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SOME INEQUALITIES FOR WEIGHTED AND INTEGRAL MEANS OF CONVEX FUNCTIONS ON LINEAR SPACES

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Abstract. Let f be a convex function on a convex subset C of a linear space and x, $y \in C$, with $x \neq y$. If $p : [0,1] \to \mathbb{R}$ is a Lebesgue integrable and symmetric function, namely $p(1-t) = p(t)$ for all $t \in [0,1]$ and such that the condition

$$0 \leq \int_0^\tau p(s)\,ds \leq \int_0^1 p(s)\,ds \text{ for all } \tau \in [0,1]$$

holds, then we have

$$\left| \frac{1}{\int_0^1 p(\tau)\,d\tau} \int_0^1 p(\tau) f((1-\tau)x + \tau y)\,d\tau - \int_0^1 f((1-\tau)x+\tau y)\,d\tau \right|$$
$$\leq \frac{1}{\int_0^1 p(\tau)\,d\tau} \int_0^1 \left(\int_0^\tau p(s)\,ds \right)(1-\tau)\,d\tau \left[\nabla_- f_y(y-x) - \nabla_+ f_x(y-x) \right]$$
$$\leq \frac{1}{2} \left[\nabla_- f_y(y-x) - \nabla_+ f_x(y-x) \right].$$

Some applications for norms and semi-inner products are also provided.

1. Introduction

Let X be a real linear space, x, $y \in X$, $x \neq y$ and let

$$[x,y] := \{(1-\lambda)x + \lambda y,\ \lambda \in [0,1]\}$$

be the *segment* generated by x and y. We consider the function $f : [x,y] \to \mathbb{R}$ and the attached function $\varphi_{(x,y)} : [0,1] \to \mathbb{R}$, $\varphi_{(x,y)}(t) := f[(1-t)x + ty]$, $t \in [0,1]$.

It is well known that f is convex on $[x,y]$ iff $\varphi(x,y)$ is convex on $[0,1]$, and the following lateral derivatives exist and satisfy

(i) $\varphi'_{\pm(x,y)}(s) = \nabla_\pm f_{(1-s)x+sy}(y-x)$, $s \in [0,1)$,
(ii) $\varphi'_{+(x,y)}(0) = \nabla_+ f_x(y-x)$,
(iii) $\varphi'_{-(x,y)}(1) = \nabla_- f_y(y-x)$,

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where $\nabla_{\pm}f_x(y)$ are the *Gâteaux lateral derivatives,* we recall that

$$\begin{aligned}\nabla_{+}f_x(y) &:= \lim_{h\to 0+}\frac{f(x+hy)-f(x)}{h},\\ \nabla_{-}f_x(y) &:= \lim_{k\to 0-}\frac{f(x+ky)-f(x)}{k},\quad x,\ y\in X.\end{aligned}$$

The following inequality is the well-known Hermite-Hadamard integral inequality for convex functions defined on a segment $[x,y]\subset X$:

$$f\left(\frac{x+y}{2}\right)\le\int_0^1 f[(1-t)x+ty]\,dt\le\frac{f(x)+f(y)}{2},\tag{HH}$$

which easily follows by the classical Hermite-Hadamard inequality for the convex function $\varphi(x,y):[0,1]\to\mathbb{R}$

$$\varphi_{(x,y)}\left(\frac{1}{2}\right)\le\int_0^1\varphi_{(x,y)}(t)\,dt\le\frac{\varphi_{(x,y)}(0)+\varphi_{(x,y)}(1)}{2}.$$

For other related results see the monograph on line [8]. For some recent results in linear spaces see [1], [2] and [9]-[12].

In the recent paper [7] we established the following refinements and reverses of Féjer's inequality for functions defined on linear spaces:

Theorem 1.1. *Let f be an convex function on C and $x,\ y\in C$ with $x\neq y$. If $p:[0,1]\to[0,\infty)$ is Lebesgue integrable and symmetric, namely $p(1-t)=p(t)$ for all $t\in[0,1]$, then*

$$\begin{aligned}0&\le\frac{1}{2}\left[\nabla_{+}f_{\frac{x+y}{2}}(y-x)-\nabla_{-}f_{\frac{x+y}{2}}(y-x)\right]\int_0^1\left|t-\frac{1}{2}\right|p(t)\,dt\\ &\le\int_0^1 f((1-t)x+ty)\,p(t)\,dt-f\left(\frac{x+y}{2}\right)\int_0^1 p(t)\,dt\\ &\le\frac{1}{2}\left[\nabla_{-}f_y(y-x)-\nabla_{+}f_x(y-x)\right]\left(\int_0^1\left|t-\frac{1}{2}\right|p(t)\,dt\right)\end{aligned}\tag{1.1}$$

and

$$\begin{aligned}0&\le\frac{1}{2}\left[\nabla_{+}f_{\frac{x+y}{2}}(y-x)-\nabla_{-}f_{\frac{x+y}{2}}(y-x)\right]\int_0^1\left(\frac{1}{2}-\left|t-\frac{1}{2}\right|\right)p(t)\,dt\\ &\le\frac{f(x)+f(y)}{2}\int_0^1 p(t)\,dt-\int_0^1 f((1-t)x+ty)\,p(t)\,dt\\ &\le\frac{1}{2}\left[\nabla_{-}f_y(y-x)-\nabla_{+}f_x(y-x)\right]\int_0^1\left(\frac{1}{2}-\left|t-\frac{1}{2}\right|\right)p(t)\,dt.\end{aligned}\tag{1.2}$$

If we take $p\equiv 1$ in (1.1), then we get

$$\begin{aligned}0&\le\frac{1}{8}\left[\nabla_{+}f_{\frac{x+y}{2}}(y-x)-\nabla_{-}f_{\frac{x+y}{2}}(y-x)\right]\\ &\le\int_0^1 f[(1-t)x+ty]\,dt-f\left(\frac{x+y}{2}\right)\\ &\le\frac{1}{8}\left[\nabla_{-}f_y(y-x)-\nabla_{+}f_x(y-x)\right]\end{aligned}\tag{1.3}$$

that was firstly obtained in [4], while from (1.2) we recapture the result obtained in [5]

$$
\begin{aligned}
0 &\le \frac{1}{8}\left[\nabla_{+} f_{\frac{x+y}{2}}(y-x) - \nabla_{-} f_{\frac{x+y}{2}}(y-x)\right] \qquad (1.4)\\
&\le \frac{f(x)+f(y)}{2} - \int_0^1 f[(1-t)x+ty]\,dt \\
&\le \frac{1}{8}\left[\nabla_{-} f_y(y-x) - \nabla_{+} f_x(y-x)\right].
\end{aligned}
$$

Motivated by the above results, we establish in this paper some upper and lower bounds for the difference

$$\int_0^1 p(\tau) f((1-\tau)x+\tau y)\,d\tau - \int_0^1 p(\tau)\,d\tau \int_0^1 f((1-\tau)x+\tau y)\,d\tau$$

where f is a convex function on C and $x,\ y \in C$, with $x \ne y$ while $p:[0,1]\to\mathbb{R}$ is a Lebesgue integrable function such that

$$0 \le \int_0^{\tau} p(s)\,ds \le \int_0^1 p(s)\,ds \text{ for all } \tau\in[0,1].$$

Some applications for norms and semi-inner products are also provided.

2. Main Results

We start to the following identity that is of interest in itself as well:

Lemma 2.1. *Let f be a convex function on C and $x,\ y \in C$, with $x \ne y$. If $g:[0,1]\to\mathbb{C}$ is a Lebesgue integrable function, then we have the equality*

$$
\begin{aligned}
&\int_0^1 g(\tau)\varphi_{(x,y)}(\tau)\,d\tau - \int_0^1 g(\tau)\,d\tau \int_0^1 \varphi_{(x,y)}(\tau)\,d\tau \qquad (2.1)\\
&= \int_0^1 \left(\int_\tau^1 g(s)\,ds\right) \tau\varphi'_{(x,y)}(\tau)\,d\tau \\
&+ \int_0^1 \left(\int_0^\tau g(s)\,ds\right) (\tau-1)\,\varphi'_{(x,y)}(\tau)\,d\tau.
\end{aligned}
$$

Proof. Integrating by parts in the Lebesgue integral, we have

$$
\begin{aligned}
&\int_0^\tau t\varphi'_{(x,y)}(t)\,dt + \int_\tau^1 (t-1)\,\varphi'_{(x,y)}(t)\,dt \\
&= \tau\varphi_{(x,y)}(\tau) - \int_0^\tau \varphi_{(x,y)}(t)\,dt - (\tau-1)\,\varphi_{(x,y)}(\tau) - \int_\tau^1 \varphi_{(x,y)}(t)\,dt \\
&= \varphi_{(x,y)}(\tau) - \int_0^1 \varphi_{(x,y)}(t)\,dt
\end{aligned}
$$

that holds for all $\tau\in[0,1]$.

If we multiply this identity by $g(\tau)$ and integrate over τ in $[0,1]$, then we get

$$\int_0^1 g(\tau)\varphi_{(x,y)}(\tau)\,d\tau - \int_0^1 g(\tau)\,d\tau\int_0^1 \varphi_{(x,y)}(t)\,dt \tag{2.2}$$
$$= \int_0^1 g(\tau)\left(\int_0^\tau t\varphi'_{(x,y)}(t)\,dt\right)d\tau + \int_0^1 g(\tau)\left(\int_\tau^1 (t-1)\varphi'_{(x,y)}(t)\,dt\right)d\tau.$$

Using integration by parts, we get

$$\int_0^1 g(\tau)\left(\int_0^\tau t\varphi'_{(x,y)}(t)\,dt\right)d\tau \tag{2.3}$$
$$= \int_0^1 \left(\int_0^\tau t\varphi'_{(x,y)}(t)\,dt\right)d\left(\int_0^\tau g(s)\,ds\right)$$
$$= \left(\int_0^\tau g(s)\,ds\right)\left(\int_0^\tau t\varphi'_{(x,y)}(t)\,dt\right)\Bigg|_0^1$$
$$- \int_0^1 \left(\int_0^\tau g(s)\,ds\right)\tau\varphi'_{(x,y)}(\tau)\,d\tau$$
$$= \left(\int_0^1 g(s)\,ds\right)\left(\int_0^1 t\varphi'_{(x,y)}(t)\,dt\right)$$
$$- \int_0^1 \left(\int_0^\tau g(s)\,ds\right)\tau\varphi'_{(x,y)}(\tau)\,d\tau$$
$$= \int_0^1 \left(\int_0^1 g(s)\,ds - \int_0^\tau g(s)\,ds\right)\tau\varphi'_{(x,y)}(\tau)\,d\tau$$
$$= \int_0^1 \left(\int_\tau^1 g(s)\,ds\right)\tau\varphi'_{(x,y)}(\tau)\,d\tau$$

and

$$\int_0^1 g(\tau)\left(\int_\tau^1 (t-1)\varphi'_{(x,y)}(t)\,dt\right)d\tau \tag{2.4}$$
$$= \int_0^1 \left(\int_\tau^1 (t-1)\varphi'_{(x,y)}(t)\,dt\right)d\left(\int_0^\tau g(s)\,ds\right)$$
$$= \left(\int_\tau^1 (t-1)\varphi'_{(x,y)}(t)\,dt\right)\left(\int_0^\tau g(s)\,ds\right)\Bigg|_0^1$$
$$+ \int_0^1 \left(\int_0^\tau g(s)\,ds\right)(\tau-1)\varphi'_{(x,y)}(\tau)\,d\tau$$
$$= \int_0^1 \left(\int_0^\tau g(s)\,ds\right)(\tau-1)\varphi'_{(x,y)}(\tau)\,d\tau,$$

which proves the identity in (2.1). □

Theorem 2.1. *Let f be an operator convex function on C and x, $y \in C$, with $x \neq y$. If $p:[0,1]\to\mathbb{R}$ is a Lebesgue integrable function such that*

$$0 \le \int_0^\tau p(s)\,ds \le \int_0^1 p(s)\,ds \text{ for all } \tau\in[0,1], \tag{2.5}$$

then we have the inequalities

$$\int_0^1 \left(\int_\tau^1 p(s)\,ds\right) \tau d\tau \nabla_+ f_x(y-x) \tag{2.6}$$
$$- \int_0^1 \left(\int_0^\tau p(s)\,ds\right) (1-\tau)\, d\tau \nabla_- f_y(y-x)$$
$$\leq \int_0^1 p(\tau) f((1-\tau)x+\tau y)\,d\tau - \int_0^1 p(\tau)\,d\tau \int_0^1 f((1-\tau)x+\tau y)\,d\tau$$
$$\leq \int_0^1 \left(\int_\tau^1 p(s)\,ds\right) \tau d\tau \nabla_- f_y(y-x)$$
$$- \int_0^1 \left(\int_0^\tau p(s)\,ds\right) (1-\tau)\, d\tau \nabla_+ f_x(y-x)$$

or, equivalently,

$$\int_0^1 (1-\tau) \left(\int_0^\tau [p(1-s)\nabla_- f_y(y-x) - p(s)\nabla_+ f_x(y-x)]\,ds\right) d\tau \tag{2.7}$$
$$\leq \int_0^1 p(\tau) f((1-\tau)x+\tau y)\,d\tau - \int_0^1 p(\tau)\,d\tau \int_0^1 f((1-\tau)x+\tau y)\,d\tau$$
$$\leq \int_0^1 (1-\tau) \left(\int_0^\tau [p(1-s)\nabla_+ f_x(y-x) - p(s)\nabla_- f_y(y-x)]\,ds\right) d\tau.$$

Proof. We have for $\varphi_{(x,y)}$ and $p:[0,1]\to\mathbb{R}$ a Lebesgue integrable function that

$$\int_0^1 p(\tau)\varphi_{(x,y)}(\tau)\,d\tau - \int_0^1 p(\tau)\,d\tau \int_0^1 \varphi_{(x,y)}(\tau)\,d\tau \tag{2.8}$$
$$= \int_0^1 \left(\int_\tau^1 p(s)\,ds\right)(\tau)\,\varphi'_{(x,y)}(\tau)\,d\tau$$
$$- \int_0^1 \left(\int_0^\tau p(s)\,ds\right)(1-\tau)\,\varphi'_{(x,y)}(\tau)\,d\tau.$$

By the gradient inequalities for $\varphi_{(x,y)}$ we have

$$\tau\nabla_- f_y(y-x) \geq \tau\varphi'_{(x,y)}(\tau) \geq \tau\nabla_+ f_x(y-x) \tag{2.9}$$

and

$$(1-\tau)\nabla_- f_y(y-x) \geq (1-\tau)\varphi'_{(x,y)}(\tau) \geq (1-\tau)\nabla_+ f_x(y-x) \tag{2.10}$$

for all $\tau \in (0,1)$.

From

$$\int_0^\tau p(s)\,ds \leq \int_0^1 p(s)\,ds = \int_0^\tau p(s)\,ds + \int_\tau^1 p(s)\,ds,$$

we get that $\int_\tau^1 p(s)\,ds \geq 0$ for all $\tau\in(0,1)$.

From (2.9) we derive that

$$\left(\int_\tau^1 p(s)\,ds\right)\tau\nabla_- f_y(y-x) \geq \left(\int_\tau^1 p(s)\,ds\right)\tau\varphi'_{(x,y)}(\tau)$$
$$\geq \left(\int_\tau^1 p(s)\,ds\right)\tau\nabla_+ f_x(y-x)$$

and from (2.10) that

$$\begin{aligned}
-\left(\int_0^\tau p(s)\,ds\right)(1-\tau)\nabla_+ f_x(y-x) &\le -\left(\int_0^\tau p(s)\,ds\right)(1-\tau)\varphi'_{(x,y)}(\tau) \\
&\le -\left(\int_0^\tau p(s)\,ds\right)(1-\tau)\nabla_- f_y(y-x)
\end{aligned}$$

all $\tau\in(0,1)$.

If we integrate these inequalities over $\tau\in[0,1]$ and add the obtained results, then we get

$$\begin{aligned}
&\int_0^1\left(\int_\tau^1 p(s)\,ds\right)\tau d\tau\nabla_- f_y(y-x)-\int_0^1\left(\int_0^\tau p(s)\,ds\right)(1-\tau)\,d\tau\nabla_+ f_x(y-x) \\
&\ge \int_0^1\left(\int_\tau^1 p(s)\,ds\right)\tau\varphi'_{(x,y)}(\tau)\,d\tau-\int_0^1\left(\int_0^\tau p(s)\,ds\right)(1-\tau)\varphi'_{(x,y)}(\tau)\,d\tau \\
&\ge \int_0^1\left(\int_\tau^1 p(s)\,ds\right)\tau d\tau\nabla_+ f_x(y-x)-\int_0^1\left(\int_0^\tau p(s)\,ds\right)(1-\tau)\,d\tau\nabla_- f_y(y-x).
\end{aligned}$$

By using the equality (2.1) we obtain

$$\begin{aligned}
&\int_0^1\left(\int_\tau^1 p(s)\,ds\right)\tau d\tau\nabla_+ f_x(y-x) \\
&-\int_0^1\left(\int_0^\tau p(s)\,ds\right)(1-\tau)\,d\tau\nabla_- f_y(y-x) \\
&\le \int_0^1 p(\tau)\varphi_{(x,y)}(\tau)\,d\tau-\int_0^1 p(\tau)\,d\tau\int_0^1\varphi_{(x,y)}(\tau)\,d\tau \\
&\le \int_0^1\left(\int_\tau^1 p(s)\,ds\right)\tau d\tau\nabla_- f_y(y-x) \\
&-\int_0^1\left(\int_0^\tau p(s)\,ds\right)(1-\tau)\,d\tau\nabla_+ f_x(y-x),
\end{aligned}\tag{2.11}$$

namely (2.6).

If we change the variable $y=1-\tau$, then we have

$$\int_0^1\left(\int_\tau^1 p(s)\,ds\right)\tau d\tau=\int_0^1\left(\int_{1-y}^1 p(s)\,ds\right)(1-y)\,dy.$$

Also by the change of variable $u=1-s$, we get

$$\int_{1-y}^1 p(s)\,ds=\int_0^y p(1-u)\,du,$$

which implies that

$$\int_0^1\left(\int_\tau^1 p(s)\,ds\right)\tau d\tau=\int_0^1\left(\int_0^\tau p(1-s)\,ds\right)(1-\tau)\,d\tau.$$

Therefore

$$\int_0^1 \left(\int_\tau^1 p(s)\,ds \right) \tau d\tau \nabla_- f_y(y-x) - \int_0^1 \left(\int_0^\tau p(s)\,ds \right) (1-\tau)\,d\tau \nabla_+ f_x(y-x)$$
$$= \int_0^1 \left(\int_0^\tau p(1-s)\,ds \right) (1-\tau)\,d\tau \nabla_- f_y(y-x)$$
$$- \int_0^1 \left(\int_0^\tau p(s)\,ds \right) (1-\tau)\,d\tau \nabla_+ f_x(y-x)$$
$$= \int_0^1 (1-\tau) \left(\int_0^\tau [p(1-s)\nabla_- f_y(y-x) - p(s)\nabla_+ f_x(y-x)]\,ds \right) d\tau$$

and

$$\int_0^1 \left(\int_\tau^1 p(s)\,ds \right) \tau d\tau \nabla_+ f_x(y-x) - \int_0^1 \left(\int_0^\tau p(s)\,ds \right) (1-\tau)\,d\tau \nabla_- f_y(y-x)$$
$$= \int_0^1 \left(\int_0^\tau p(1-s)\,ds \right) (1-\tau)\,d\tau \nabla_+ f_x(y-x)$$
$$- \int_0^1 \left(\int_0^\tau p(s)\,ds \right) (1-\tau)\,d\tau \nabla_- f_y(y-x)$$
$$= \int_0^1 (1-\tau) \left(\int_0^\tau [p(1-s)\nabla_+ f_x(y-x) - p(s)\nabla_- f_y(y-x)]\,ds \right) d\tau,$$

and by (2.11) we get (2.7). □

We say that the function $p : [0,1] \to \mathbb{R}$ is symmetric on $[0,1]$ if

$$p(1-t) = p(t) \text{ for all } t \in [0,1].$$

Corollary 2.1. *Let f be a convex function on C and x, $y \in C$, with $x \neq y$. If $p : [0,1] \to \mathbb{R}$ is a Lebesgue integrable and symmetric function such that the condition (2.5) holds, then we have*

$$\left| \frac{1}{\int_0^1 p(\tau)\,d\tau} \int_0^1 p(\tau) f((1-\tau)x + \tau y)\,d\tau - \int_0^1 f((1-\tau)x + \tau y)\,d\tau \right| \tag{2.12}$$
$$\leq \frac{1}{\int_0^1 p(\tau)\,d\tau} \int_0^1 \left(\int_0^\tau p(s)\,ds \right) (1-\tau)\,d\tau \left[\nabla_- f_y(y-x) - \nabla_+ f_x(y-x) \right]$$
$$\leq \frac{1}{2} \left[\nabla_- f_y(y-x) - \nabla_+ f_x(y-x) \right].$$

Proof. Since p is symmetric, then $p(1-s) = p(s)$ for all $s \in [0,1]$ and by (2.7) we get

$$\int_0^1 \left(\int_0^\tau p(s)\,ds \right) (1-\tau)\,d\tau \left[\nabla_+ f_x(y-x) - \nabla_- f_y(y-x) \right]$$
$$\leq \int_0^1 p(\tau)\varphi_{(x,y)}(\tau)\,d\tau - \int_0^1 p(\tau)\,d\tau \int_0^1 \varphi_{(x,y)}(\tau)\,d\tau$$
$$\leq \left[\nabla_- f_y(y-x) - \nabla_+ f_x(y-x) \right] \int_0^1 \left(\int_0^\tau p(s)\,ds \right) (1-\tau)\,d\tau,$$

which is equivalent to the first inequality in (2.12).

Since $0 \leq \int_0^\tau p(s)\,ds \leq \int_0^1 p(\tau)\,d\tau$, hence

$$\int_0^1 \left(\int_0^\tau p(s)\,ds\right)(1-\tau)\,d\tau \leq \int_0^1 p(\tau)\,d\tau \int_0^1 (1-\tau)\,d\tau = \frac{1}{2}\int_0^1 p(\tau)\,d\tau$$

and the last part of (2.12) is proved. $\square$

Remark 2.1. If the function p is nonnegative and symmetric then the inequality (2.12) holds true.

If we consider the weight $p:[0,1]\to[0,\infty)$, $p(s)=\left|s-\frac{1}{2}\right|$, then

$$\begin{aligned}
&\int_0^1 \left(\int_0^\tau p(s)\,ds\right)(1-\tau)\,d\tau \\
&= \int_0^1 \left(\int_0^\tau \left|s-\frac{1}{2}\right| ds\right)(1-\tau)\,d\tau \\
&= \int_0^{\frac{1}{2}} \left(\int_0^\tau \left|s-\frac{1}{2}\right| ds\right)(1-\tau)\,d\tau \\
&+ \int_{\frac{1}{2}}^1 \left(\int_0^\tau \left|s-\frac{1}{2}\right| ds\right)(1-\tau)\,d\tau \\
&= \int_0^{\frac{1}{2}} \left(\int_0^\tau \left(\frac{1}{2}-s\right) ds\right)(1-\tau)\,d\tau \\
&+ \int_{\frac{1}{2}}^1 \left(\int_0^{\frac{1}{2}} \left(\frac{1}{2}-s\right) ds + \int_{\frac{1}{2}}^\tau \left(s-\frac{1}{2}\right)\right)(1-\tau)\,d\tau \\
&= \int_0^{\frac{1}{2}} \left(\frac{1}{2}\tau-\frac{\tau^2}{2}\right)(1-\tau)\,d\tau \\
&+ \int_{\frac{1}{2}}^1 \left(\int_0^{\frac{1}{2}} \left(\frac{1}{2}-s\right) ds + \int_{\frac{1}{2}}^\tau \left(s-\frac{1}{2}\right) ds\right)(1-\tau)\,d\tau.
\end{aligned}$$

We have

$$\begin{aligned}
&\int_0^{\frac{1}{2}} \left(\frac{1}{2}\tau-\frac{\tau^2}{2}\right)(1-\tau)\,d\tau = \frac{1}{2}\int_0^{\frac{1}{2}} (1-\tau)\,\tau\,(1-\tau)\,d\tau \\
&= \frac{1}{2}\int_0^{\frac{1}{2}} (1-\tau)^2\,\tau d\tau = \frac{11}{384}
\end{aligned}$$

and

$$\begin{aligned}
&\int_{\frac{1}{2}}^1 \left(\int_0^{\frac{1}{2}} \left(\frac{1}{2}-s\right) ds + \int_{\frac{1}{2}}^\tau \left(s-\frac{1}{2}\right) ds\right)(1-\tau)\,d\tau \\
&= \int_{\frac{1}{2}}^1 \left(\frac{1}{8}+\frac{1}{2}\left(\tau-\frac{1}{2}\right)^2\right)(1-\tau)\,d\tau \\
&= \frac{1}{8}\int_{\frac{1}{2}}^1 (1-\tau)\,d\tau + \frac{1}{2}\int_{\frac{1}{2}}^1 \left(\tau-\frac{1}{2}\right)^2 (1-\tau)\,d\tau = \frac{7}{384}.
\end{aligned}$$

Therefore

$$\int_0^1 \left(\int_0^\tau p(s)\,ds\right)(1-\tau)\,d\tau = \frac{3}{64}.$$

Since $\int_0^1 \left|\tau - \frac{1}{2}\right| d\tau = \frac{1}{4}$, hence

$$\frac{1}{\int_0^1 p(\tau)\,d\tau}\int_0^1 \left(\int_0^\tau p(s)\,ds\right)(1-\tau)\,d\tau = \frac{3}{16}.$$

Utilising (2.12) for symmetric weight $p:[0,1]\to[0,\infty)$, $p(s)=\left|s-\frac{1}{2}\right|$, we get

$$\left|4\int_0^1 \left|\tau-\frac{1}{2}\right| f((1-\tau)x+\tau y)\,d\tau - \int_0^1 f((1-\tau)x+\tau y)\,d\tau\right| \tag{2.13}$$
$$\le \frac{3}{16}\left[\nabla_- f_y(y-x) - \nabla_+ f_x(y-x)\right]$$

where f is a convex function on C and x, $y \in C$, with $x \neq y$.

Consider now the symmetric function $p(s)=(1-s)s$, $x\in[0,1]$. Then

$$\int_0^\tau p(s)\,ds = \int_a^\tau (1-s)\,sds = -\frac{1}{6}\tau^2(2\tau-3),\ \ \tau\in[0,1]$$

and

$$\int_0^1 \left(\int_0^\tau p(s)\,ds\right)(1-\tau)\,d\tau = -\frac{1}{6}\int_0^1 \tau^2(2\tau-3)(1-\tau)\,d\tau = \frac{1}{40}.$$

Also

$$\int_0^1 p(\tau)\,d\tau = \int_0^1 (1-\tau)\,\tau d\tau = \frac{1}{6}$$

and

$$\frac{1}{\int_0^1 p(\tau)\,d\tau}\int_0^1 \left(\int_0^\tau p(s)\,ds\right)(1-\tau)\,d\tau = \frac{3}{20}$$

and by (2.12) we obtain

$$\left|6\int_0^1 (1-\tau)\,\tau f((1-\tau)x+\tau y)\,d\tau - \int_0^1 f((1-\tau)x+\tau y)\,d\tau\right| \tag{2.14}$$
$$\le \frac{3}{20}\left[\nabla_- f_y(y-x) - \nabla_+ f_x(y-x)\right],$$

where f is a convex function on C and x, $y\in C$, with $x\neq y$.

3. Examples for Norms

Now, assume that $(X,\|\cdot\|)$ is a normed linear space. The function $f_0(s)=\frac{1}{2}\|x\|^2$, $x\in X$ is convex and thus the following limits exist

(iv) $\langle x,y\rangle_s := \nabla_+ f_{0,y}(x) = \lim_{t\to 0+}\frac{\|y+tx\|^2-\|y\|^2}{2t}$;

(v) $\langle x,y\rangle_i := \nabla_- f_{0,y}(x) = \lim_{s\to 0-}\frac{\|y+sx\|^2-\|y\|^2}{2s}$;

for any x, $y \in X$. They are called the *lower* and *upper semi-inner* products associated to the norm $\|\cdot\|$.

For the sake of completeness we list here some of the main properties of these mappings that will be used in the sequel (see for example [2] or [6]), assuming that p, $q \in \{s, i\}$ and $p \neq q$:

(a) $\langle x, x\rangle_p = \|x\|^2$ for all $x \in X$;
(aa) $\langle \alpha x, \beta y\rangle_p = \alpha\beta \langle x, y\rangle_p$ if α, $\beta \geq 0$ and x, $y \in X$;
(aaa) $\left|\langle x, y\rangle_p\right| \leq \|x\| \|y\|$ for all x, $y \in X$;
(av) $\langle \alpha x + y, x\rangle_p = \alpha \langle x, x\rangle_p + \langle y, x\rangle_p$ if x, $y \in X$ and $\alpha \in \mathbb{R}$;
(v) $\langle -x, y\rangle_p = -\langle x, y\rangle_q$ for all x, $y \in X$;
(va) $\langle x + y, z\rangle_p \leq \|x\| \|z\| + \langle y, z\rangle_p$ for all x, y, $z \in X$;
(vaa) The mapping $\langle \cdot, \cdot\rangle_p$ is continuous and subadditive (superadditive) in the first variable for $p = s$ (or $p = i$);
(vaaa) The normed linear space $(X, \|\cdot\|)$ is smooth at the point $x_0 \in X \setminus \{0\}$ if and only if $\langle y, x_0\rangle_s = \langle y, x_0\rangle_i$ for all $y \in X$; in general $\langle y, x\rangle_i \leq \langle y, x\rangle_s$ for all x, $y \in X$;
(ax) If the norm $\|\cdot\|$ is induced by an inner product $\langle \cdot, \cdot\rangle$, then $\langle y, x\rangle_i = \langle y, x\rangle = \langle y, x\rangle_s$ for all x, $y \in X$.

The function $f_r(x) = \|x\|^r$ ($x \in X$ and $1 \leq r < \infty$) is also convex. Therefore, the following limits, which are related to the superior (inferior) semi-inner products,

$$\nabla_{\pm} f_{r,y}(x) := \lim_{t \to 0\pm} \frac{\|y + tx\|^r - \|y\|^r}{t} = r \|y\|^{r-1} \lim_{t \to 0\pm} \frac{\|y + tx\| - \|y\|}{t} = r \|y\|^{r-2} \langle x, y\rangle_{s(i)}$$

exist for all x, $y \in X$ whenever $r \geq 2$; otherwise, they exist for any $x \in X$ and nonzero $y \in X$. In particular, if $r = 1$, then the following limits

$$\nabla_{\pm} f_{1,y}(x) := \lim_{t \to 0\pm} \frac{\|y + tx\| - \|y\|}{t} = \frac{\langle x, y\rangle_{s(i)}}{\|y\|}$$

exist for x, $y \in X$ and $y \neq 0$.

If $p : [0, 1] \to \mathbb{R}$ is a Lebesgue integrable and symmetric function such that the condition

$$0 \leq \int_0^\tau p(s)\, ds \leq \int_0^1 p(s)\, ds \text{ for all } \tau \in [0, 1],$$

is valid, then by (2.12) we get

$$\left|\frac{1}{\int_0^1 p(\tau)\, d\tau} \int_0^1 p(\tau) \|(1-\tau)x + \tau y\|^r d\tau - \int_0^1 \|(1-\tau)x + \tau y\|^r d\tau\right| \tag{3.1}$$
$$\leq \frac{r}{\int_0^1 p(\tau)\, d\tau} \int_0^1 \left(\int_0^\tau p(s)\, ds\right)(1-\tau)\, d\tau$$
$$\times \left[\|y\|^{r-2} \langle y - x, y\rangle_i - \|x\|^{r-2} \langle y - x, x\rangle_s\right].$$

If $r \geq 2$, then the inequality (3.1) holds for all x, $y \in X$. If $r \in [1, 2)$, then the inequality (3.1) holds for all x, $y \in X$ with x, $y \neq 0$.

For $r=2$ we get

$$\left|\frac{1}{\int_0^1 p(\tau)\,d\tau}\int_0^1 p(\tau)\left\|(1-\tau)x+\tau y\right\|^2 d\tau-\int_0^1\left\|(1-\tau)x+\tau y\right\|^2 d\tau\right| \tag{3.2}$$
$$\leq \frac{2}{\int_0^1 p(\tau)\,d\tau}\int_0^1\left(\int_0^\tau p(s)\,ds\right)(1-\tau)\,d\tau\left[\langle y-x,y\rangle_i-\langle y-x,x\rangle_s\right]$$

for all x, $y\in X$.

If we take $p(\tau)=\left|\tau-\frac{1}{2}\right|$, $\tau\in[0,1]$ in (3.1), then we obtain

$$\left|4\int_0^1\left|\tau-\frac{1}{2}\right|\left\|(1-\tau)x+\tau y\right\|^r d\tau-\int_0^1\left\|(1-\tau)x+\tau y\right\|^r d\tau\right| \tag{3.3}$$
$$\leq\frac{3r}{16}\left[\|y\|^{r-2}\langle y-x,y\rangle_i-\|x\|^{r-2}\langle y-x,x\rangle_s\right].$$

If $X=H$ a real inner product space, then from (3.2) we get

$$\left|\frac{1}{\int_0^1 p(\tau)\,d\tau}\int_0^1 p(\tau)\left\|(1-\tau)x+\tau y\right\|^2 d\tau-\int_0^1\left\|(1-\tau)x+\tau y\right\|^2 d\tau\right| \tag{3.4}$$
$$\leq \frac{2}{\int_0^1 p(\tau)\,d\tau}\int_0^1\left(\int_0^\tau p(s)\,ds\right)(1-\tau)\,d\tau\left\|y-x\right\|^2$$

for all x, $y\in H$.

4. Examples for Functions of Several Variables

Now, let $\Omega\subset\mathbb{R}^n$ be an open convex set in $\mathbb{R}^n$. If $F:\Omega\to\mathbb{R}$ is a differentiable convex function on Ω, then, obviously, for any $\bar{c}\in\Omega$ we have

$$\nabla F_{\bar{c}}(\bar{y})=\sum_{i=1}^n\frac{\partial F(\bar{c})}{\partial x_i}\cdot y_i,\ \ \bar{y}=(y_1,...,y_n)\in\mathbb{R}^n,$$

where $\frac{\partial F}{\partial x_i}$ are the partial derivatives of F with respect to the variable x_i ($i=1,...,n$).

If $p:[0,1]\to\mathbb{R}$ is a Lebesgue integrable and symmetric function such that the condition (2.5) holds, then we have for all $\bar{a}$, $\bar{b}\in\Omega$ that

$$\left|\frac{1}{\int_0^1 p(\tau)\,d\tau}\int_0^1 p(\tau)F\left((1-\tau)\bar{a}+\tau\bar{b}\right)d\tau-\int_0^1 f\left((1-\tau)\bar{a}+\tau\bar{b}\right)d\tau\right| \tag{4.1}$$
$$\leq\frac{1}{\int_0^1 p(\tau)\,d\tau}\int_0^1\left(\int_0^\tau p(s)\,ds\right)(1-\tau)\,d\tau$$
$$\times\sum_{i=1}^n\left(\frac{\partial F(\bar{b})}{\partial x_i}-\frac{\partial F(\bar{a})}{\partial x_i}\right)(b_i-a_i)$$
$$\leq\frac{1}{2}\sum_{i=1}^n\left(\frac{\partial F(\bar{b})}{\partial x_i}-\frac{\partial F(\bar{a})}{\partial x_i}\right)(b_i-a_i).$$

If we take $p(\tau) = \left|\tau - \frac{1}{2}\right|$, $\tau \in [0,1]$ in (4.1), then we get

$$\left|4\int_0^1 \left|\tau - \frac{1}{2}\right| F\left((1-\tau)\bar{a} + \tau\bar{b}\right) d\tau - \int_0^1 f\left((1-\tau)\bar{a} + \tau\bar{b}\right) d\tau\right| \tag{4.2}$$
$$\leq \frac{3}{16}\sum_{i=1}^{n}\left(\frac{\partial F\left(\bar{b}\right)}{\partial x_i} - \frac{\partial F\left(\bar{a}\right)}{\partial x_i}\right)(b_i - a_i)$$

for all $\bar{a}$, $\bar{b} \in \Omega$.

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