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ON FREDHOLM PROPERTY OF A BOUNDARY VALUE PROBLEM FOR THIRD ORDER OPERATOR-DIFFERENTIAL EQUATIONS

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Abstract. In the paper we obtain conditions of Fredholm property for third order operator-differential equations in a finite domain. All conditions were expressed by the coefficients of the given equation.

1. Introduction

Let H be Hilbert space, A be a positive definite self-adjoint operator in H. Obviously, the domain of definition of the operator A^γ $(\gamma \geq 0)$ - $D(A^\gamma)$ becomes a Hilbert space H_γ with respect to the scalar product $(x,y)_\gamma = (A^\gamma x, A^\gamma y)$. We consider $H_0 = H$.

Denote by $L_2((0,1):H)$ a Hilbert space of functions $f(t)$ determined on the interval $(0,1)$ with the values in H, moreover

$$\|f\|_{L_2((0;1);H)} = \left(\int_0^1 \|f(t)\|^2 dt \right)^{\frac{1}{2}}.$$

We introduce the Hilbert space

$$W_2^3((0,1):H) = \{u, u''' \in L_2((0,1):H), A^3 u \in L^2((0,1):H)\}$$

with the norm

$$\|u\|_{W_2^3((0,1):H)} = \left(\|u'''\|^2_{L_2((0,1):H)} + \|A^3 u\|^2_{L_2((0,1:H))} \right)^{1/2}$$

From the theorem on traces [3, Theorem 3.1] it follows that

$$\overset{\circ}{W}{}_2^3((0,1):H) = \{u : u \in W_2^3((0,1):H), u(0) = 0, u'(1) = 0, u''(1) = 0\}$$

is a complete subspace of the space $W_2^3((0,1):H)$.

Note that here and in the sequel, the derivatives are understood in the sense of distribution theory [3,p.14,15]. The space $W_2^3(R,H)$ is defined in the same way, where $R = (-\infty, +\infty)$.

From the theorem on intermediate derivatives and on traces [3, Theorem 2.3, Theorem 3.1], it follows that, if $u \in W_2^3((0,1):H)$, then $A^{3-k}u^{(k)} \in L_2((0,1):$

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$H)$, $u^{(k)}(0) \in H_{3-k-\frac{1}{2}}$, $u^{(k)}(1) \in H_{3-k-\frac{1}{2}}$ $(k = 0,1,2)$ and $\|A^{3-k}u^{(k)}\|_{L_2((0,1):H)} \leq const\|u\|_{W_2^3((0,1):H)}$, $\|u^{(k)}(0)\|_{3-k-\frac{1}{2}} \leq const\|u\|_{W_2^3((0,1):H)}$, $\|u^{(k)}(1)\|_{3-k-\frac{1}{2}} \leq$ $\leq const\|u\|_{W_2^3((0,1):H)}$.

It is known that $e^{-At}\varphi \in W_2^3((0,1):H)$, if and only if $\varphi \in H_{\frac{5}{2}}$, where e^{-At} is a semigroup of bounded operators generated by the operator $(-A)$.

In the space $\overset{\circ}{W}{}_2^3((0,1):H)$ consider the following operator

$$Lu = u''' + A^3 u + \sum_{j=0}^{2}(A_{3-j} + T_{3-j})u^{(j)}$$

where the coefficients of the operator L satisfy the conditions:

(1) A is a positive-definite self-adjoint operator with completely continuous inverse: A^{-1};
(2) The operators $B_j = A_j A^{-j}$ $(j = 1,2,3)$ are bounded in H;
(3) The operators $K_j = T_j A^{-j}$ $(j = 1,2,3)$ are completely continuous in H.

Obviously, from the theorem on intermediate derivatives it follows that, subject to the conditions 1)-3), L is a bounded operator acting from $\overset{\circ}{W}{}_2^3((0,1):H)$ to $L_2((0,1):H)$.

In this paper, we find some sufficient conditions that provide Fredholm property of the operator L, i.e., $\dim Ker\, L = \operatorname{co}\dim ImL < \infty$ and ImL is a closed set in $L_2((0,1):H)$.

Then we apply this theorem on Fredholm solvability to the boundary value problem

$$u'''(t) + A^3 u(t) + \sum_{j=0}^{2}(A_{3-j} + T_{3-j})u^{(j)}(t) = f(t),\ t \in (0,1) \tag{1.1}$$

$$u(0) = 0, u'(1) = 0, u''(1) = 0. \tag{1.2}$$

Note that, unique solvability of the problem of type (1.1),(1.2) was studied in [8] for $A_j = T_j = 0$, $j = 1,2,3$ and for $T_j = 0$ in [1,2,4].

For second order operator-differential equations the unique solvability of the problem was considered for example in [5,6,7].

2. Some results

In the space $\overset{\circ}{W}{}_2^3((0,1):H)$ we determine the operators

$$P_0 u = u''' + A^3 u,\ P_1 u = \sum_{j=0}^{2} A_{3-j}u^{(j)},\ Tu = \sum_{j=0}^{2} T_{3-j}u^{(j)}.$$

The operator

$$Lu = P_0 u + P_1 u + Tu \tag{2.1}$$

from the space $\overset{\circ}{W}{}_2^3((0,1):H)$ to $L_2((0,1):H)$ is bounded. We have the following theorem.

Theorem 2.1. *The operator P_0 isomorphically maps the space $\overset{\circ}{W}{}_2^3((0,1):H)$ onto $L_2((0,1):H)$.*

Proof. First we show that $KerP_0=\{0\}$. Indeed, for $u\in\overset{\circ}{W}{}_2^3((0,1):H)$ we have

$$\|P_0u\|^2_{L_2((0,1):H)}=\|u'''+A^3u\|^2_{L_2((0,1):H)}=$$

$$=\|u'''\|^2_{L_2((0,1):H)}+\|A^3u\|^2_{L_2((0,1):H)}+2Re(u''',A^3u)_{L_2((0,1):H)}. \tag{2.2}$$

Since for $u\in\overset{\circ}{W}{}_2^3((0,1):H)$ by integration by parts we get

$$(u''',A^3u)_{L_2((0,1):H)}=\|A^{3/2}u'(0)\|^2-(A^3u,u''')_{L_2((0,1):H)},$$

we have $2Re(u''',A^3u)_{L_2((0,1):H)}=\|A^{\frac{3}{2}}u'(0)\|^2$. Then taking into account this equality in equality (2.2), we have

$$|P_0u\|^2_{L_2((0,1):H)}=\|u'''\|^2_{L_2((0,1):H)}+\|A^3u\|^2_{L_2((0,1):H)}+\|u'(0)\|^2_{\frac{3}{2}}. \tag{2.3}$$

From the condition $P_0u=0$ it follows that $u=0$, i.e. $KerP_0=\{0\}$. We now prove that $ImP_0=L_2((0,1):H)$, i.e. the equation $P_0u=f$ has a solution for any $f\in L_2((0,1):H)$. Let $f_1(t)=f(t)$ $t\in(0,1)$ and $f_1(t)=0$ for $R\backslash(0,1)$. Then $f_1(t)\in L_2(R:H)$ and $\|f_1\|_{L_2(R:H)}=\|f\|_{L_2((0,1):H)}$. It is easy to see that,

$$u_1(t)=\frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{+\infty}((i\xi)^3+A^3)^{-1}\hat{f}_1(\xi)e^{i\xi t}d\xi,$$

where $\hat{f}_1(\xi)$ is a Fourier transformation of the function $f_1(t)$ and satisfies the equation $u'''(t)+A^3u(t)=f_1(t)$ almost everywhere in $R=(-\infty,\infty)$. On the other hand, from the Plancherel theorem it follows that

$$\|u_1(t)\|^2_{W_2^3(R:H)}=\|u_1'''\|^2_{L_2((0,1):H)}+\|A^3u_1\|^2_{L_2(R:H)}=\|\xi^3\hat{u}_1(\xi)\|^2_{L_2(R:H)}+$$

$$+\|A^3\hat{u}_1(\xi)\|^2_{L_2((0,1):H)}=\|\xi^3((i\xi)^3+A^3)^{-1}f_1(\xi)\|^2_{L_2(R)}+\|A^3((i\xi)^3+A^3)^{-1}\hat{f}_1(\xi)\|^2_{L_2(R)}\leq$$

$$\leq\left(\sup_{\xi\in R}\|\xi^3((i\xi)^3+A^3)^{-1}\|^2+\sup_{\xi\in R}\|A^3((i\xi)^3+A^3)^{-1}\|^2\right)\cdot\|\hat{f}_1(\xi)\|\|^2_{L_2(R;H)}=$$

$$=const\|f\|^2_{L_2((0,1):H)}.$$

Here we use the fact that if the operator A is a self-adjoint positive-definite operator, then for any $\xi\in R$

$$\|\xi^3((i\xi)^3+A^3)^{-1}\|=\sup_{\mu\in\sigma(A)}|\xi^3((i\xi)^3+\mu^3)^{-1}|\leq 1$$

and

$$\|A^3((i\xi)^3+A^3)^{-1}\|=\sup_{\mu\in\sigma(A)}|\mu^3((i\xi)^3+\mu^3)^{-1}|\leq 1.$$

Let us denote by $\alpha(t)=u_1(t)$ for $t\in[0,1]$ and $\alpha(t)=0$ for $t\in R\backslash[0,1]$.

Obviously, $\alpha(t)$ satisfies the equation $u'''(t)+A^3u(t)=f(t)$ almost everywhere in $(0,1)$ and $\alpha(t)\in W_2^3((0,1):H)$. Then by the theorem on traces, $\alpha^{(k)}(0)\in$

$H_{3-k-\frac{1}{2}}$, $\alpha^{(k)}(1) \in H_{3-k-\frac{1}{2}}$ $(k = 0, 1, 2)$. Now we will look for the solution of the equation $P_0 u = f$ in the form

$$u(t) = \alpha(t) + e^{-tA}x_1 + e^{\omega(1-t)A}x_2 + e^{\bar{\omega}(1-t)A}x_3,$$

where x_1, x_2 and $x_3 \in H_{\frac{5}{2}}$ are unknown vectors, $\omega = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$. From the condition $u \in \overset{\circ}{W}{}_2^3((0,1):H)$, it follows that

$$x_1 + e^{\omega A}x_2 + e^{\bar{\omega}A}x_3 = -\alpha(0), \quad e^{-A}x_1 + \omega x_2 + \bar{\omega}x_3 = A^{-1}\alpha'(1),$$

$e^{-A}x_1 + \omega^2 x_2 + \bar{\omega}^2 x_3 = -A^{-2}\alpha''(1)$.

Taking into account that the vectors $\alpha(0), A^{-1}\alpha'(1), -A^{-2}\alpha''(1) \in H_{\frac{5}{2}}$, then solving this system we get x_1, x_2 *and* $x_3 \in H_{\frac{5}{2}}$, i.e. $u(t) \in \overset{\circ}{W}{}_2^3((0,1):H)$ Consequently, $ImP_0 = L_2((0,1):H)$. The theorem is proved. □

From this theorem and theorem on intermediate derivatives it follows that the following norms

$$N_j = \sup_{\substack{u \neq 0, \\ u \in \overset{\circ}{W}{}_2^3((0,1):H)}} \|A^{3-j}u^{(j)}\| \cdot \|P_0^{-1}u\|^{-1} \ \ j = 0, 1, 2; \tag{2.4}$$

are finite since the norms $\|u\|_{W_2^3((0,1):H)}$ and $\|P_0^{-1}u\|_{L_2((0,1):H)}$ are equivalent in $\overset{\circ}{W}{}_2^3((0,1):H)$. Next we prove the following result

Theorem 2.2. *We have the following estimations:*

$$N_0 \leq 1; \ \ N_1 \leq 2^{\frac{1}{2}} \cdot 3^{-\frac{1}{4}}, \ \ N_2 \leq 2 \cdot 3^{-\frac{1}{2}}. \tag{2.5}$$

Proof. From equality (2.3) it follows that $N_0 \leq 1$. Prove that $N_2 \leq 2 \cdot 3^{-\frac{1}{2}}$. Obviously, for $u \in \overset{\circ}{W}{}_2^3((0,1):H)$ $(u(0) = 0, u'(1) = 0, u''(1) = 0)$.

$$\|A^2u'\|^2_{L_2((0,1):H)} = (A^2u', A^2u')_{L_2((0,1):H)} = -(A^3u, Au'')_{L_2((0,1):H)} \leq$$
$$\leq \|A^3u\|_{L_2((0,1):H)} \cdot \|Au''\|_{L_2((0,1):H)}. \tag{2.6}$$

Let $\varepsilon > 0$ be any positive number. Then for any $u \in \overset{\circ}{W}{}_2^3((0,1):H)$ we have the equality

$$\|\varepsilon u''' + Au'' + \frac{1}{\varepsilon}Au'\|^2_{L_2((0,1):H)} = \varepsilon^2\|u'''\|^2_{L_2((0,1):H)} + \|Au''\|^2_{L_2((0,1):H)} +$$
$$+ \frac{1}{\varepsilon^2}\|Au'\|^2_{L_2((0,1):H)} + 2\varepsilon Re(u''', A^2u')_{L_2((0,1):H)} + 2Re(u''', Au')_{L_2((0,1):H)} +$$
$$+ \frac{1}{\varepsilon} \cdot 2Re(Au'', A^2u')_{L_2((0,1):H)}. \tag{2.7}$$

Integrating by parts, we have :

$$(u''', Au'')_{L_2((0,1):H)} = -\|u''(0)\|^2 - (Au'', u''')$$

i.e.

$$2\varepsilon Re(u''', Au')_{L_2((0,1):H)} = -\varepsilon\|u''(0)\|^2_{\frac{1}{2}}. \tag{2.8}$$

In the same way we have

$$2 \cdot \frac{1}{\varepsilon}(Au'', A^2u')_{L_2((0,1):H)} = -\frac{1}{\varepsilon}\|u'(0)\|^2_{\frac{3}{2}}. \tag{2.9}$$

Taking into account the equalities (2.8),(2.9), from equality (2.7) we get:

$$\|\varepsilon u''' + Au'' + \frac{1}{\varepsilon}Au'\|^2_{L_2((0,1):H)} = \varepsilon^2\|u'''\|^2_{L_2((0,1):H)} +$$

$$+\frac{1}{\varepsilon^2}\|Au'\|^2_{L_2((0,1):H)} - \|Au''\|^2_{L_2((0,1):H)} - \|\varepsilon^{\frac{1}{2}}A^{\frac{1}{2}}u''(0) + \varepsilon^{-\frac{1}{2}}A^{\frac{3}{2}}u'(0)\|^2.$$

Hence we have

$$\|Au''\|^2_{L_2((0,1):H)} \leq \varepsilon^2\|u'''\|^2_{L_2((0,1):H)} + \frac{1}{\varepsilon^2}\|A^2u'\|^2_{L_2((0,1):H)}.$$

Assuming $\varepsilon = \|A^2u'\|^{\frac{1}{2}}_{L_2((0,1):H)} \cdot \|u'''\|^{-\frac{1}{2}}_{L_2((0,1):H)}$, we get

$$\|Au''\|^2_{L_2((0,1):H)} \leq 2\|u'''\|^2_{L_2((0,1):H)} \cdot \|A^2u'\|^2_{L_2((0,1):H)}. \tag{2.10}$$

Now, substituting the inequality (2.6) in (2.10), we get

$$\|Au''\|^2_{L_2((0,1):H)} \leq 2 \cdot \|u'''\|_{L_2((0,1):H)} \cdot \|A^3u\|^{1/2}_{L_2((0,1):H)} \cdot \|Au''\|^{\frac{1}{2}}_{L_2((0,1):H)}$$

i.e.

$$\|Au''\|_{L_2((0,1):H)} \leq 2^{\frac{2}{3}}\|A^3u\|^{\frac{1}{3}}_{L_2((0,1):H)} \cdot \|u'''\|^{\frac{2}{3}}_{L_2((0,1):H)}.$$

Then

$$\|Au''\|^2 \leq 2^{\frac{4}{3}} \cdot \left(\frac{1}{\eta^2}\|A^3u\|^2_{L_2((0,1):H)}\right)^{\frac{1}{3}} \cdot \left(\tau\|u'''\|^2_{L_2((0,1):H)}\right)^{\frac{2}{3}} \leq$$

$$\leq 2^{\frac{4}{3}}\left(\frac{1}{3}\frac{1}{\eta^2}\|A^3u\|^2_{L_2((0,1):H)} + \frac{2}{3}\tau\|u'''\|^2_{L_2((0,1):H)}\right)$$

for $\tau > 0$.

Assuming $\eta = 2^{-\frac{1}{3}}$, we have

$$\|Au''\|^2_{L_2((0,1):H)} \leq \frac{4}{3}\|u\|^2_{\overset{\circ}{W}{}_2^3((0,1):H)}.$$

Consequently,

$$\|Au''\|_{L_2((0,1):H)} \leq 2 \cdot 3^{-\frac{1}{2}}\|u\|_{W_2^3((0,1):H)}.$$

Now, substituting the inequality in (2.6), we get

$$\|A^2u'\|_{L_2((0,1):H)} \leq 2^{\frac{1}{2}} \cdot 3^{-\frac{1}{4}}\|P_0u\|_{L_2((0,1):H)}.$$

i.e.

$$N_1 \leq 2^{\frac{1}{2}} \cdot 3^{-\frac{1}{4}}; \;\; N_2 \leq 2 \cdot 3^{-\frac{1}{2}}.$$

The Theorem is proved. □

3. Main results

We now prove the following theorem.

Theorem 3.1. *Let conditions 1) and 2) be fulfilled, moreover the operator $B_j = A_j A^{-j}$ $(j = 1, 2, 3)$ satisfy the condition*

$$q = \alpha_0 \|B_3\| + \alpha_1 \|B_2\| + \alpha_2 \|B_1\| < 1, \tag{3.1}$$

where $\alpha_0 = 1$ $\alpha_1 = 2^{\frac{1}{2}} \cdot 3^{-\frac{1}{4}}$, $\alpha_2 = 2 \cdot 3^{-\frac{1}{2}}$. Then the operator $P_0 + P_1$ isomorphically maps the space $\overset{\circ}{W}{}_2^3((0,1):H)$ onto $L_2((0,1):H)$.

Proof. Since P_0 is an isomorphism, after substitution of $P_0 u = v$ we can write the equation $(P_0 + P_1)u = f$ in the form $v + P_1 P_0^{-1} v = f$. Since

$$\|P_1 P_0^{-1} v\|_{L_2((0,1):H)} = \|P_1 u\|_{L_2((0,1):H)} \leq \sum_{j=0}^{2} \|A_{3-j} u^{(j)}\|_{L_2((0,1):H)} \leq$$

$$\leq \sum_{j=0}^{2} \|A_{3-j} A^{-3+j}\| \|A^{3-j} u^{(j)}\|_{L_2((0,1):H)},$$

taking into account the result of theorem 2.2, we get

$$\|P_1 P_0^{-1} v\|_{L_2((0,1):H)} \leq \sum_{j=0}^{2} \|B_{3-j}\| \cdot N_j \cdot \|P_0 u\|_{L_2((0,1):H)} \leq$$

$$\leq \sum_{j=0}^{2} \alpha_j \|B_{3-j}\| \cdot \|v\|_{L_2((0,1):H)} = q \|v\|_{L_2((0,1):H)}.$$

Since $q < 1$, the operator $E + P_1 P_0^{-1}$ is invertible in the space $L_2((0,1):H)$. Then the operator $(P_0 + P_1)$ is invertible in $L_2((0,1):H)$ and $(P_0 + P_1)^{-1} = P_0^{-1}(E + P_1 P_0^{-1})^{-1}$. Hence it follows that $\|u\|_{\overset{\circ}{W}{}_2^3((0,1):H)} \leq const \|f\|_{L_2((0,1):H)}$. The theorem is proved. ☐

Theorem 3.2. *Let conditions 1) and 3) be fulfilled. Then the operator T is a completely continuous operator from the space $\overset{\circ}{W}{}_2^3((0,1):H)$ to $L_2((0,1):H)$.*

Proof. Since for $u \in \overset{\circ}{W}{}_2^3((0,1):H)$

$$Tu = T_2 u' + T_1 u'' + T_3 u = K_2 A^2 u' + K_1 A u'' + K_0 A^3 u,$$

where K_0, K_1 and K_2 are completely continuous operators, then it suffices to prove that $K_0 A^3$, $K_2 A^2 \frac{d}{dt}$ and $K_1 A \frac{d^2}{dt^2}$ are completely continuous operators from $\overset{\circ}{W}{}_2^3((0,1):H)$ to $L_2((0,1):H)$. For definiteness we consider the operator $K_1 A \frac{d^2}{dt^2}$. Since K_1 is a completely continuous operator, we can represent it in the form of a finite-dimensional operator and an operator with sufficiently small norm $K_1 = \sum_{i=1}^{n} (\cdot, x_i) y_i + K_1'$: $\|K_1'\| < \varepsilon$ for sufficiently small $\varepsilon > 0$. Obviously, if e_k $(k = 1, 2, ...)$ is an orthonormal basis of the system of eigen-vectors of the operator A, (i.e. $Ae_k = \lambda_k e_k$) then, $x_i = \sum\limits_{p=1}^{n} c_{p_i} e_i + x_i(\varepsilon)$, moreover $\|x_i(\varepsilon)\| \le \varepsilon$. Then for one - dimensional operator $K_{1,i} = (,e_i)y_i$ we have

$$\|K_{1,i}Au''\|_{L_2((0,1):H)} = \lambda_k \|y_i\| \cdot |(u''(t), e_k)|_{L_2((0,1)}.$$

Since for the function $\psi(t) = (u(t), e_k)$ we get

$$\|\psi''(t)\|^2_{L_2((0,1):H)} = \int\limits_0^1 (\psi''(t), \bar{\psi}''(t))dt = \psi'(t) \cdot \bar{\psi}''(t)\big|_0^1 - (\psi'(t) \cdot \bar{\psi}'''(t))_{L_2((0,1)} =$$
$$= (\psi'(t) \cdot \bar{\psi}'''(t))_{L_2((0,1)} \le \|\psi'(t)\| \|\psi'''(t)\|$$

and

$$\|\psi\prime(t)\|^2 = \int\limits_0^1 (\psi'(t), \bar{\psi}'(t))dt = \psi(t) \cdot \psi'(t)\big|_0^1 - (\psi(t), \psi''(t))_{L_2} \le \|\psi(t)\| \cdot \|\psi''(t)\|,$$

we obtain

$$\|\psi\prime(t)\|^2_{L_2((0,1):H)} \le \|\psi(t)\| \cdot \|\psi'(t)\|^{\frac{1}{2}} \cdot \|\psi'''(t)\|^{\frac{1}{2}}$$

i.e.

$$\|\psi'(t)\|_{L_2((0,1):H)} \le \|\psi'''(t)\|^{\frac{1}{3}} \cdot \|\psi(t)\|^{\frac{2}{3}} \le \varepsilon \|\psi'''(t)\| + M(\varepsilon)\|\psi(t)\|_{L_2((0,1)}.$$

Thus,

$$\|K_{1,i}Au''\|_{L_2((0,1):H)} \le \varepsilon \|\psi\|_{W_2^3((0,1):H)} + M(\varepsilon)\|\psi\|_{L_2((0,1):H)}. \tag{3.2}$$

Now, we can show that for any $\varepsilon > 0$

$$\|K_{1,i}Au''\|_{L_2((0,1):H)} \le \varepsilon \|u\|_{W_2^3((0,1):H)} + M(\varepsilon)\|u\|_{L_2((0,1):H)}.$$

Since the embedding $W_2^3((0,1) : H) \subset L_2((0,1) : H)$ is compact, $K_{1,i}A\frac{d^2}{dt^2}$ is a compact operator. Indeed, if $\{u_k\}_{k=1}^{\infty} \subset \overset{\circ}{W}{}_2^3((0,1) : H)$ and $\|u_k\|_{W_2^3((0,1):H)} \le M$, then there exists a sequence $u_{k,n} \subset \{u_k\}$, moreover, $u_{k,n} \to u_0$ in $L_2((0,1) : H)$. Then from inequality 3.2 it follows that

$$\|K_{1i}A(u''_n - u''_m)\|_{L_2((0,1):H)} \le 2\varepsilon \cdot M + M(\varepsilon)\|u_n - u_m\|_{L_2((0,1):H)}.$$

Since as $n, m \to \infty$ $\|u_n - u_m\| \to 0$, and ε is any positive number, we conclude that the sequence $\{K_{1,i}Au''_n\}_{n=1}^{\infty}$ converges in $L_2((0,1) : H)$ i.e. the operator $K_{1,i}$ is a compact operator. Consequently, the operator $K_1 A\frac{d^2}{dt^2}$ is a Fredholm operator from the space $\overset{\circ}{W}{}_2^3((0,1) : H)$ to $L_2((0,1) : H)$.

The theorem has been proved. □

We now prove the basic theorem.

Theorem 3.3. *Let conditions 1)-3) and* (3.1) *be fulfilled. Then* $L : \overset{\circ}{W}{}_2^3((0,1) : H) \to L_2((0,1) : H)$ *is a Fredholm operator.*

Proof. Obviously, $L = P_0 + P_1 + T = (E + T(P_0 + P_1)^{-1})(P_0 + P_1)$.

Since the operator $(P_0 + P_1)$ is an isomorphism, $T(P_0 + P_1)^{-1}$ is a compact operator, then L is a Fredholm operator.

Corollary 3.1. *Let the conditions of theorem 3.3 be fulfilled. Then problem* (1.1),(1.2) *is Fredholm solvable.*

□

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