac{t}{2}\right)^{\frac{(\alpha-4\lambda)p-4\lambda\delta}{p-1}}}dt\right)^{\frac{p-1}{p}}$$

$$\lesssim \frac{1}{\left(sh\frac{r}{2}\right)^{\frac{(\alpha-6\lambda+1)p-4\lambda\delta+2\lambda-1-(2\lambda+1-\alpha)p}{p}}}$$

$$\lesssim \frac{1}{\left(sh\frac{r}{2}\right)^{\frac{(2\alpha-8\lambda)p+2\lambda-1-4\lambda\delta}{p}}} \lesssim \frac{1}{\left(sh\frac{r}{2}\right)^{\frac{(\alpha-4\lambda)p+2\lambda-1-4\lambda\delta}{p}}} \lesssim 1,$$

From this and (5.8), we have

$$|I_G^\alpha f_2(chx)| \lesssim |H(0,r)|_\lambda^{-\frac{1}{q}}\left(\int_r^\infty A_{cht}^\lambda |f_2chx|^p(M_G\chi_H(cht))^\delta sh^{2\lambda}tdt\right)^{\frac{1}{p}}. \tag{5.9}$$

Combining(5.5), (5.7) and (5.9), by Lemma 4.4 we obtain

$$|I_G^\alpha f_2(chx)| \lesssim |H(0,r)|_\lambda^{-\frac{1}{q}} \left(\int_r^\infty A_{cht}^\lambda |fchx|^p (M_G\chi_H(cht))^\delta sh^{2\lambda}tdt\right)^{\frac{1}{p}}$$
$$\lesssim |H(0,r)|_\lambda^{-\frac{1}{q}} w(r)^{\frac{1}{p}} \|f\|_{M_{p,\lambda,w}}, \text{ for } x \in H(0,r)$$

and

$$\left\{w(r)^{-\frac{q}{p}} \int_{H(0,r)} |I_G^\alpha f_2(chx)|^q sh^{2\lambda}xdx\right\}^{\frac{1}{q}} \lesssim \|f\|_{M_{p,\lambda,w}}. \tag{5.10}$$

By (5.1) and (5.10) we get (3.5).

(*ii*) For $f \in L_{1,\lambda,w}(\mathbb{R}_+, G)$ and for $f \in H(0,r)$ let $f = f_1 + f_2$, $f_1 = f\chi_H$. By Theorem 5.1, $I_G^\alpha$ is bounded from $L_{1,\lambda}(\mathbb{R}_+, G)$ to $WL_{q,\lambda}(\mathbb{R}_+, G)$

$$|\{x \in H(0,r) : |I_G^\alpha f_1(chx)| > \beta\}|_\lambda \lesssim \left(\frac{1}{\beta}\|f_1\|_{L_{1,\lambda}}\right)^q \lesssim \left(\frac{w(r)}{\beta}\|f\|_{M_{1,\lambda,w}}\right)^q. \tag{5.11}$$

It follows from (5.2) and Lemma 4.4 with $p = 1$, $\delta = 1 - \frac{\alpha}{2\lambda+1} = \frac{1}{q}$ that at $0 < r < 2$

$$|I_G^\alpha f_2(chx)| \lesssim |H(0,r)|_\lambda^{-\frac{1}{q}} \int_r^\infty A_{cht}^\lambda |f_2(ch\,x)|^p (M_G\chi_H(cht))^{\frac{1}{q}} sh^{2\lambda}tdt$$
$$\lesssim |H(0,r)|_\lambda^{-\frac{1}{q}} w(r)\|f\|_{M_{1,\lambda,w}} \text{ for } x \in H(0,r). \tag{5.12}$$

And suppose $\delta = 1 - \frac{\alpha}{4\lambda} = \frac{1}{q}$ at $2 \le r < \infty$

$$|I_G^\alpha f_2(chx)| \lesssim |H(0,r)|_\lambda^{-\frac{1}{q}} w(r)\|f\|_{M_{1,\lambda,w}}, \text{ for } x \in H(0,r). \tag{5.13}$$

From (5.12) and (5.13) we have

$$|\{x \in H(0,r)\} : |I_G^\alpha f_2(chx)| > \beta|_\lambda \lesssim \int_{H(0,r)} \left(\frac{A_{cht}^\lambda |I_G^\alpha f_2(chx)|}{\beta}\right)^q sh^{2\lambda}tdt$$
$$\lesssim \left(\frac{w(r)}{\beta}\|f\|_{M_{1,\lambda,w}}\right)^q. \tag{5.14}$$

Combining (5.11) and (5.14) we obtain (3.6).

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Elman J. Ibrahimov
*Institute of Mathematics and Mechanics of NAS of Azerbaijan, Baku, Azerbaijan*
E-mail address: elmanibrahimov@yahoo.com

Saadat A. Jafarova
*Azerbaijan State Economic University, AZ1001, Baku, Azerbaijan*
E-mail address: sada-jafarova@rambler.ru

S. Elifnur Ekincioglu
*Department of Mathematics, Dumlupinar University, 43100 Kutahya, Turkey*
E-mail address: elifnurekincioglu@gmail.com

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