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SOME FIXED POINT THEOREMS ON COMPACT MODULAR METRIC SPACES

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Abstract. In this article, we prove some fixed point theorems for contractive mappings in compact modular metric spaces. Then we illustrate our main theorem with an example. The application of the obtained result is also discussed.

1. Introduction

Fixed point theory involves many fields of mathematics such as functional analysis, mathematical analysis, general topology and operator theory. In addition, applications of this theory are crucial in various disciplines of engineering, statistics, economics, computer sciences, and in work on problems related to approximation theory, game theory, dynamic programming, differential equations and integral equations. Many authors studied on this theory and wrote a lot of papers related to it, see [2, 4, 9, 23, 24, 25] and others.

In literature, for the first time, the concept of modular spaces was introduced by Nakano in 1950 [18]. This concept was considered as an generalization of the concept of metric spaces. Afterwards, many fixed point theorems on modular spaces were given by some authors [10, 12, 14, 16, 26]. In 2008, by using properties of modular spaces, Chistyakov [5, 6, 7] introduced a new metric structure, which has a physical interpretation. This structure was called modular metric space. Many researchers examined properties of this structure and proved some well known fixed point theorems on modular metric space [1, 11, 13, 15, 17, 20, 21].

Let (X, d) be a metric space and T be a selfmapping on X. T is said to be contractive mapping if

$$d(Tx, Ty) < d(x, y) \text{ where } x \neq y$$

for every $x, y \in X$. Contractiveness of a mapping on complete metric spaces does not ensure existence of a fixed point. To ensure this, authors as Bailey [3], Edelstein [8], Nemytzki [19], Suzuki [22] proved some fixed point theorems on compact metric spaces.

In this article, we prove some fixed point theorems for contractive mappings in compact modular metric spaces. Then we illustrate our main theorem with an example. The application of the obtained result is also discussed.

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2. Preliminaries

Definition 2.1. [16] Let X be a linear space on $\mathbb{R}$. A functional $\rho : X \to [0, \infty]$ is called a modular on X if the following conditions hold:

(A1) $\rho(0) = 0$;
(A2) If $x \in X$ and $\rho(\alpha x) = 0$ for all numbers $\alpha > 0$, then $x = 0$;
(A3) $\rho(-x) = \rho(x)$ for all $x \in X$;
(A4) $\rho(\alpha x + \beta y) \leq \rho(x) + \rho(y)$ for all $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$ and $x, y \in X$.

Let $X \neq \emptyset$, $\lambda \in (0, \infty)$. For all $\lambda > 0$ and $x, y \in X$, the function $\omega : (0, \infty) \times X \times X \to [0, \infty]$ will be written as $\omega_\lambda(x, y) = \omega(\lambda, x, y)$.

Definition 2.2. [6] Let $X \neq \emptyset$. A function $\omega : (0, \infty) \times X \times X \to [0, \infty]$ is said to be a metric modular on X, if the following conditions hold for all $x, y, z \in X$:
(i) $\omega_\lambda(x, y) = 0$ for all $\lambda > 0$ if and only if $x = y$;
(ii) $\omega_\lambda(x, y) = \omega_\lambda(y, x)$ for all $\lambda > 0$;
(iii) $\omega_{\lambda+\mu}(x, y) \leq \omega_\lambda(x, z) + \omega_\mu(z, y)$ for all $\lambda, \mu > 0$.

If $0 < \mu < \lambda$, from properties of metric modular, we have

$$\omega_\lambda(x, y) \leq \omega_{\lambda-\mu}(x, x) + \omega_\mu(x, y) = \omega_\mu(x, y)$$

for all $x, y \in X$.

From [6, 7], we know that for a fixed $a_0 \in X$, the two sets

$$X_\omega = X_\omega(a_0) = \{x \in X : \lim_{\lambda\to\infty} \omega_\lambda(x, a_0) = 0\}$$

and

$$X^*_\omega = X^*_\omega(a_0) = \{a \in X : \exists\lambda = \lambda(a) > 0 \text{ such that } \omega_\lambda(a, a_0) < \infty\}$$

are said to be modular spaces. As ω is a modular on X, X_ω can be equipped with a metric

$$d_\omega(a, b) = \inf\{\lambda > 0 : \omega_\lambda(a, b) \leq \lambda\}$$

which is generated by ω for any $a, b \in X_\omega$.

Definition 2.3. [15] Let X_ω be a modular metric space.
(1) The sequence $(x_n)_{n\in\mathbb{N}}$ in X_ω is called (modular) convergent to $x \in X_\omega$ if $\omega_\lambda(x_n, x) \to 0$, as $n \to \infty$ for all $\lambda > 0$.
(2) The sequence $(x_n)_{n\in\mathbb{N}}$ in X_ω is called (modular) Cauchy if $\omega_\lambda(x_n, x_m) \to 0$, as $m, n \to \infty$ for all $\lambda > 0$.
(3) $C \subseteq X_\omega$ is called complete (modular) if every (modular) Cauchy sequence in C is a (modular) convergent is in C.
(4) $C \subseteq X_\omega$ is called bounded if for all $\lambda > 0$,

$$\delta_\omega(C) = \sup\{\omega_\lambda(x, y) : \ x, y \in C\} < \infty.$$

Definition 2.4. [20, 21] Let X_ω be a modular metric space. Then it is said that $T : X_\omega \to X_\omega$ is (modular) continuous if

$$x_n \to x \Rightarrow Tx_n \to Tx$$

for each $(x_n)_{n\in\mathbb{N}} \in X_\omega$ as $n \to \infty$.

3. Main Results

Here, we give a definition of compactness by using [10, 12, 14].

Definition 3.1. Let X_ω be a modular metric space. $B \subseteq X_\omega$ is (sequentially) compact if and only if every sequence in B has a convergent subsequence in B. If X_ω is a compact, we will say that X_ω is compact modular metric space.

The following lemma is given by using [14, 15]

Lemma 3.1. *Let X_ω be a modular metric space. If $A \subseteq X_\omega$ is compact, then it is bounded.*

Lemma 3.2. *Let X_ω be a modular metric space. If $T : X_\omega \to X_\omega$ is (modular) contractive, then T is a (modular) continuous mapping.*

Proof. Let $T : X_\omega \to X_\omega$ be a contractive mapping and $\omega_\lambda(x_n, x) \to 0$ for each $(x_n)_{n\in\mathbb{N}} \in X_\omega$ and all $\lambda > 0$ as $n \to \infty$. From contractiveness of T, we get

$$\omega_\lambda(Tx_n, Tx) < \omega_\lambda(x_n, x) \to 0$$

as $n \to \infty$. So, T is a continuous mapping. □

Theorem 3.1. *Let X_ω be a compact modular metric space and given a mapping $T : X_\omega \to X_\omega$. If*

$$\omega_\lambda(Tx, Ty) < \omega_\lambda(x, y) \text{ for } x \neq y \text{ and all } \lambda > 0,$$

then T has a unique fixed point.

Proof. We define a function $F : X_\omega \to \mathbb{R}^+$ by $Fx = \omega_\lambda(x, Tx)$. Since the mapping T is contractive, from Lemma 3.2, it is continuous. Then F is continuous. From Lemma 3.1, F is bounded. Let $\inf_{x\in X_\omega} \omega_\lambda(x, Tx) = c_\lambda$ for any $\lambda > 0$. We choose $x_n \in X_\omega$ such that

$$\omega_\lambda(x_n, Tx_n) = c_\lambda + \frac{1}{n}$$

for all $n \in \mathbb{N}$. So, we set a sequence $(x_n)_{n\in\mathbb{N}}$. Since X_ω is compact modular, there exists a subsequence (x_{n_i}) such that $(x_{n_i}) \to u \in X_\omega$. Thus,

$$Fu = \omega_\lambda(u, Tu) = c_\lambda = \inf_{x\in X_\omega} \omega_\lambda(x, Tx) = \min_{x\in X_\omega} \omega_\lambda(x, Tx)$$

for any $\lambda > 0$. Since this expression is satisfied for any $\lambda > 0$, it is also satisfied for all $\lambda > 0$. We assume that $u \neq Tu$. Since the mapping T is contractive, we have

$$FTu = \omega_\lambda(Tu, T^2u) < \omega_\lambda(u, Tu) = Fu.$$

This is a contradiction. Hence, u is a fixed point of T. To show the uniqueness of fixed point of T, we assume that v is another fixed point of T with $u \neq v$. Then

$$\omega_\lambda(u, v) = \omega_\lambda(Tu, Tv) < \omega_\lambda(u, v).$$

So, this is a contradiction. Therefore, u is a unique fixed point of T. □

Now, we extend Theorem 3.1.

Theorem 3.2. *Let X_ω be a modular metric space and given a mapping $T : X_\omega \to X_\omega$ which satisfies the following condition for $x \neq y$ and all $\lambda > 0$*

$$\omega_\lambda(Tx, Ty) < \omega_\lambda(x, y).$$

If there exists a point $x_0 \in X_\omega$ such that sequence of iterates $T^n x_0$ involves a convergent subsequence $T^{n_i} x_0$, then $u = lim_{n\to\infty} T^{n_i} x_0$ is a unique fixed point of T.

Proof. We consider a sequence of reals $\{\omega_\lambda(T^n x_0, T^{n+1} x_0)\}$. If we take $T^{n_0+1} x_0 = T^{n_0} x_0$ for some $n_0 \in \mathbb{N}$, then $\{T^n x_0\}$ is a stationary sequence for $n \geq n_0$. Hence, $T^{n_0} x_0 = u$. So, $T^{n_0+1} x_0 = T^{n_0} x_0$ implies $Tu = u$. Now, we assume that $T^{n+1} x_0 \neq T^n x_0$ for all $n \in \mathbb{N}$. From contractiveness of T, it is obvious that $\{\omega_\lambda(T^n x_0, T^{n+1} x_0)\}$ is a strictly decreasing sequence of positive reals. So, it converges. Since $\lim_{n\to\infty} T^{n_i} x_0 = u$ and from Lemma 3.2, we have

$$\lim_{n\to\infty} T^{n_i+1} x_0 = \lim_{n\to\infty} T T^{n_i} x_0 = Tu$$

and

$$\lim_{n\to\infty} T^{n_i+2} x_0 = \lim_{n\to\infty} T^2 T^{n_i} x_0 = T^2 u.$$

Thus,

$$\lim_{n\to\infty} \omega_\lambda(T^{n_i} x_0, T^{n_i+1} x_0) = \omega_\lambda(u, Tu)$$

and

$$\lim_{n\to\infty} \omega_\lambda(T^{n_i+1} x_0, T^{n_i+2} x_0) = \omega_\lambda(Tu, T^2 u)$$

for all $\lambda > 0$. $\{T^{n_i} x_0, T^{n_i+1} x_0\}$ and $\{T^{n_i+1} x_0, T^{n_i+2} x_0\}$ are subsequences of the convergent sequence $\{T^n x_0, T^{n+1} x_0\}$. Then they have the same limit. Therefore, we obtain that $\omega_\lambda(u, Tu) = \omega_\lambda(Tu, T^2 u)$ for all $\lambda > 0$. Then we have $Tu = u$. Otherwise, if $Tu \neq u$, since T is contractive, we get $\omega_\lambda(Tu, T^2 u) < \omega_\lambda(u, Tu)$. Then we get a contradiction. So, T has a fixed point.

Uniqueness of the fixed point of T can be shown in a similar way with the proof of Theorem 3.1. □

Theorem 3.3. *Let X_ω be a compact modular metric space and given a continuous mapping $T : X_\omega \to X_\omega$. Let there exists a positive integer $k(x, y)$ such that*

$$0 < \omega_\lambda(x, y) \Rightarrow \omega_\lambda(T^{k(x,y)} x, T^{k(x,y)} y) < \omega_\lambda(x, y) \tag{3.1}$$

for every $x, y \in X_\omega$. Then T has a unique fixed point.

Proof. We give a proof in a similar way with the proof of Theorem 3.1. We begin by defining a function $F : X_\omega \to \mathbb{R}^+$ by $Fx = \omega_\lambda(x, Tx)$ for all $\lambda > 0$. Since T is continuous, F is continuous. From compactness of X_ω, there exists a $u \in X_\omega$ where minimum of F arise.

Now, we show that $Fu = 0$. Then we assume that $Fu = \omega_\lambda(u, Tu) > 0$. Thus, from (3.1), there exists a positive integer $k(u, Tu)$ such that

$$\omega_\lambda(T^{k(u,Tu)} u, T^{k(u,Tu)} Tu) < \omega_\lambda(u, Tu).$$

We obtain that $FT^{k(u,Tu)} u < Fu$. This is a contradiction. Finally, we get

$$Fu = \omega_\lambda(u, Tu) = 0 \Rightarrow Tu = u.$$

□

Theorem 3.4. *Let X_ω be a compact modular metric space and $T : X_\omega \to X_\omega$ be a mapping on X_ω. If for $x, y \in X_\omega$ and all $\lambda > 0$*

$$\frac{1}{2}\omega_\lambda(x, Tx) < \omega_{\frac{\lambda}{2}}(x, y) \Rightarrow \omega_\lambda(Tx, Ty) < \omega_\lambda(x, y), \tag{3.2}$$

then T has a unique fixed point.

Proof. We consider

$$\gamma = \inf\{\omega_\lambda(x, y) : x \in X\}$$

and set a sequence $\{x_n\}$ in X_ω with $\lim_{n\to\infty} \omega_\lambda(x_n, Tx_n) = \gamma$. Since X_ω is compact modular, there exist $v, w \in X_\omega$ such that the sequences $\{x_n\}$ and $\{Tx_n\}$ converge them, respectively. To show that $\gamma = 0$, we assume the contrary. Let $\gamma > 0$. Then we get

$$\lim_{n\to\infty} \omega_\lambda(x_n, w) = \omega_\lambda(v, w) = \lim_{n\to\infty} \omega_\lambda(x_n, Tx_n) = \gamma$$

for $\lambda > 0$. We take $n_0 \in \mathbb{N}$ such that

$$\frac{2}{3}\gamma < \omega_{\frac{\lambda}{2}}(x_n, w) \text{ and } \omega_{\frac{\lambda}{2}}(x_n, Tx_n) < \frac{4}{3}\gamma$$

for $\lambda > 0$ and $n \in \mathbb{N}$ with $n \geq n_0$. Therefore,

$$\frac{1}{2}\omega_\lambda(x_n, Tx_n) < \omega_{\frac{\lambda}{2}}(x_n, w)$$

for $n \geq n_0$. From the equation (3.2), we get $\omega_\lambda(Tx_n, Tw) < \omega_\lambda(x_n, w)$ for $n \geq n_0$ and $\lambda > 0$. Then we have

$$\omega_\lambda(w, Tw) = \lim_{n\to\infty} \omega_\lambda(Tx_n, Tw) \leq \lim_{n\to\infty} \omega_\lambda(x_n, w) = \gamma$$

for all $\lambda > 0$. Using definition of γ, we get $\omega_\lambda(w, Tw) = \gamma$. Since

$$\frac{1}{2}\omega_\lambda(w, Tw) < \frac{1}{2}\omega_{\frac{\lambda}{2}}(w, Tw) < \omega_{\frac{\lambda}{2}}(w, Tw),$$

we have

$$\omega_\lambda(Tw, T^2w) < \omega_\lambda(w, Tw) = \gamma.$$

This contradicts with the definition of γ. Then we can say that $\gamma = 0$. To show the existence of a fixed point of T, we assume that T does not have any fixed point. Since

$$0 < \frac{1}{2}\omega_\lambda(x_n, Tx_n) < \omega_{\frac{\lambda}{2}}(x_n, Tx_n)$$

for every $n \in \mathbb{N}$ and all $\lambda > 0$, we get

$$\omega_\lambda(Tx_n, T^2x_n) < \omega_\lambda(x_n, Tx_n).$$

Then we have

$$\lim_{n\to\infty} \omega_\lambda(v, Tx_n) = \omega_\lambda(v, w) = \lim_{n\to\infty} \omega_\lambda(x_n, Tx_n) = \gamma = 0.$$

So, the sequence $\{Tx_n\}$ also converges to v. Moreover, we get

$$\begin{aligned}
\lim_{n\to\infty} \omega_\lambda(v, T^2x_n) &\leq \lim_{n\to\infty} (\omega_{\frac{\lambda}{2}}(v, Tx_n), \omega_{\frac{\lambda}{2}}(Tx_n, T^2x_n)) \\
&\leq \lim_{n\to\infty} (\omega_{\frac{\lambda}{2}}(v, Tx_n), \omega_{\frac{\lambda}{2}}(x_n, Tx_n)) = 0.
\end{aligned}$$

Therefore, $\{T^2x_n\}$ also converges to v. We assume that

$$\frac{1}{2}\omega_\lambda(x_n, Tx_n) \geq \omega_{\frac{\lambda}{2}}(x_n, v)$$

and

$$\frac{1}{2}\omega_\lambda(Tx_n, T^2x_n) \geq \omega_{\frac{\lambda}{2}}(Tx_n, v)$$

for all $\lambda > 0$. Then we get

$$\begin{aligned}
\omega_\lambda(x_n, Tx_n) &\leq \omega_{\frac{\lambda}{2}}(x_n, v) + \omega_{\frac{\lambda}{2}}(Tx_n, v) \\
&\leq \frac{1}{2}\omega_{\frac{\lambda}{2}}(x_n, Tx_n) + \frac{1}{2}\omega_{\frac{\lambda}{2}}(Tx_n, T^2x_n) \\
&< \frac{1}{2}\omega_{\frac{\lambda}{2}}(x_n, Tx_n) + \frac{1}{2}\omega_{\frac{\lambda}{2}}(x_n, Tx_n) \\
&= \omega_\lambda(x_n, Tx_n).
\end{aligned}$$

This is a contradiction. So, either

$$\frac{1}{2}\omega_\lambda(x_n, Tx_n) < \omega_{\frac{\lambda}{2}}(x_n, v)$$

or

$$\frac{1}{2}\omega_\lambda(Tx_n, T^2x_n) < \omega_{\frac{\lambda}{2}}(Tx_n, v)$$

holds for every $n \in \mathbb{N}$ and all $\lambda > 0$. Then from (3.2), either

$$\omega_\lambda(Tx_n, Tv) < \omega_\lambda(x_n, v)$$

or

$$\omega_\lambda(T^2x_n, Tv) < \omega_\lambda(Tx_n, v)$$

holds for every $n \in \mathbb{N}$ and all $\lambda > 0$.

(A) There exists infinite set $I \subset \mathbb{N}$ with $\omega_\lambda(Tx_n, Tv) < \omega_\lambda(x_n, v)$ for all $n \in I$ and $\lambda > 0$,
or

(B) There exists an infinite set $J \subset \mathbb{N}$ with $\omega_\lambda(T^2x_n, Tv) < \omega_\lambda(Tx_n, v)$ for all $n \in J$ and $\lambda > 0$, holds.

If (A) holds, then for $n \in I$ and $\lambda > 0$

$$\omega_\lambda(v, Tv) = \lim_{n\to\infty} \omega_\lambda(Tx_n, Tv) \leq \lim_{n\to\infty} \omega_\lambda(x_n, v) = 0 \Rightarrow Tv = v.$$

And, if (B) holds, then for $n \in J$ and $\lambda > 0$

$$\omega_\lambda(v, Tv) = \lim_{n\to\infty} \omega_\lambda(T^2x_n, Tv) \leq \lim_{n\to\infty} \omega_\lambda(Tx_n, v) = 0 \Rightarrow Tv = v.$$

Thus, in both cases, v is a fixed point of T. This is a contradiction. Then there exists an $a \in X_\omega$ such that $Ta = a$. We show the uniqueness of fixed point. We assume existence of another fixed point $b \in X_\omega$ with $a \neq b$. Since

$$\frac{1}{2}\omega_\lambda(Ta, a) = 0 < \omega_{\frac{\lambda}{2}}(a, b),$$

we get

$$\omega_\lambda(a, b) = \omega_\lambda(Ta, Tb) < \omega_\lambda(a, b)$$

which is a contradiction. Then a is a unique fixed point of T. $\square$

Example 3.1. Let $X = [0,1]$. We take a mapping $\omega : (0,\infty) \times [0,1] \times [0,1] \to [0,\infty]$ which is defined by $\omega_\lambda(x,y) = \frac{|x-y|}{1+\lambda}$ for all $x,y \in X = X_\omega$ and $\lambda > 0$. Since all sequences taken on X_ω have convergent subsequences, X_ω is a compact modular metric space. We define the mapping $T : X_\omega \to X_\omega$ with $Tx = \frac{2x+1}{4}$. For all $x,y \in X_\omega$ and $\lambda > 0$, we have

$$\omega_\lambda(Tx,Ty) = \frac{|\frac{2x+1}{4} - \frac{2y+1}{4}|}{1+\lambda} = \frac{|x-y|}{2(1+\lambda)} < \frac{|x-y|}{1+\lambda} = \omega_\lambda(x,y).$$

From Theorem 3.2, the mapping T has a unique fixed point. Moreover, it is $\frac{1}{2} \in X_\omega$.

4. Application

Here, by using Theorem 3.1, we show that system of linear equations has a unique solution.

Theorem 4.1. *Let X be a vector space and X_ω be a compact modular metric space with*

$$\omega_\lambda(x,y) = \max_{1\le i\le n} \frac{|x_i - y_i|}{\lambda}$$

for $x,y \in X_\omega$ and all $\lambda > 0$. If $\sum_{j=1}^{n} |a_{ij}| = \beta \neq 0$ and $0 < \alpha < \frac{1}{\beta}$ for $i = 1,2,\cdots,n$, then the system of n-linear equations

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n + b_1 &= 0 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n + b_2 &= 0 \\ \vdots \quad &\quad \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n + b_n &= 0 \end{aligned} \tag{4.1}$$

has a unique solution.

Proof. We define a mapping $T : X_\omega \to X_\omega$ with $Tx = x - \alpha(Ax+b)$ for

$$x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \in X_\omega, \quad b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} \in X_\omega$$

and

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \ldots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \ldots & a_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \ldots & a_{nn} \end{pmatrix}.$$

Since $\max_{1\le i\le n}\left|1-\alpha\sum_{j=1}^{n} a_{ij}\right| < 1$, we get

$$
\begin{aligned}
\omega_\lambda(Tx,Ty) &= \frac{1}{\lambda}\left(\max_i |(x_j-\alpha(Ax_j+b)-y_j+\alpha(Ay_j+b))|\right)\\
&= \frac{1}{\lambda}\left(\max_i \left|(x_j-y_j)-\alpha\left(\sum_{j=1}^{n} a_{ij}(x_j-y_j)\right)\right|\right)\\
&\le \frac{1}{\lambda}\max_i |x_j-y_j|\left|1-\alpha\sum_{j=1}^{n} a_{ij}\right|\\
&\le \max_i \left|1-\alpha\sum_{j=1}^{n} a_{ij}\right|\frac{1}{\lambda}\left(\max_j |x_j-y_j|\right)\\
&= \max_i \left|1-\alpha\sum_{j=1}^{n} a_{ij}\right|\omega_\lambda(x,y)\\
&< \omega_\lambda(x,y).
\end{aligned}
$$

for all $\lambda > 0$ and $i = 1, 2, \cdots, n$. So, T is a contractive mapping. Then from Theorem 3.1, we can say that the system of linear equations (4.1) has a unique solution. □

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