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SOME PROPERTIES OF THE SPECTRUM OF THE DIRAC OPERATOR WITH A SPECTRAL PARAMETER IN THE BOUNDARY CONDITION

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Abstract. In this paper, we consider the Dirac operator subject to non-separated boundary conditions of which one contains the spectral parameter. The representations of some entire functions are given, the reality eigenvalues and the absence of associated vector functions to the vector eigenfunctions are proved, an asymptotic formula for the eigenvalues of the Dirac operator are obtained.

1. Introduction

The one-dimensional stationary Dirac system (related to the behavior of a relativistic electron in an electrostatic field) has the following canonical form:

$$BY'(x) + Q(x)\,Y(x) = \lambda Y(x)\,, \tag{1.1}$$

where λ is the spectral parameter, $B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $Q(x) = \begin{pmatrix} p(x) & q(x) \\ q(x) & -p(x) \end{pmatrix}$, $Y(x) = \begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix}$.

Suppose that the elements $p(x)$ and $q(x)$ matrices $Q(x)$ in (1.1) are real functions belonging to the space $W_2^1[0,\pi]$. By $W_2^1[0,\pi]$ we denote the space consisting of absolutely continuous functions defined on a segment $[0,\pi]$, which have a derivative, summable with a square on $[0,\pi]$. Consider a boundary problem generated on a segment $[0,\pi]$ by the Dirac equation (1.1) and the boundary conditions of the form

$$\begin{aligned} &(\alpha\lambda + \beta)\,y_1(0) + y_2(0) + \omega y_1(\pi) = 0\,, \\ &-\overline{\omega} y_1(0) + \gamma y_1(\pi) + y_2(\pi) = 0, \end{aligned} \tag{1.2}$$

where α, β, γ are real numbers, ω is a complex number, and $\alpha\omega \neq 0$. The boundary-value problem (1.1), (1.2) will be denoted by D.

At present, many questions of direct and inverse problems for Dirac operators in the case of separated and non-separated boundary conditions are well studied. Thus, direct problems for such operators were studied in [5, 9, 11, 12, 21], and questions on the recovery of operators — in [1, 2, 3, 4, 6, 7, 8, 10, 13, 14, 15, 17, 18,

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19, 20, 23] and other works. In particular, in papers [3, 4, 6, 7, 8, 13, 14, 15, 23] a number of important results were obtained in the theory of inverse problems for systems of differential equations under separated boundary conditions. [1, 2, 10, 17, 18, 19, 20] are devoted to the study of inverse spectral problems for equation (1.1) with various types of nonseparated boundary conditions, including periodic, antiperiodic, and quasiperiodic [10, 17, 19, 20] boundary conditions.

In this paper, we consider the Dirac operator subject to non-separated boundary conditions of which one contains the spectral parameter. The representations of some entire functions are given, the reality eigenvalues and the absence of associated vector functions to the vector eigenfunctions are proved, an asymptotic formula for the eigenvalues of the boundary value problem D are obtained.

2. Some spectral properties of the boundary value problem

Denote by $C(x,\lambda)=\begin{pmatrix} c_1(x,\lambda) \\ c_2(x,\lambda)\end{pmatrix}$ and $S(x,\lambda)=\begin{pmatrix} s_1(x,\lambda) \\ s_2(x,\lambda)\end{pmatrix}$ are solutions of equation (1.1) satisfying the initial conditions

$$C(0,\lambda)=\begin{pmatrix} 1 \\ 0\end{pmatrix}, S(0,\lambda)=\begin{pmatrix} 0 \\ 1\end{pmatrix}. \tag{2.1}$$

Lemma 2.1. *The following representations hold:*

$$c_1(\pi,\lambda)=\cos\lambda\pi+\psi_1(\lambda), c_2(\pi,\lambda)=\sin\lambda\pi+\psi_2(\lambda),$$

$$s_1(\pi,\lambda)=-\sin\lambda\pi+A_1\frac{\cos\lambda\pi}{\lambda}+B_1\frac{\sin\lambda\pi}{\lambda}+\frac{\psi_3(\lambda)}{\lambda},$$

$$s_2(\pi,\lambda)=\cos\lambda\pi+A_2\frac{\sin\lambda\pi}{\lambda}+B_2\frac{\cos\lambda\pi}{\lambda}+\frac{\psi_4(\lambda)}{\lambda},$$

where

$$A_1=A+Q_1,\ \ A_2=A+Q_2, B_1=-\frac{p(0)+p(\pi)}{2},\ \ B_2=\frac{p(0)-p(\pi)}{2},$$

$$A=\frac{1}{2}\int_0^\pi\left[p^2(x)+q^2(x)\right]dx, Q_1=\frac{q(\pi)-q(0)}{2},\ \ Q_2=-\frac{q(0)+q(\pi)}{2},$$

$$\psi_j(\lambda)=\int_{-\pi}^{\pi}\tilde{\psi}_j(t)\,e^{i\lambda t}dt, \tilde{\psi}_j(t)\in L_2[-\pi,\pi]\ ,\ \ j=\overline{1,4}.$$

The proof of the lemma follows from [16, p. 66].

Definition 2.1. A complex number λ_0 is called an eigenvalue of boundary value problem D if equation (1.1) with $\lambda=\lambda_0$ has a nontrivial solution $Y_0(x)$ satisfying the boundary conditions (1.2); in this case the vector function $Y_0(x)$ is called the vector eigenfunction of problem D corresponding to the eigenvalue λ_0. The set of eigenvalues is called the spectrum of D. The vector functions

$$Y_1(x)=\begin{pmatrix} y_{1,1}(x) \\ y_{2,1}(x)\end{pmatrix}, Y_2(x)=\begin{pmatrix} y_{1,2}(x) \\ y_{2,2}(x)\end{pmatrix}, ..., Y_r(x)=\begin{pmatrix} y_{1,r}(x) \\ y_{2,r}(x)\end{pmatrix}$$

are called as the associated vector functions to the vector eigenfunction $Y_0(x) = \begin{pmatrix} y_{1,0}(x) \\ y_{2,0}(x) \end{pmatrix}$, if they are absolutely continuous, they satisfy the differential equations

$$BY_j'(x) + Q(x)Y_j(x) - Y_{j-1}(x) = \lambda_0 Y_j(x) \tag{2.2}$$

and boundary conditions

$$\begin{gathered} (\alpha\lambda+\beta)\,y_{1,j}(0) + y_{2,j}(0) + \omega y_{1,j}(\pi) + \alpha y_{1,j-1}(0) = 0, \\ -\overline{\omega} y_{1,j}(0) + \gamma y_{1,j}(\pi) + y_{2,j}(\pi) = 0, \end{gathered} \tag{2.3}$$

$$j = 1, 2, 3, ..., r.$$

Theorem 2.1. *When $\alpha < 0$ the eigenvalues of the problem D are real.*

Proof. Let λ be an eigenvalue of the boundary value problem D, and let $Y(x) = \begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix}$ be the corresponding vector eigenfunction. Multiply the equality (1.1) by $\left(\overline{y_1(x)},\ \overline{y_2(x)}\right)$ on the left,

$$B\overline{Y'(x)} + Q(x)\,\overline{Y(x)} = \bar{\lambda}\overline{Y(x)}$$

by $(y_1(x),\ y_2(x))$ and subtract one product from the other. As a result, we get

$$\left(\lambda - \bar{\lambda}\right)\ \left[|y_1(x)|^2 + |y_2(x)|^2\right] = \frac{d}{dx}\ \left[\overline{y_1(x)}\,y_2(x) - y_1(x)\,\overline{y_2(x)}\right].$$

Integrating this equality from zero to π, we find

$$2\,i\,\mathrm{Im}\lambda \int_0^\pi \left[|y_1(x)|^2 + |y_2(x)|^2\right] dx =$$

$$= \overline{y_1(\pi)}\,y_2(\pi) - y_1(\pi)\,\overline{y_2(\pi)} - \overline{y_1(0)}\,y_2(0) + y_1(0)\,\overline{y_2(0)}. \tag{2.4}$$

It follows from the boundary conditions (1.2) that

$$\begin{aligned} &y_2(0) = -\omega y_1(\pi) - (\alpha\lambda+\beta)\,y_1(0)\ , \\ &y_2(\pi) = \overline{\omega} y_1(0) - \gamma y_1(\pi)\,. \end{aligned}$$

Let's consider these at the equality (2.4):

$$2\,i\,\mathrm{Im}\lambda \int_0^\pi \left[|y_1(x)|^2 + |y_2(x)|^2\right] dx =$$

$$= \overline{y_1(\pi)}\ \left[\overline{\omega} y_1(0) - \gamma y_1(\pi)\right] - y_1(\pi)\left[\omega\overline{y_1(0)} - \gamma\overline{y_1(\pi)}\right] +$$

$$+\overline{y_1(0)}\ \left[\omega y_1(\pi) + (\alpha\lambda+\beta)\,y_1(0)\right] - y_1(0)\left[\overline{\omega}\,\overline{y_1(\pi)} + \left(\alpha\bar{\lambda}+\beta\right)\overline{y_1(0)}\right] =$$

$$= 2\,i\alpha\ |y_1(0)|^2\,\mathrm{Im}\lambda.$$

Consequently,

$$\mathrm{Im}\lambda\left\{\int_0^\pi \left[|y_1(x)|^2 + |y_2(x)|^2\right]\ dx - \alpha\ |y_1(0)|^2\right\} = 0.$$

Since $\alpha < 0$, the expression in braces is distinct from zero. Therefore, $\mathrm{Im}\lambda = 0$.
Theorem 2.1 is proved. □

Theorem 2.2. *If $\alpha < 0$, then the vector eigenfunctions of the boundary value problem D have no associated vector functions.*

Proof. Let‘s assume the opposite. Suppose that, the boundary value problem D has an vector eigenfunction of $Y_1(x) = \begin{pmatrix} y_{1,1}(x) \\ y_{2,1}(x) \end{pmatrix}$ associated to the $Y_0(x) = \begin{pmatrix} y_{1,0}(x) \\ y_{2,0}(x) \end{pmatrix}$ vector eigenfunction, corresponding to the eigenvalue λ_0. Then, by virtue of (1.1), (2.2), the following equalities hold:

$$B\overline{Y_0'(x)} + Q(x)\overline{Y_0(x)} = \lambda_0\overline{Y_0(x)}, \tag{2.5}$$

$$BY_1'(x) + Q(x)Y_1(x) - Y_0(x) = \lambda_0 Y_1(x). \tag{2.6}$$

Multiply the equality (2.5) by $(y_{1,1}(x),\ y_{2,1}(x))$ – from the left and equality (2.6) by $\left(\overline{y_{1,0}(x)},\ \overline{y_{2,0}(x)}\right)$ and subtract them side by side, we will get

$$|y_{1,0}(x)|^2 + |y_{2,0}(x)|^2 = \frac{d}{dx}\left[y_{2,1}(x)\overline{y_{1,0}(x)} - y_{1,1}(x)\overline{y_{2,0}(x)}\right].$$

Integrating this equality from zero to π for x, we obtain

$$\int_0^\pi \left[|y_{1,0}(x)|^2 + |y_{2,0}(x)|^2\right] dx =$$

$$= y_{2,1}(\pi)\overline{y_{1,0}(\pi)} - y_{1,1}(\pi)\overline{y_{2,0}(\pi)} - y_{2,1}(0)\overline{y_{1,0}(0)} + y_{1,1}(0)\overline{y_{2,0}(0)}. \tag{2.7}$$

According to the boundary conditions (1.2) and (2.3), we have

$$\begin{aligned} y_{2,0}(0) &= -\omega y_{1,0}(\pi) - (\alpha\lambda + \beta)\, y_{1,0}(0), \\ y_{2,0}(\pi) &= \overline{\omega} y_{1,0}(0) - \gamma y_{1,0}(\pi), \end{aligned}$$

$$\begin{aligned} y_{2,1}(0) &= -(\alpha\lambda + \beta)\, y_{1,1}(0) - \omega y_{1,1}(\pi) - \alpha y_{1,0}(0), \\ y_{2,1}(\pi) &= \overline{\omega} y_{1,1}(0) - \gamma y_{1,1}(\pi). \end{aligned}$$

By substituting these expressions into (2.7), we find

$$\int_0^\pi \left[|y_{1,0}(x)|^2 + |y_{2,0}(x)|^2\right] dx =$$

$$= \overline{y_{1,0}(\pi)}\left[\overline{\omega} y_{1,1}(0) - \gamma y_{1,1}(\pi)\right] - y_{1,1}(\pi)\left[\omega\overline{y_{1,0}(0)} - \gamma\overline{y_{1,0}(\pi)}\right] +$$

$$+\overline{y_{1,0}(0)}\left[(\alpha\lambda + \beta)\, y_{1,1}(0) + \omega y_{1,1}(\pi) + \alpha y_{1,0}(0)\right] -$$

$$-y_{1,1}(0)\left[\overline{\omega}\overline{y_{1,0}(\pi)} + (\alpha\lambda + \beta)\,\overline{y_{1,0}(0)}\right] = \alpha\, |y_{1,0}(0)|^2.$$

Hence

$$\int_0^\pi \left[|y_{1,0}(x)|^2 + |y_{2,0}(x)|^2\right] dx - \alpha\, |y_{1,0}(0)|^2 = 0.$$

This contradicts the fact that the left side of this relation is positive according to the inequality $\alpha < 0$. Theorem 2.2 is proved. □

3. Asymptotics of eigenvalues

Theorem 3.1. *The boundary value problem D has a countable set of eigenvalues γ_k $(k = \pm 0, \pm 1, \pm 2, ...)$. These eigenvalues satisfy the asymptotic formula*

$$\gamma_k = k + a + \frac{A}{\pi k} +$$
$$+ \frac{4(-1)^k b \mathrm{Re}\omega + \alpha q(\pi)(\gamma^2 - 1) - \alpha b^2 q(0) - 2\alpha\gamma p(\pi) - 2b^2 - 2|\omega|^2}{2\pi\alpha b^2 k} + \frac{\xi_k}{k}, \quad (3.1)$$

as $|k| \to \infty$, where $a = \frac{1}{\pi}\mathrm{arcctg}\gamma$, $b = \sqrt{1+\gamma^2}$, $\{\xi_k\} \in l_2$.

Proof. The general solution of the equation (1.1) has a form

$$y(x, \lambda) = M_1 S(x, \lambda) + M_2 C(x, \lambda),$$

where M_1, M_2 are arbitrary constants. Considering the boundary conditions (1.2), the initial conditions (2.1) and taking into account the identity

$$c_1(x, \lambda)[s_2(x, \lambda) + \gamma s_1(x, \lambda)] - s_1(x, \lambda)[c_2(x, \lambda) + \gamma c_1(x, \lambda)] = 1$$

it is easy to verify that the characteristic function of the boundary value problem D will be

$$d(\lambda) = 2\mathrm{Re}\omega - \varphi(\lambda) + |\omega|^2 s_1(\pi, \lambda) + (\alpha\lambda + \beta)\theta(\lambda), \quad (3.2)$$

where

$$\varphi(\lambda) = c_2(\pi, \lambda) + \gamma c_1(\pi, \lambda), \quad \theta(\lambda) = s_2(\pi, \lambda) + \gamma s_1(\pi, \lambda).$$

The zeros of the function $d(\lambda)$ are the eigenvalues of the problem D.

Using the representations in the lemma 2.1, we transform the characteristic function (3.2) to the form

$$d(\lambda) = 2\mathrm{Re}\omega + \alpha\lambda(\cos\lambda\pi - \gamma\sin\lambda\pi) + (\alpha\gamma A_1 + \alpha B_2 + \beta - \gamma)\cos\lambda\pi +$$
$$+ \left(\alpha A_2 + \alpha\gamma B_1 - 1 - |\omega|^2 - \beta\gamma\right)\sin\lambda\pi + \psi_5(\lambda), \quad (3.3)$$

where $\psi_5(\lambda) = \int_{-\pi}^{\pi} \tilde{\psi}_5(t)\, e^{i\lambda t} dt$, $\tilde{\psi}_5(t) \in L_2[-\pi, \pi]$.

Let denote the contour bounded a square

$$K_n = \left\{\lambda : |\mathrm{Re}\lambda - a| \le n + \frac{1}{2}, |\mathrm{Im}\lambda| \le n + \frac{1}{2}\right\}$$

by Γ_n. According to the relation (3.3), we have

$$d(\lambda) = F(\lambda) + G(\lambda),$$

where

$$F(\lambda) = \alpha\lambda(\cos\lambda\pi - \gamma\sin\lambda\pi),$$
$$G(\lambda) = 2\mathrm{Re}\omega + (\alpha\gamma A_1 + \alpha B_2 + \beta - \gamma)\cos\lambda\pi +$$
$$+ \left(\alpha A_2 + \alpha\gamma B_1 - 1 - |\omega|^2 - \beta\gamma\right)\sin\lambda\pi + \psi_5(\lambda).$$

By the standart method (see, for example, [16, p. 43]), it can be shown that for sufficiently large n on Γ_n, the inequality $|F(\lambda)| > |G(\lambda)|$ holds. Then, by Rouche's therom [22, p. 263], the square of K_n contains the same number of zeros of the functions of $d(\lambda)$ and $F(\lambda)$, i.e. $2n + 2$. Denote these zeros in non-decreasing order of their modules by

$$\gamma_{-n}, \gamma_{-n+1}, \ldots, \gamma_{-1}, \gamma_{-0}, \gamma_{+0}, \gamma_1, \ldots, \gamma_{n-1}, \gamma_n.$$

Hence, the problem D has a counting number of eigenvalue.

Applying Rouché's theorem, it is easy to verify that the zeros of function (3.3) when $k \to \pm\infty$ obey the asymptotic formula

$$\gamma_k = k + a + \varepsilon_k, \tag{3.4}$$

where $\varepsilon_k = O\left(k^{-1}\right)$. Let's precise the asymptotic of number ε_k. We will obtain the following equalities if we use from the separation of the functions $\sin x$ and $\cos x$ for this.

$$\cos \gamma_k \pi = (-1)^k \cos\left(a + \varepsilon_k\right)\pi = \frac{(-1)^k \gamma}{\sqrt{1+\gamma^2}} - \frac{(-1)^k}{\sqrt{1+\gamma^2}} \varepsilon_k \pi + O\left(\frac{1}{k^2}\right),$$

$$\sin \gamma_k \pi = (-1)^k \sin\left(a + \varepsilon_k\right)\pi = (-1)^k \left(\frac{1}{\sqrt{1+\gamma^2}} + \frac{\gamma}{\sqrt{1+\gamma^2}} \varepsilon_k \pi\right) + O\left(\frac{1}{k^2}\right).$$

Then, we obtain,

$$\frac{2\mathrm{Re}\omega}{\gamma_k} = \frac{2\mathrm{Re}\omega}{k} + o\left(\frac{1}{k^2}\right),$$

$$\frac{\cos \gamma_k \pi}{\gamma_k} = (-1)^k \frac{\gamma}{\sqrt{1+\gamma^2}} \cdot \frac{1}{k} + o\left(\frac{1}{k^2}\right),$$

$$\frac{\sin \gamma_k \pi}{\gamma_k} = \frac{(-1)^k}{\sqrt{1+\gamma^2}} \cdot \frac{1}{k} + o\left(\frac{1}{k^2}\right),$$

$$\frac{\psi_5\left(\gamma_k\right)}{\gamma_k} = \frac{\tau_k}{k}, \{\tau_k\} \in l_2.$$

By plugging this expression into the equation $d(\lambda) = 0$, we obtain the asymptotic formula

$$\frac{2\mathrm{Re}\omega}{k} + \alpha\left(\frac{(-1)^2 \gamma}{\sqrt{1+\gamma^2}} - \frac{(-1)^k}{\sqrt{1+\gamma^2}} \varepsilon_k \pi - \gamma\left(\frac{(-1)^k}{\sqrt{1+\gamma^2}} + \frac{(-1)^k \gamma}{\sqrt{1+\gamma^2}} \varepsilon_k \pi\right)\right) +$$

$$+ \left(\alpha\gamma A_1 + \alpha B_2 - \gamma + \beta\right) \cdot \frac{(-1)^k \gamma}{k\sqrt{1+\gamma^2}} +$$

$$+ \left(\alpha A_2 - 1 - |\omega|^2 + \alpha\gamma B_1 - \beta\gamma\right) \cdot \frac{(-1)^k}{k\sqrt{1+\gamma^2}} + \frac{\eta_k}{k} = 0, \{\eta_k\} \in l_2.$$

Then,

$$\varepsilon_k = \frac{4\left(-1\right)^k b\mathrm{Re}\omega + \alpha q\left(\pi\right)\left(\gamma^2 - 1\right) - \alpha b^2 q\left(0\right) - 2\alpha\gamma p(\pi) - 2b^2 - 2\left|\omega\right|^2}{2\pi\alpha b^2 k} + \frac{\xi_k}{k},$$

Taking into account this formula in expression (3.4), we obtain the desired formula (3.1). Theorem 3.1 is proved. □

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References

[1] N.V. Abbasli, I.M. Nabiev, The uniqueness of the reconstruction of the Dirac system by three spectra, *J. Contemporary Appl. Math.* **8** (2018), 3-8.

[2] T.Sh. Abdullaev, I.M. Nabiev, An Algorithm for Reconstructing the Dirac Operator with a Spectral Parameter in the Boundary Condition, *Comput. Math. Math. Phys.* **56** (2016), no.2, 256–262.

[3] S. Albeverio, R. Hryniv, Y. Mykytyuk, Inverse spectral problemsfor Dirac operators with summable potentials, *Russian J.Math. Phys.* **12** (2005), no.4, 406–423.

[4] R. Kh. Amirov, B. Keskin, A. S. Ozkan, Direct and inverse problems for the Dirac operator with spectral parameter linearly contained in boundary condition, *Ukrainian Math. J.* **61** (2009), no.9, 1365-1379.

[5] S. Currie, T.T. Roth, B.A. Watson, Eigenvalue interlacing for first order differential systems with periodic 2×2 matrix potentials and quasi–periodic boundary conditions, *Operators and Matrices* **12** (2018), no.2, 489 –499.

[6] M.G. Gasymov, T.T. Dzhabiev, Determination of a system of Dirac differential equations using two spectra, in: Proceedings of School-Seminar on the Spectral Theory of Operators and Representations of Group Theory, *Baku, Elm,* 1975, 46-71 (in Russian).

[7] O. Gorbunov, V. Yurko, Inverse problem for Dirac system with singularities in interior points, *Anal. Math. Phys.* **6** (2016), no.1, 1–29.

[8] Y. Guo, G. Wei, R. Yao, Inverse problems for Dirac operator with the potential known on an interior subinterval, *Anal. Math. Phys.* **9** (2019), no.1, 155–163.

[9] H.M. Huseynov, A.R. Latifova, On eigenvalues and eigenfunctions of one class of Dirac operators with discontinuous coefficients, *Trans. Natl. Acad. Sci. Azerb.* **24** (2004), no.1, 103-112.

[10] E. Korotyaev, Inverse problem and estimates for periodic Zakharov–Shabat systems, *J. Reine Angew. Math.* **583** (2005), 87–115.

[11] V.M. Kurbanov, A.M. Abdullayeva, Bessel property and basicity of the system of root vector–functions of Dirac operator with summable coefficient, *Operators and Matrices.* **12** (2018), no.4, 943-954.

[12] A.A. Lunyov, M. M. Malamud, On the Riesz basis property of root vectors system for 2×2 Dirac type operators, *J. Math. Anal. Appl.* **441** (2016), 57–103.

[13] Kh.R. Mamedov, O. Akcay, Inverse eigenvalue problem for a class of Dirac operators with discontinuous Coefficient, *Boundary Value Problems* **2014** (2014), 1-20.

[14] Kh.R. Mamedov, O. Akcay, Inverse problem for a class of Dirac operators by the Weyl function, *Dynamic Systems and Appl.* **26** (2017), 183 –196.

[15] Kh.R. Mamedov, O. Akcay, Inverse problem for a class of Dirac operator, *Taiwanese J. Math.* **13** (2014), no.3, 753-772.

[16] V.A. Marchenko, *Sturm-Liouville operators and their applications,* Naukova Dumka, Kiev, 1977 (in Russian).

[17] T. V. Misyura, A characteristic of the spectra of periodic and antiperiodic boundary value problems for the Dirac operator. II*, Teor. Funk. Funk. Anal. Prilozh.* (1979), no.31, 102–109.

[18] I.M. Nabiev, Solution of a class of inverse problems for the Dirac operator, *Trans. Acad. sci. Azerb., ser. phys.-tech. math. sci.* **21** (2001), no.1, 146-157.

[19] I.M. Nabiev, Solution of the inverse quasi-periodic problem for the Dirac system, *Math. Notes* **89** (2011), no.6, 845–852.

[20] I.M. Nabiev, The inverse periodic problem for the Dirac operator,*Proc. IMM of NAS of Azerb.* **19** (2003), 177-180.

[21] A.M. Savchuk, A. A. Shkalikov, Dirac operator with complex-valued summable potential, *Math. Notes* **96** (2014), 777–810.

[22] Yu.V. Sidorov, M.V. Fedoryuk, M.I. Shabunin, *Lectures on the theory of functions of a complex variable,* Nauka, Moscow, 1982 (in Russian).
[23] Ch.-Fu Yang, D.-Q. Liu, Half-inverse problem for the Dirac operator, *Appl. Math. Letters* **87** (2019), 172–178.

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