

ON NEGATIVE EIGENVALUES OF THE SCHRÖDINGER OPERATOR

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Abstract. In the paper, we consider the Schrödinger operator in an electric field, depending on the intensity of the magnetic field. Under certain constraints imposed on the electric potential, we prove its self-adjointness. It is proved that if the exact lower bound of the electric field is negative, then with an increase in the intensity of the magnetic field below the threshold of the essential spectrum, negative eigenvalues appear. We establish that as the magnetic field intensity increases, the number of negative eigenvalues of the Schrödinger operator increases.

1. Introduction

From Kato's magnetic inequality (see, for example, [6, p. 28]), it follows that under the influence of an external magnetic field, the energy of the system increases. For this reason, a superconducting material in a magnetic field loses its superconducting quality. We can expect that if the exact lower bound of the electric potential $V(x)$ is negative, then at sufficiently large values of the magnetic field strength μ , due to the potential $\mu^2 V(x)$ of the inductive field, the material will again restore its superconducting quality. This means that the Euler-Lagrange equation corresponding to the Ginsburg-Landau functional, i.e., Schrödinger equation

$$-\Delta u(x) + \mu^2 V(x)u(x) = \lambda u(x),$$

for some negative values of the spectral parameter λ , will have a nontrivial solution from the space $L_2(R^n)$.

The objective of the paper is to give a rigorous mathematical proof of the above-mentioned heuristic reasoning.

In the space $L_2(R^n)$ we introduce the operator H_μ^0 with the domain $D(H_\mu^0) = C_0^\infty(R^n)$, operating according to the rule

$$H_\mu^0 \psi(x) = -\Delta \psi(x) + \mu^2 V(x)\psi(x),$$

where Δ is the Laplace operator; $C_0^\infty(R^n)$ is the class of infinitely differentiable complex-valued functions with compact support in R^n ; μ the magnetic field

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strength; and $V(x)$ the real function defined in the n -dimensional Euclidean space R^n .

We impose the following constraints on the function $V(x)$:

- a) $V(x) \in C^2(R^n)$; b) $\lim_{x \rightarrow \infty} V(x) = 0$; c) $\inf_{x \in R^n} V(x) < 0$.

We denote the closure of the operator H_μ^0 by H_μ and call it the Schrödinger operator. In the work, we establish the self-adjointness of the Schrödinger operator H_μ (i.e., the essentially self-adjointness of the operator H_μ^0), study its spectrum and prove that as the magnetic field intensity μ increases below the threshold of the essential spectrum, negative eigenvalues of the operator H_μ appear.

Note that recently there has been great interest in the Schrödinger magnetic operator (see [1, 2, 3, 5, 6, 7, 11-16]). Many works are dedicated to issues of self-adjointness (see [1, 6, 11]), to the study of negative eigenvalues (see [2, 4, 5, 8, 10, 14, 15]), which play a key role in various fields of physics and chemistry, as well as to the study of the essential spectrum of the Schrödinger operator (see [3, 7]).

2. Self -Adjointness

First, using the self-adjointness criterion (see [13, p. 283, Theorem VIII. 3]) and the Fournais-Helffer method (see [6, p. 7, Theorem 1.2.2]), we prove the essential self-adjointness of the operator H_μ^0 .

Theorem 2.1. *Let conditions a) and b) be satisfied. Then the operator H_μ^0 is essentially self-adjoint on $C_0^\infty(R^n)$.*

Proof. From conditions a) and b) it follows that the operator H_μ^0 is bounded below on $C_0^\infty(R^n)$. Therefore, one can find a positive number t such for any function $\psi(x)$ from the space $C_0^\infty(R^n)$ the following inequality will hold true

$$(H_\mu^0 + tE)\psi(x), \psi(x) \geq \|\psi(x)\|^2, \tag{2.1}$$

where E is the identity operator, (\cdot, \cdot) and $\|\cdot\|$ the scalar product and norm, respectively, in the space $L_2(R^n)$.

Let's show that the domain of the operator $H_\mu^0 + tE$ is everywhere dense in $L_2(R^n)$. Suppose that there exists a function $f(x)$ from $L_2(R^n)$ such that

$$(f, (H_\mu^0 + tE) \psi(x)) = 0, \quad \forall \psi(x) \in C_0^\infty(R^n). \tag{2.2}$$

It follows from condition (2.2) that, in the sense of generalized functions, a regular generalized function $f(x)$ satisfies the equation

$$-\Delta f(x) + \mu^2 V(x)f(x) + tf(x) = 0. \tag{2.3}$$

From this equality and condition a) it follows that $f(x) \in W_{2,loc}^2(R^n)$, where $W_{2,loc}^2(R^n)$ is the second order local Sobolev space.

Let's consider a sequence $\{C_k(x)\}_{k=1}^\infty$ from the space $C_0^\infty(R^n)$ elements of which are defined as follows:

$$C_k(x) = \begin{cases} 0, & \text{if } |x| \geq 2k, \\ 1, & \text{if } |x| \leq k, \\ \text{sufficiently smooth,} & \text{if } k \leq |x| \leq 2k, \end{cases} \quad k = 1, 2, \dots,$$

where $|x| = \sqrt{x_1^2 + x_2^2 + \dots + x_n^2}$.

It is evident that

$$\lim_{k \rightarrow \infty} C_k(x) = 1 \quad (\textit{pointwise}) \tag{2.4}$$

and

$$\lim_{k \rightarrow \infty} \nabla C_k(x) = 0 \quad (\textit{pointwise}) \tag{2.5}$$

where $\nabla = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \dots, \frac{\partial}{\partial x_n} \right)$.

Using the Green's formula, we calculate the integral

$$\int_{R^n} [-\Delta(C_k^2(x)u(x))] \overline{f(x)} dx,$$

where $u(x) \in C_0^\infty(R^n)$. We then have:

$$\begin{aligned} & \int_{R^n} [-\Delta(C_k^2(x)u(x))] \overline{f(x)} dx = -\sum_{j=1}^n \int_{R^n} \frac{\partial^2(C_k^2(x)u(x))}{\partial x_j^2} \overline{f(x)} dx = \\ & = -\sum_{j=1}^n \int_{R^n} \frac{\partial}{\partial x_j} \left[\frac{\partial(C_k^2(x)u(x))}{\partial x_j^2} \overline{f(x)} \right] dx + \sum_{j=1}^n \int_{R^n} \frac{\partial(C_k^2(x)u(x))}{\partial x_j} \frac{\partial \overline{f(x)}}{\partial x_j} dx = \\ & = \sum_{j=1}^n \int_{R^n} \frac{\partial(C_k^2(x)u(x))}{\partial x_j} \frac{\partial \overline{f(x)}}{\partial x_j} = 2 \sum_{j=1}^n \int_{R^n} C_k(x) \frac{\partial C_k(x)}{\partial x_j} \frac{\partial \overline{f(x)}}{\partial x_j} u(x) dx + \\ & \quad + \sum_{j=1}^n \int_{R^n} C_k^2(x) \frac{\partial u(x)}{\partial x_j} \frac{\partial \overline{f(x)}}{\partial x_j} = \\ & = 2 \int_{R^n} C_k(x) \left[\nabla C_k(x) \cdot \nabla \overline{f(x)} \right] u(x) dx + 2 \int_{R^n} C_k^2(x) \left[\nabla \overline{f(x)} \cdot \nabla u \right] dx. \end{aligned} \tag{2.6}$$

A simple calculation shows that the following formula holds true:

$$\begin{aligned} & \int_{R^n} \nabla C_k(x) \overline{f(x)} \cdot \nabla(C_k(x)u(x)) dx = \int_{R^n} (\nabla C_k(x))^2 \overline{f(x)} u(x) dx + \\ & \quad + \int_{R^n} C_k(x) \nabla C_k(x) \cdot \left[\overline{f(x)} \nabla u(x) - u(x) \nabla \overline{f(x)} \right] dx + \\ & \quad + 2 \int_{R^n} C_k(x) \nabla C_k(x) \cdot \nabla \overline{f(x)} u(x) dx + \int_{R^n} C_k^2(x) \nabla \overline{f(x)} \cdot \nabla u(x) dx. \end{aligned} \tag{2.7}$$

By adding to each side of equality (2.7) the integral

$$\int_{R^n} (\mu^2 V(x) + t) C_k^2(x) \overline{f(x)} u(x) dx$$

and taking into account equalities (2.3) and (2.6), we obtain

$$\begin{aligned} & \int_{R^n} \nabla(C_k(x) \overline{f(x)}) \cdot \nabla(C_k(x) u(x)) dx + \int_{R^n} [\mu^2 V(x) + t] C_k^2(x) \overline{f(x)} u(x) dx = \\ & = \int_{R^n} (\nabla C_k(x))^2 \overline{f(x)} u(x) dx + \int_{R^n} C_k(x) \nabla C_k(x) \cdot [2i \operatorname{Im}(\overline{f(x)} \nabla u(x))] dx. \end{aligned} \tag{2.8}$$

It is evident that $f(x) \in W_{2,loc}^1(R^n)$, as $f(x) \in W_{2,loc}^2(R^n)$. From everywhere density of $C_0^\infty(R^n)$ in $W_{2,loc}^1(R^n)$ it follows that equality (2.8) holds true even for $u(x) = f(x)$. Substituting the function $f(x)$ into $u(x)$ in equality (2.8), we obtain

$$\begin{aligned} & \int_{R^n} |\nabla(f(x) C_k(x))|^2 dx + \int_{R^n} [\mu^2 V(x) + t] C_k^2(x) |f(x)|^2 dx = \\ & = \int_{R^n} (\nabla C_k(x))^2 |f(x)|^2 dx + \int_{R^n} C_k(x) \nabla C_k(x) [2i \operatorname{Im}(\overline{f(x)} \nabla f(x))] dx. \end{aligned}$$

Hence we have

$$\begin{aligned} & \int_{R^n} |\nabla(f(x) C_k(x))|^2 dx + \int_{R^n} [\mu^2 V(x) + t] C_k^2(x) |f(x)|^2 dx = \\ & = \int_{R^n} (\nabla C_k(x))^2 |f(x)|^2 dx. \end{aligned} \tag{2.9}$$

From inequality (2.1) we obtain:

$$\begin{aligned} & \|C_k(x) f(x)\|^2 \leq ((H_\mu^0 + tE) C_k(x) f(x), C_k(x) f(x)) = \\ & = \int_{R^n} |\nabla(f(x) C_k(x))|^2 dx + \int_{R^n} [\mu^2 V(x) + t] C_k^2(x) |f(x)|^2 dx. \end{aligned} \tag{2.10}$$

Using equality (2.9), inequality (2.10) can be rewritten as follows:

$$\|C_k(x) f(x)\|^2 \leq \int_{R^n} (\nabla C_k(x))^2 |f(x)|^2 dx. \tag{2.11}$$

From the properties (2.4) and (2.5) of the sequence $\{C_k(x)\}_{k=1}^\infty$ and inequality (2.11) we obtain:

$$\begin{aligned} \|f(x)\|^2 &= \lim_{k \rightarrow \infty} \|C_k(x) f(x)\|^2 \leq \overline{\lim}_{k \rightarrow \infty} \|C_k(x) f(x)\|^2 \leq \\ & \leq \overline{\lim}_{k \rightarrow \infty} \int_{R^n} (\nabla C_k(x))^2 |f(x)|^2 dx = 0. \end{aligned}$$

Hence we have that $f(x) = 0$ almost everywhere in R^n . Therefore, the operator $H_\mu^0 + tE$ is an essentially self-adjoint operator. It follows that the operator H_μ^0

is essentially self-adjoint on $C_0^\infty(R^n)$, as tE is the bounded self-adjoint operator. This completes the proof of the theorem. \square

From theorem II .4 of [9] we get that the following lemma is true

Lemma 2.1. *If conditions a) and b) are satisfied, then the essential spectrum of the Schrödinger operator H_μ coincides with the positive half-axis, i.e.,*

$$\sigma_{ess}(H_\mu) = [0, +\infty).$$

3. Proof of the main result

Now we show that as the magnetic field strength μ increases below the threshold of the essential spectrum, i.e., below zero, the eigenvalues of the operator H_μ appear.

Theorem 3.1. *Let the function $V(x)$ satisfy conditions a)-c). Then, for sufficiently large μ , the Schrödinger operator H_μ has at least one negative eigenvalue.*

Proof. It is known that (see [6, p. 9]) the ground state of the operator H_μ is defined as follows:

$$\inf \sigma(H_\mu) = \inf_{u(x) \in W_2^1(R^n) \setminus \{0\}} \frac{\int_{R^n} \left\{ |\nabla u(x)|^2 + \mu^2 V(x) |u(x)|^2 \right\} dx}{\int_{R^n} |u(x)|^2 dx}. \tag{3.1}$$

Note that the quantity (3.1) is important in classical quantum mechanics (see [16]), since it is the lowest energy that a particle can reach in R^n , interacting with the force field generated by the potential $\mu^2 V(x)$ (and which will ultimately be achieved due to radiation of energy).

To prove the theorem, it suffices to prove that there exists a function $\varphi_\mu(x)$ from $W_2^1(R^n) \setminus \{0\}$ such that

$$\frac{\int_{R^n} \left\{ |\nabla \varphi_\mu(x)|^2 + \mu^2 V(x) |\varphi_\mu(x)|^2 \right\} dx}{\int_{R^n} |\varphi_\mu(x)|^2 dx} < 0. \tag{3.2}$$

To prove inequality (3.2), we follow the path indicated in Example 1.3.2 of [6]. Let the function $V(x)$ reach its minimum at the point $x^{(0)} = (x_1^{(0)}, x_2^{(0)}, \dots, x_n^{(0)})$, i.e.

$$\inf_{x \in R^n} V(x) = \min_{x \in R^n} V(x) = V(x^{(0)}).$$

Consider the function

$$\varphi_\mu(x) = e^{-\mu|x-x^{(0)}|^2}.$$

Let us prove that

$$\lim_{\mu \rightarrow +\infty} \frac{\int_{R^n} \left\{ |\nabla \varphi_\mu(x)|^2 + \mu^2 V(x) |\varphi_\mu(x)|^2 \right\} dx}{\int_{R^n} |\varphi_\mu(x)|^2 dx} = V(x^{(0)}). \tag{3.3}$$

Passing to the polar coordinate system, we obtain:

$$\int_{R^n} |\varphi_\mu(x)|^2 dx = \int_{R^n} \bar{e}^{-2\mu|x-x^{(0)}|^2} dx = \sigma_n \int_{R^n} \bar{e}^{-2\mu r^2} r^{n-1} dr = \frac{\sigma_n A_n}{(2\mu)^{\frac{n}{2}}}, \tag{3.4}$$

where σ_n is the area of $(n - 1)$ -dimensional unit sphere,

$$A_n = \int_0^{+\infty} \bar{e}^{s^2} s^{n-1} ds.$$

In order to calculate the integral

$$\int_{R^n} |\nabla\varphi_\mu(x)|^2 dx$$

we find the partial derivatives $\frac{\partial\varphi_\mu(x)}{\partial x_i}$, $i = 1, 2, \dots, n$.

We have :

$$\frac{\partial\varphi_\mu(x)}{\partial x_i} = \frac{\partial}{\partial x_i} \left(\bar{e}^{2\mu|x-x^{(0)}|^2} \right) = -2\mu e^{-\mu|x-x^{(0)}|^2} \left(x_i - x_i^{(0)} \right), \quad i = 1, 2, \dots, n.$$

From here we obtain

$$|\nabla\varphi_\mu(x)|^2 dx = 4\mu^2 e^{-2\mu|x-x^{(0)}|^2} |x - x^{(0)}|^2 dx. \tag{3.5}$$

Using (3.5) we have:

$$\begin{aligned} \int_{R^n} |\nabla\varphi_\mu(x)|^2 dx &= \int_{R^n} 4\mu^2 |x - x^{(0)}|^2 e^{-2\mu|x-x^{(0)}|^2} dx = \\ &= 4\mu^2 \int_{R^n} \bar{e}^{2\mu|t|^2} |t|^2 dt = 4\mu^2 \sigma_n \int_0^{+\infty} e^{-2\mu r^2} r^{n+1} dr = \frac{\sigma_n B_n}{(2\mu)^{\frac{n-2}{2}}}, \end{aligned} \tag{3.6}$$

where

$$B_n = \int_0^{+\infty} \bar{e}^{s^2} s^{n+1} ds.$$

From (3.4) and (3.6) we obtain

$$\begin{aligned} &\frac{\int_{R^n} \left\{ |\nabla\varphi_\mu(x)|^2 + \mu^2 V(x) |\varphi_\mu(x)|^2 \right\} dx}{\int_{R^n} |\varphi_\mu(x)|^2 dx} = \\ &= \frac{\frac{\sigma_n B_n}{(2\mu)^{\frac{n-2}{2}}} + \mu^2 \int_{R^n} V(x) \bar{e}^{2\mu|x-x^{(0)}|^2} dx}{\frac{\sigma_n A_n}{(2\mu)^{\frac{n}{2}}}}. \end{aligned} \tag{3.7}$$

Using the Taylor formula for the function $V(x)$ at the minimum point $x^{(0)}$ with a remainder in the Lagrange form, we have:

$$\int_{R^n} V(x) e^{-2\mu|x-x^{(0)}|^2} dx = \int_{R^n} V(x^{(0)}) e^{-2\mu|x-x^{(0)}|^2} dx + \frac{1}{2} \sum_{i,j=1}^n \int_{R^n} e^{-2\mu|x-x^{(0)}|^2} \frac{\partial^2 V(\xi(x), x^{(0)})}{\partial x_i \partial x_j} (x_i - x_i^{(0)}) (x_j - x_j^{(0)}) dx. \tag{3.8}$$

From the boundedness of the function $\frac{\partial^2 V(\xi(x), x^{(0)})}{\partial x_i \partial x_j}$ in R^n , it follows that there exists a continuous and bounded function $M(x)$ in R^n such that the following equality is true:

$$\frac{1}{2} \sum_{i,j=1}^n \frac{\partial^2 V(\xi(x), x^{(0)})}{\partial x_i \partial x_j} (x_i - x_i^{(0)}) (x_j - x_j^{(0)}) = M(x) |x - x^{(0)}|^2. \tag{3.9}$$

From (3.4),(3.7),(3.8) and (3.9) we obtain

$$\frac{\int_{R^n} \left\{ |\nabla \varphi_\mu(x)|^2 + \mu^2 V(x) |\varphi_\mu(x)|^2 \right\} dx}{\int_{R^n} |\varphi_\mu(x)|^2 dx} = \frac{\frac{\sigma_n B_n}{(2\mu)^{\frac{n-2}{2}}} + \mu^2 V(x^{(0)}) \frac{\sigma_n A_n}{(2\mu)^{\frac{n}{2}}} + \mu^2 \int_{R^n} e^{-2\mu|x-x^{(0)}|^2} M(x) |x - x^{(0)}|^2 dx}{\frac{\sigma_n A_n}{(2\mu)^{\frac{n}{2}}}}. \tag{3.10}$$

From the boundedness and continuity of the function $M(x)$ in R^n , it follows that there exists a positive number M_0 such that

$$\int_{R^n} e^{-2\mu|x-x^{(0)}|^2} M(x) |x - x^{(0)}|^2 dx = M_0 \int_{R^n} e^{-2\mu|x-x^{(0)}|^2} |x - x^{(0)}|^2 dx. \tag{3.11}$$

As

$$\int_{R^n} e^{-2\mu|x-x^{(0)}|^2} |x - x^{(0)}|^2 dx = \frac{\sigma_n B_n}{(2\mu)^{\frac{n+2}{2}}},$$

then from (3.10) and (3.11) we obtain

$$\frac{\int_{R^n} \left\{ |\nabla \varphi_\mu(x)|^2 + \mu^2 V(x) |\varphi_\mu(x)|^2 \right\} dx}{\int_{R^n} |\varphi_\mu(x)|^2 dx} = \mu^2 \frac{\frac{2B_n}{\mu} + A_n V(x^{(0)}) + \frac{M_0 B_n}{2\mu}}{A_n}. \tag{3.12}$$

It follows from (3.12) that

$$\lim_{\mu \rightarrow \infty} \frac{1}{\mu^2} \left\{ \frac{\int_{R^n} \left\{ |\nabla \varphi_\mu(x)|^2 + \mu^2 V(x) |\varphi_\mu(x)|^2 \right\} dx}{\int_{R^n} |\varphi_\mu(x)|^2 dx} \right\} = V(x^{(0)}). \tag{3.13}$$

As $V(x^{(0)}) < 0$ then it follows from (3.13) that for sufficiently large μ the following inequality holds true:

$$\frac{\int_{R^n} \left\{ |\nabla \varphi_\mu(x)|^2 + \mu^2 V(x) |\varphi_\mu(x)|^2 \right\} dx}{\int_{R^n} |\varphi_\mu(x)|^2 dx} < 0.$$

□

This completes the proof of the theorem.

Corollary 3.1. *From this theorem and from the minimax principle (see [6] or [10]), it follows that as the magnetic field strength μ increases, the number of negative eigenvalues of the operator H_μ increases.*

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