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INVESTIGATION OF THE PERIODICITY AND STABILITY IN THE NEUTRAL DIFFERENTIAL SYSTEMS BY USING KRASNOSELSKII'S FIXED POINT THEOREM

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Abstract. In this paper, we study the periodicity and stability of solutions for a neutral differential system. In the process we use the fundamental matrix solution to convert the given differential system into an equivalent integral system. Then we employ Krasnoselskii's fixed point theorem to show the existence and stability of periodic solutions of this neutral differential system. Our results extend and complement some earlier publications.

1. Introduction

Delay differential equations have received increasing attention during recent years since these equations have been proved to be valuable tools in the modeling of many phenomena in various fields of science and engineering, see the papers [1]–[17] and the references therein.

Ding and Li [8] discussed the existence and asymptotic stability of periodic solutions for the following neutral functional differential equation

$$\begin{aligned}&\frac{d}{dt}u(t) - q\frac{d}{dt}u(t-r)\\&= p(t) - au(t) - aqu(t-r) - bf(u(t)) + bqf(u(t-r)).\end{aligned}\tag{1.1}$$

By employing Krasnoselskii's fixed point theorem, the authors obtained the existence and asymptotic stability results for periodic solutions.

In this paper, we are interested on the existence and asymptotic stability of periodic solutions of the following neutral differential system

$$\begin{aligned}&\frac{d}{dt}u(t) - q\frac{d}{dt}u(t-r)\\&= P(t) + A(t)u(t) + A(t)qu(t-r) - bf(u(t)) + bqf(u(t-r)),\end{aligned}\tag{1.2}$$

where $b > 0$, $|q| < 1$, $r > 0$ and A is nonsingular $n \times n$ matrix with continuous real-valued functions as its elements. The functions $P : \mathbb{R} \to \mathbb{R}^n$ and $f : \mathbb{R}^n \to \mathbb{R}^n$ are continuously differentiable.

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In the analysis we use the fundamental matrix solution coupled with Floquet theory to invert the differential system (1.2) into an integral system. Then, we employ Krasnoselskii's fixed point theorem to show the existence and asymptotic stability of periodic solutions of the system (1.2). The obtained integral system is the sum of two mappings, one is a compact operator and the other is a contraction. It is easy to see that (1.2) reduce to (1.1) when $n = 1$ and $A(t) = -a$. Then, the results obtained here extend some results of the work of Ding and Li [8].

2. Existence of periodic solutions

In this section, $C^1(\mathbb{R}, \mathbb{R}^n)$ and $C(\mathbb{R}, \mathbb{R}^n)$ denote the set of all continuously differentiable functions and all continuous functions $\phi : \mathbb{R} \to \mathbb{R}^n$ respectively. For $T > 0$, $C_T = \{\phi \in C(\mathbb{R}, \mathbb{R}^n),\ \phi(t+T) = \phi(t)\}$ is a Banach space with the supremum norm

$$\|\phi\|_0 = \sup_{t\in\mathbb{R}} |\phi(t)| = \sup_{t\in[0,T]} |\phi(t)|,$$

where $|.|$ denotes the infinity norm for $x \in \mathbb{R}^n$ and $C_T^1 = C^1(\mathbb{R}, \mathbb{R}^n) \cap C_T$ is a Banach space with the norm $\|\phi\|_1 = \|\phi\|_0 + \|\phi'\|_0$ in a period interval. Also, if A is an $n \times n$ real matrix, then we define the norm of A by $|A| = \max_{1\le i\le n} \sum_{j=1}^{n} |a_{ij}|$.

For a sufficiently small positive L, (1.2) can be transformed as

$$\begin{aligned}&\frac{d}{dt}v(t) - q\frac{d}{dt}v(t-\tau)\\&= LP_1(t) + LA_1(t)v(t) + LA_1(t)qv(t-\tau) - Lbf(v(t)) + Lbqf(v(t-\tau)),\end{aligned} \tag{2.1}$$

where $v(t) = u(Lt)$, $\tau = \frac{r}{L}$, $P_1(t) = P(Lt)$ and $A_1(t) = A(Lt)$.

First we make the following definition.

Definition 2.1. If the matrix A_1 is periodic of period $\omega = \frac{T}{L}$, then the linear system

$$y'(t) = LA_1(t)y(t), \tag{2.2}$$

is said to be noncritical with respect to ω if it has no periodic solution of period ω except the trivial solution $y = 0$.

Throughout this paper it is assumed that system (2.2) is noncritical. Next we state some known results [7] about system (2.2). Let K represent the fundamental matrix of (2.2) with $K(0) = I$, where I is the $n \times n$ identity matrix. Then

- (a) $\det K(t) \neq 0$.
- (b) There exists a constant matrix B such that $K(t+\omega) = K(t)e^{B\omega}$, by Floquet theory.
- (c) System (2.2) is noncritical if and only if $\det(I - K(\omega)) \neq 0$.

Lemma 2.1. *If the matrix LA_1 is periodic of period ω and $h \in C_\omega$, then the linear system*

$$x'(t) = LA_1(t)x(t) + h(t), \tag{2.3}$$

has a unique ω-periodic solution

$$x(t) = K(t)\left(K^{-1}(\omega) - I\right)^{-1} \int_t^{t+\omega} K^{-1}(s)h(s)\,ds.$$

Proof. Since $K(t)K^{-1}(t) = I$, it follows that

$$\begin{aligned}0 &= \frac{d}{dt}\left(K(t)K^{-1}(t)\right) = \frac{d}{dt}\left(K(t)\right)K^{-1}(t) + K(t)\frac{d}{dt}\left(K^{-1}(t)\right) \\ &= \left(LA_1(t)K(t)\right)K^{-1}(t) + K(t)\frac{d}{dt}\left(K^{-1}(t)\right) \\ &= LA_1(t) + K(t)\frac{d}{dt}\left(K^{-1}(t)\right).\end{aligned}$$

This implies

$$\frac{d}{dt}\left(K^{-1}(t)\right) = -K^{-1}(t)LA_1(t). \tag{2.4}$$

If x is a solution of (2.3) with $x(0) = x_0$, then

$$\begin{aligned}\frac{d}{dt}\left[K^{-1}(t)x(t)\right] &= \frac{d}{dt}\left(K^{-1}(t)\right)x(t) + K^{-1}(t)\frac{d}{dt}x(t) \\ &= -K^{-1}(t)LA_1(t)x(t) + K^{-1}(t)\left[LA_1(t)x(t) + h(t)\right] \\ &= K^{-1}(t)h(t),\end{aligned}$$

by (2.4). An integration of the above equation from 0 to t yields

$$x(t) = K(t)x(0) + K(t)\int_0^t K^{-1}(s)h(s)\,ds. \tag{2.5}$$

Since $x(\omega) = x_0 = x(0)$, we get

$$x(0) = (I - K(\omega))^{-1}\int_0^\omega K(\omega)K^{-1}(s)h(s)\,ds. \tag{2.6}$$

A substitution of (2.6) into (2.5) yields

$$\begin{aligned}x(t) = K(t)(I - K(\omega))^{-1}\int_0^\omega K(\omega)K^{-1}(s)h(s)\,ds \\ + K(t)\int_0^t K^{-1}(s)h(s)\,ds.\end{aligned} \tag{2.7}$$

Since

$$(I - K(\omega))^{-1} = \left(K(\omega)\left(K^{-1}(\omega) - I\right)\right)^{-1} = \left(K^{-1}(\omega) - I\right)^{-1}K^{-1}(\omega),$$

(2.7) becomes

$$\begin{aligned}x(t) &= K(t)\left(K^{-1}(\omega) - I\right)^{-1}\int_0^\omega K^{-1}(s)h(s)\,ds + K(t)\int_0^t K^{-1}(s)h(s)\,ds \\ &= K(t)\left(K^{-1}(\omega) - I\right)^{-1}\left\{\int_0^\omega K^{-1}(s)h(s)\,ds\right. \\ &\quad \left.+K^{-1}(\omega)\int_0^t K^{-1}(s)h(s)\,ds - \int_0^t K^{-1}(s)h(s)\,ds\right\}. \\ &= K(t)\left(K^{-1}(\omega) - I\right)^{-1}\left\{\int_t^\omega K^{-1}(s)h(s)\,ds\right. \\ &\quad \left.+K^{-1}(\omega)\int_0^t K^{-1}(s)h(s)\,ds\right\}\end{aligned}$$

By letting $s = \mu - \omega$, the above expression implies

$$x(t) = K(t)\left(K^{-1}(\omega) - I\right)^{-1}\left\{\int_t^\omega K^{-1}(s)h(s)\,ds + K^{-1}(\omega)\int_\omega^{t+\omega} K^{-1}(\mu-\omega)h(\mu-\omega)\,d\mu\right\}. \tag{2.8}$$

By (b) we have $K(t-\omega) = K(t)e^{-B\omega}$and $K(\omega) = e^{B\omega}$. Hence,

$$K^{-1}(\omega)K^{-1}(\mu-\omega) = K^{-1}(\mu).$$

Consequently, (2.8) becomes

$$x(t) = K(t)\left(K^{-1}(\omega) - I\right)^{-1}\left\{\int_t^\omega K^{-1}(s)h(s)\,ds + \int_\omega^{t+\omega} K^{-1}(s)h(s)\,ds\right\}.$$

□

Theorem 2.1 (Krasnoselskii's fixed point theorem [18]). *Let $\mathbb{M}$ be a closed convex nonempty subset of a Banach space $(\mathbb{S}, \|.\|)$. Suppose that $\mathcal{A}$ and $\mathcal{B}$ map $\mathbb{M}$ into $\mathbb{S}$ such that*

(i) $\mathcal{A}x + \mathcal{B}y \in \mathbb{M}$, $\forall x, y \in \mathbb{M}$,

(ii) $\mathcal{A}$ is continuous and $\mathcal{A}\mathbb{M}$ is contained in a compact set,

(iii) $\mathcal{B}$ is a contraction with constant $\alpha < 1$.

Then there is a $z \in \mathbb{M}$ with $z = \mathcal{A}z + \mathcal{B}z$.

By applying Lemma 2.1 and Theorem 2.1, we obtain in this section the existence of periodic solutions of (1.2).

Theorem 2.2. *Suppose that $f \in C^1(\mathbb{R}^n)$ and $P_1, A_1 \in C^1_\omega$. If there exists a constant $H > 0$ such that*

$$\frac{\sup_{|u|\le H}|f(u)|}{H} < \frac{1}{(1 + (1 + L\|A_1\|)c\omega)Lb}, \tag{2.9}$$

and that

$$|q| < \frac{1 - (1 + (1 + L\|A_1\|)c\omega)Lb\frac{\sup_{|u|\le H}|f(u)|}{H}}{1 + 2\|A_1\|(1 + (1 + L\|A_1\|)c\omega)L + (1 + (1 + L\|A_1\|)c\omega)Lb\frac{\sup_{|u|\le H}|f(u)|}{H}}, \tag{2.10}$$

and

$$\|P_1\|_0 \le \frac{(1-|q|)H}{(1 + (1 + L\|A_1\|)c\omega)L} - 2\|A_1\||q|H - b(1+|q|)\sup_{|u|\le H}|f(u)|, \tag{2.11}$$

where $\|A_1\| = \sup_{t\in[0,\omega]}|A_1(t)|$ and

$$c = \sup_{t\in[0,\omega]}\big(\sup_{t\le s\le t+\omega}|[K(s)(K^{-1}(\omega) - I)K^{-1}(t)]^{-1}|\big).$$

Then (1.2) has a T-periodic solution.

Proof. According to the condition (2.11), we get

$$\begin{aligned}
&(1+(1+L\|A_1\|)c\omega)L\|P_1\|_0+[1+(1+(1+L\|A_1\|)c\omega)2L\|A_1\|]|q|H\\
&+(1+(1+L\|A_1\|)c\omega)Lb(1+|q|)\sup_{|u|\le H}|f(u)|\\
&\le (1+(1+L\|A_1\|)c\omega)L\\
&\times\left\{\frac{(1-|q|)H}{(1+(1+L\|A_1\|)c\omega)L}-2\|A_1\||q|H-b(1+|q|)\sup_{|u|\le H}|f(u)|\right\}\\
&+[1+(1+(1+L\|A_1\|)c\omega)2L\|A_1\|]|q|H\\
&+(1+(1+L\|A_1\|)c\omega)Lb(1+|q|)\sup_{|u|\le H}|f(u)|\\
&=H. \qquad (2.12)
\end{aligned}$$

We need to prove that (2.1) has a ω-periodic solution. Let

$$\mathbb{S}=\left\{\phi\in C^1(\mathbb{R},\mathbb{R}^n),\ \|\phi\|_1=\|\phi\|_0+\|\phi'\|_0<+\infty\right\},$$

and

$$\mathbb{M}=\left\{\phi\in C^1_\omega,\ \|\phi\|_1\le H\right\},$$

then $\mathbb{M}$ is a bounded closed convex set of the Banach space $\mathbb{S}$.

Consider the system

$$\begin{aligned}
\frac{d}{dt}v(t)&=LA_1(t)v(t)+LP_1(t)+LA_1(t)qv(t-\tau)\\
&\quad-Lbf(v(t))+Lbqf(v(t-\tau))+q\frac{d}{dt}v(t-\tau).
\end{aligned}$$

According to Lemma 2.1, this equation has a unique ω-periodic solution

$$\begin{aligned}
v(t)=K(t)\left(K^{-1}(\omega)-I\right)^{-1}\int_t^{t+\omega}&K^{-1}(s)\Big[LP_1(s)+LA_1(s)qv(s-\tau)\\
&-Lbf(v(s))+Lbqf(v(s-\tau))+q\frac{\partial}{\partial s}v(s-\tau)\Big]ds,
\end{aligned}$$

Performing an integration by part and the fact that $v(t+\omega-\tau)=v(t-\tau)$, we obtain

$$\begin{aligned}
&K(t)\left(K^{-1}(\omega)-I\right)^{-1}\int_t^{t+\omega}K^{-1}(s)q\frac{\partial}{\partial s}v(s-\tau)ds\\
&=K(t)\left(K^{-1}(\omega)-I\right)^{-1}\left\{\left[K^{-1}(t+\omega)-K^{-1}(t)\right]qv(t-\tau)\right.\\
&\left.-\int_t^{t+\omega}\frac{\partial}{\partial s}\left[K^{-1}(s)\right]qv(s-\tau)ds\right\} \qquad (2.13)
\end{aligned}$$

Noting that $K^{-1}(t+\omega)=e^{-B\omega}K^{-1}(t)$, we have

$$K^{-1}(t+\omega)-K^{-1}(t)=e^{-B\omega}K^{-1}(t)-K^{-1}(t)=\left(K^{-1}(\omega)-I\right)K^{-1}(t). \qquad (2.14)$$

Since

$$\frac{d}{dt}K^{-1}(t)=-K^{-1}(t)LA_1(t), \qquad (2.15)$$

then, a substitution of (2.14) and (2.15) into (2.13) yields

$$\begin{aligned} &K(t)\left(K^{-1}(\omega)-I\right)^{-1}\int_t^{t+\omega}K^{-1}(s)q\frac{\partial}{\partial s}v\left(s-\tau\right)ds \\ &= qv\left(t-\tau\right)+K(t)\left(K^{-1}(\omega)-I\right)^{-1}\int_t^{t+\omega}K^{-1}(s)LA_1\left(s\right)qv\left(s-\tau\right)ds. \end{aligned}$$

Therefore

$$\begin{aligned} v\left(t\right) = qv\left(t-\tau\right)+K(t)\left(K^{-1}(\omega)-I\right)^{-1}\int_t^{t+\omega}K^{-1}(s)\left[LP_1\left(s\right)\right. \\ \left.+2LA_1\left(s\right)qv\left(s-\tau\right)-Lbf\left(v\left(s\right)\right)+Lbqf\left(v\left(s-\tau\right)\right)\right]ds. \end{aligned}$$

Define the operators $\mathcal{A}$ and $\mathcal{B}$ by

$$\begin{aligned} \left(\mathcal{A}\varphi\right)(t) = K(t)\left(K^{-1}(\omega)-I\right)^{-1}\int_t^{t+\omega}K^{-1}(s)\left[LP_1\left(s\right)\right. \\ \left.+2LA_1\left(s\right)q\varphi\left(s-\tau\right)-Lbf\left(\varphi\left(s\right)\right)+Lbqf\left(\varphi\left(s-\tau\right)\right)\right]ds, \end{aligned}$$

and

$$\left(\mathcal{B}\varphi\right)(t) = q\varphi\left(t-\tau\right).$$

In order to prove (2.1) has a ω-periodic solution, we shall make sure that $\mathcal{A}$ and $\mathcal{B}$ satisfy the conditions of Theorem 2.1.

For all $x, y \in \mathbb{M}$, we have $x\left(t+\omega\right) = x\left(t\right)$, $y\left(t+\omega\right) = y\left(t\right)$ and $\left\|x\right\|_1 \leq H$, $\left\|y\right\|_1 \leq H$. Now let us discuss $\mathcal{A}x+\mathcal{B}y$. We have

$$\begin{aligned} \left(\mathcal{A}x\right)(t+\omega) = K(t+\omega)\left(K^{-1}(\omega)-I\right)^{-1}\int_{t+\omega}^{t+2\omega}K^{-1}(s)\left[LP_1\left(s\right)\right. \\ \left.+2LA_1\left(s\right)qx\left(s-\tau\right)-Lbf\left(x\left(s\right)\right)+Lbqf\left(x\left(s-\tau\right)\right)\right]ds. \end{aligned}$$

By letting $s = \mu+\omega$, the above expression implies

$$\begin{aligned} \left(\mathcal{A}x\right)(t+\omega) = K(t+\omega)\left(K^{-1}(\omega)-I\right)^{-1}\int_t^{t+\omega}K^{-1}(\mu+\omega)\left[LP_1\left(\mu+\omega\right)\right. \\ +2LA_1\left(\mu+\omega\right)qx\left(\mu+\omega-\tau\right) \\ \left.-Lbf\left(x\left(\mu+\omega\right)\right)+Lbqf\left(x\left(\mu+\omega-\tau\right)\right)\right]d\mu. \end{aligned}$$

By (b) we have $K(t+\omega) = K(t)e^{B\omega}$and $K(\omega) = e^{B\omega}$. Hence

$$K(t+\omega)\left(K^{-1}(\omega)-I\right)^{-1}K^{-1}(\mu+\omega) = K(t)\left(K^{-1}(\omega)-I\right)^{-1}K^{-1}(\mu).$$

Consequently, the above expression implies

$$\begin{aligned} \left(\mathcal{A}x\right)(t+\omega) &= K(t)\left(K^{-1}(\omega)-I\right)^{-1}\int_t^{t+\omega}K^{-1}(s)\left[LP_1\left(s\right)\right. \\ &\quad \left.+2LA_1\left(s\right)qx\left(s-\tau\right)-Lbf\left(x\left(s\right)\right)+Lbqf\left(x\left(s-\tau\right)\right)\right]ds \\ &= \left(\mathcal{A}x\right)(t), \end{aligned}$$

and

$$\left(\mathcal{B}y\right)(t+\omega) = qy\left(t+\omega-\tau\right) = qy\left(t-\tau\right) = \left(\mathcal{B}y\right)(t),$$

therefore $(\mathcal{A}x+\mathcal{B}y)(t+\omega)=(\mathcal{A}x+\mathcal{B}y)(t)$. Meanwhile, we get

$$\begin{aligned}(\mathcal{A}x)'(t) &= K'(t)\left(K^{-1}(\omega)-I\right)^{-1}\int_t^{t+\omega}K^{-1}(s)\left[LP_1(s)\right.\\ &\quad \left.+2LA_1(s)\,qx(s-\tau)-Lbf(x(s))+Lbqf(x(s-\tau))\right]ds\\ &\quad +K(t)\left(K^{-1}(\omega)-I\right)^{-1}\left[K^{-1}(t+\omega)-K^{-1}(t)\right]\left[LP_1(t)\right.\\ &\quad \left.+2LA_1(t)\,qx(t-\tau)-Lbf(x(t))+Lbqf(x(t-\tau))\right]. \end{aligned} \tag{2.16}$$

Since

$$K'(t)=LA_1(t)\,K(t), \tag{2.17}$$

and noting that $K^{-1}(t+\omega)=e^{-B\omega}K^{-1}(t)$, we have

$$K^{-1}(t+\omega)-K^{-1}(t)=e^{-B\omega}K^{-1}(t)-K^{-1}(t)=\left(K^{-1}(\omega)-I\right)K^{-1}(t). \tag{2.18}$$

A substitution of (2.17) and (2.18) into (2.16) yields

$$\begin{aligned}(\mathcal{A}x)'(t) &= LA_1(t)(\mathcal{A}x)(t)+LP_1(t)+2LA_1(t)\,qx(t-\tau)\\ &\quad -Lbf(x(t))+Lbqf(x(t-\tau)).\end{aligned}$$

Thus,

$$\begin{aligned}&\|\mathcal{A}x\|_1\\ &=\|\mathcal{A}x\|_0+\left\|(\mathcal{A}x)'\right\|_0\\ &=\sup_{t\in[0,\omega]}\left|K(t)\left(K^{-1}(\omega)-I\right)^{-1}\int_t^{t+\omega}K^{-1}(s)\left[LP_1(s)\right.\right.\\ &\left.\left.+2LA_1(s)\,qx(s-\tau)-Lbf(x(s))+Lbqf(x(s-\tau))\right]ds\right|\\ &+\sup_{t\in[0,\omega]}\left|LA_1(t)\,K(t)\left(K^{-1}(\omega)-I\right)^{-1}\int_t^{t+\omega}K^{-1}(s)\left[LP_1(s)\right.\right.\\ &\left.+2LA_1(s)\,qx(s-\tau)-Lbf(x(s))+Lbqf(x(s-\tau))\right]ds\\ &\left.+LP_1(t)+2LA_1(t)\,qx(t-\tau)-Lbf(x(t))+Lbqf(x(t-\tau))\right|\\ &\le c\omega\left[2L\|A_1\|\,|q|\,H+Lb(1+|q|)\sup_{|u|\le H}|f(u)|+L\|P_1\|_0\right]\\ &+(1+L\|A_1\|\,c\omega)\left[2L\|A_1\|\,|q|\,H+Lb(1+|q|)\sup_{|u|\le H}|f(u)|+L\|P_1\|_0\right]\\ &=(1+(1+L\|A_1\|)\,c\omega)\left[2L\|A_1\|\,|q|\,H+Lb(1+|q|)\sup_{|u|\le H}|f(u)|+L\|P_1\|_0\right],\end{aligned}$$

and

$$\begin{aligned}\|\mathcal{B}y\|_1 &=\|\mathcal{B}y\|_0+\left\|(\mathcal{B}y)'\right\|_0\le|q|\,\|y\|_0+|q|\,\left\|y'\right\|_0=|q|\,\|y\|_1\\ &\le|q|\,H.\end{aligned}$$

Therefore

$$\begin{aligned}
&\|\mathcal{A}x+\mathcal{B}y\|_1 \\
&\le \|\mathcal{A}x\|_1+\|\mathcal{B}y\|_1 \\
&\le (1+(1+L\|A_1\|)c\omega)\Big[2L|A_1||q|H \\
&+Lb(1+|q|)\sup_{|u|\le H}|f(u)|+L\|P_1\|_0\Big]+|q|H \\
&=[1+(1+(1+L\|A_1\|)c\omega)2L\|A_1\|]|q|H+(1+(1+L\|A_1\|)c\omega)L\|P_1\|_0 \\
&+(1+(1+L\|A_1\|)c\omega)Lb(1+|q|)\sup_{|u|\le H}|f(u)|.
\end{aligned}$$

By (2.12), $\|\mathcal{A}x+\mathcal{B}y\|_1\le H$. Accordingly, $\mathcal{A}x+\mathcal{B}y\in\mathbb{M}$.

For all $x\in\mathbb{M}$, $\|\mathcal{A}x\|_0\le H$, $\|(\mathcal{A}x)'\|_0\le H$. According to Ascoli Arzela lemma, the subset $\mathcal{A}\mathbb{M}$ of C_ω is a precompact set, therefore for all subsequence $\{x_n\}$ of $\mathbb{M}$, there exists the subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\mathcal{A}x_{n_k}\to x_0\in C_\omega$ as $k\to+\infty$.

Meanwhile, we get

$$\begin{aligned}
(\mathcal{A}x)''(t)&=LA_1'(t)(\mathcal{A}x)(t)+L^2A_1^2(t)(\mathcal{A}x)(t)+LA_1(t)[LP_1(t)\\
&+2LA_1(t)qx(t-\tau)-Lbf(x(t))+Lbqf(x(t-\tau))]\\
&+\big[LP_1'(t)+2Lq\left[A_1'(t)x(t-\tau)+A_1(t)x'(t-\tau)\right]\\
&-Lbf'(x(t))x'(t)+Lbqf'(x(t-\tau))x'(t-\tau)\big].
\end{aligned}$$

Thus,

$$\begin{aligned}
\sup_{t\in[0,\omega]}\left|(\mathcal{A}x)''(t)\right|\le{}&\Big(L\|A_1\|+\Big(L\|A_1'\|+L^2\|A_1\|^2\Big)c\omega\Big)\Big[2L\|A_1\||q|H\\
&+Lb(1+|q|)\sup_{|u|\le H}|f(u)|+L\|P_1\|_0\Big]\\
&+\Big[2L\left(\|A_1\|+\|A_1'\|\right)|q|H\\
&+LbH(1+|q|)\sup_{|u|\le H}|f(u)|+L\|P_1'\|_0\Big].
\end{aligned}$$

Therefore there is a constant $H_1>0$ such that

$$\sup_{t\in[0,\omega]}\left|(\mathcal{A}x)''(t)\right|\le H_1 \text{ and } \left\{(\mathcal{A}x)':x\in\mathbb{M}\right\}\subset C_\omega.$$

According to Ascoli Arzela lemma, $\{x_{n_k}\}$ has a subsequence, for simplicity, written as $\{x_{n_k}\}$, such that $(\mathcal{A}x_{n_k})'\to z_0\in C_\omega$. Since $\frac{d}{dt}$ is a closed operator, $z_0=(x_0)'$. Hence, $x_0\in C_\omega^1$ and $\{\mathcal{A}x_n\}$ is contained in a compact set. Then, $\mathcal{A}$ is a compact operator.

Suppose that $\{x_n\} \in \mathbb{M}$, $x \in \mathbb{S}$, $x_n \to x$, then $\|x_n - x\|_0 \to 0$ and $\|x_n' - x'\|_0 \to 0$ as $n \to +\infty$. And we get

$$\begin{aligned}
&\|\mathcal{A}x_n - \mathcal{A}x\|_0 \\
&= \sup_{t\in[0,\omega]} \left| K(t) \left(K^{-1}(\omega) - I\right)^{-1} \int_t^{t+\omega} K^{-1}(s) \right. \\
&\times [2LA_1(s)\, q\,(x_n(s-\tau) - x(s-\tau)) - Lb\,(f(x_n(s)) - f(x(s))) \\
&+ Lbq\,(f(x_n(s-\tau)) - f(x(s-\tau)))]\, ds| \\
&\le \omega c \left[2L\|A_1\|\,|q|\,\|x_n - x\| + Lb\,(1+|q|) \sup_{t\in[0,\omega]} |f(x_n(t)) - f(x(t))| \right],
\end{aligned}$$

and

$$\begin{aligned}
&\left\|(\mathcal{A}x_n)' - (\mathcal{A}x)'\right\|_0 \\
&= \sup_{t\in[0,\omega]} |LA_1(t)\,((\mathcal{A}x_n)(t) - (\mathcal{A}x)(t)) \\
&+ 2LA_1(t)\, q\,(x_n(t-\tau) - x(t-\tau)) - Lb\,(f(x_n(t)) - f(x(t))) \\
&+ Lbq\,(f(x_n(t-\tau)) - f(x(t-\tau)))]| \\
&\le (1 + L\|A_1\|\,\omega c)\,[2L\|A_1\|\,|q|\,\|x_n - x\| \\
&\left. + Lb\,(1+|q|) \sup_{t\in[0,\omega]} |f(x_n(t)) - f(x(t))| \right].
\end{aligned}$$

When $\|x_n - x\|_1 \to 0$ as $n \to +\infty$, $|x_n(t) - x(t)| \to 0$ for $t \in [0,\omega]$ uniformly. And since f is continuous, $\|\mathcal{A}x_n - \mathcal{A}x\|_0 \to 0$, $\left\|(\mathcal{A}x_n)' - (\mathcal{A}x)'\right\|_0 \to 0$. Consequently, $\mathcal{A}$ is continuous.

For all $x, y \in \mathbb{M}$, $\|\mathcal{B}x - \mathcal{B}y\|_1 \le |q|\,\|x - y\|_1$ and $|q| < 1$, therefore $\mathcal{B}$ is a contraction operator.

Thus, the conditions of Theorem 2.1 are satisfied and there is a $\phi \in \mathbb{M}$ such that $\phi = \mathcal{A}\phi + \mathcal{B}\phi$. It is a ω-periodic solution for (2.1). Since $v(t) = u(Lt)$, $P_1(t) = P(Lt)$ and $A_1(t) = A(Lt)$, then (1.2) has a T-periodic solution. □

Example 2.1. Consider the following neutral differential system

$$\begin{aligned}
&\frac{d}{dt}u(t) - q\frac{d}{dt}u(t-r) \\
&= P(t) + A(t)\,u(t) + A(t)\,qu(t-r) - bf(u(t)) + bqf(u(t-r)),
\end{aligned} \tag{2.19}$$

where $T = 2\pi$, $b = 1$, $q = \dfrac{1}{80}$, $r = 2$, $A(t) = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$, $P(t) = \begin{pmatrix} 0 \\ 0.01\cos(t) \end{pmatrix}$ and $f(u(t)) = \begin{pmatrix} 0 \\ \sin(u(t)) \end{pmatrix}$. For $L = 0.25$, (2.19) can be transformed as

$$\begin{aligned}
&\frac{d}{dt}v(t) - q\frac{d}{dt}v(t-\tau) \\
&= LP_1(t) + LA_1(t)\,v(t) + LA_1(t)\,qv(t-\tau) - Lbf(v(t)) + Lbqf(v(t-\tau)),
\end{aligned}$$

where $v(t) = u(0.25t)$, $\omega = 8\pi$, $\tau = 8$, $P_1(t) = \begin{pmatrix} 0 \\ 0.01\cos(0.25t) \end{pmatrix}$ and $A_1(t) = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$. Since the matrix A_1 has eigenvalues with non-zero real parts, the system $\frac{d}{dt}v(t) = LA_1(t)v(t)$ is noncritical. Let $H = 30$, then all conditions of Theorem 2.2 are satisfied and hence (2.19) has a 2π-periodic solution.

3. Asymptotic stability of periodic solutions

This section concerned with the asymptotic stability of periodic solutions. When the conditions of Theorem 2.2 are satisfied, there is a T-periodic solution u^* for (1.2). Let $v(t) = u(t) - u^*(t)$, then (1.2) is transformed as

$$\begin{aligned} v'(t) - qv'(t-r) &= A(t)v(t) + A(t)qv(t-r) - b[f(v(t)+u^*(t)) - f(u^*(t))] \\ &\quad + bq[f(v(t-r)+u^*(t-r)) - f(u^*(t-r))]. \end{aligned} \tag{3.1}$$

Obviously, (3.1) has the zero solution. Now we use Krasnoselskii's fixed point theorem to prove the zero solution for (3.1) is asymptotically stable. We set $\mathbb{S}$ as the Banach space of bounded continuous function $\phi : [-r,\infty) \to \mathbb{R}^n$ with the supremum norm $\|.\|$. Also, Given the initial function ψ, denote the norm of ψ by $\|\psi\| = \sup_{t\in[-r,0]}|\psi(t)|$, which should not cause confusion with the same symbol for the norm in $\mathbb{S}$.

Proposition 3.1 ([7], Proposition 2.14)**.** *If $t \to \varPhi(t)$ is a fundamental matrix solution for the system*

$$y'(t) = A(t)y(t), \tag{3.2}$$

defined on an open interval J, then $\Phi(t,r) := \varPhi(t)\varPhi^{-1}(r)$ is the state transition matrix. Also, the state transition matrix satisfies the Chapman-Kolmogorov identities

$$\Phi(r,r) = I, \ \ \Phi(t,s)\Phi(s,r) = \Phi(t,r),$$

and the identities

$$\Phi(t,s)^{-1} = \Phi(s,t), \ \ \frac{\partial \Phi(t,s)}{\partial s} = -\Phi(t,s)A(s).$$

Theorem 3.1. *If all conditions of Theorem 2.2 are satisfied, f satisfies the locally Lipschitz condition. Further assume that*

$$\varPhi(t) \to 0 \text{ as } t \to \infty,$$

and there exists $Q > H$ such that

$$\sup_{|u|\le H+Q}|f(u)| + \sup_{|u|\le H}|f(u)| < \frac{Q}{\lambda b}, \tag{3.3}$$

and that

$$|q| < \frac{Q - \lambda b\left(\sup\limits_{|u|\le H+Q}|f(u)| + \sup\limits_{|u|\le H}|f(u)|\right)}{(1+2\lambda\|A\|)Q + \lambda b\left(\sup\limits_{|u|\le H+Q}|f(u)| + \sup\limits_{|u|\le H}|f(u)|\right)}, \tag{3.4}$$

and

$$\|\psi\| \leq \frac{(1-(1+2\lambda\|A\|)|q|)Q - \lambda b(1+|q|)\left(\sup_{|u|\leq H+Q}|f(u)| + \sup_{|u|\leq H}|f(u)|\right)}{\theta(1+|q|)}, \tag{3.5}$$

where $\theta = \sup_{t\geq 0}|\Phi(t,0)|$ *and* $\lambda = \sup_{t\geq 0}\left|\int_0^t \Phi(t,s)ds\right|$. *Then the solution of (3.1)* $v(t) \to 0$ *as* $t \to \infty$.

Proof. According to the conditions (3.3), (3.4) and (3.5), we have

$$\begin{aligned}&(1+2\lambda\|A\|)|q|Q + \theta(1+|q|)\|\psi\| \\ &+ \lambda b(1+|q|)\left(\sup_{|u|\leq H+Q}|f(u)| + \sup_{|u|\leq H}|f(u)|\right) \leq Q.\end{aligned} \tag{3.6}$$

Given the initial function ψ, there exists a unique solution v for (3.1). Let

$$\mathbb{M}_\psi = \{\phi \in \mathbb{S},\ \|\phi\| \leq Q,\ \phi(t) = \psi(t) \text{ if } t \in [-r,0],\ |\phi(t)| \to 0 \text{ as } t \to \infty\},$$

then $\mathbb{M}_\psi$ is a bounded convex closed set of $\mathbb{S}$.

Let v be a solution of (3.1). We write (3.1) as

$$\begin{aligned}&\frac{d}{dt}\{v(t) - qv(t-r)\} \\ &= A(t)v(t) + A(t)qv(t-r) - b[f(v(t)+u^*(t)) - f(u^*(t))] \\ &+ bq[f(v(t-r)+u^*(t-r)) - f(u^*(t-r))]\end{aligned}$$

Since Φ is a fundamental matrix solution for the system (3.2). We have

$$\begin{aligned}&\frac{d}{dt}\left\{\Phi^{-1}(t)(v(t) - qv(t-r))\right\} \\ &= \left\{\frac{d}{dt}\Phi^{-1}(t)\right\}(v(t) - qv(t-r)) + \Phi^{-1}(t)\frac{d}{dt}\{(v(t) - qv(t-r))\}.\end{aligned}$$

By the Proposition 3.1, it follows that

$$\frac{d}{dt}\Phi^{-1}(t) = -\Phi^{-1}(t)A(t).$$

Then

$$\begin{aligned}&\frac{d}{dt}\left\{\Phi^{-1}(t)(v(t) - qv(t-r))\right\} \\ &= -\Phi^{-1}(t)A(t)(v(t) - qv(t-r)) + \Phi^{-1}(t)\{A(t)v(t) \\ &+ A(t)qv(t-r) - b[f(v(t)+u^*(t)) - f(u^*(t))] \\ &+ bq[f(v(t-r)+u^*(t-r)) - f(u^*(t-r))]\} \\ &= \Phi^{-1}(t)\{2A(t)qv(t-r) - b[f(v(t)+u^*(t)) - f(u^*(t))] \\ &+ bq[f(v(t-r)+u^*(t-r)) - f(u^*(t-r))]\}.\end{aligned}$$

An integration of the above equation from 0 to t yields

$$\begin{aligned}
&\Phi^{-1}(t)\left(v\left(t\right)-qv\left(t-r\right)\right)-\Phi^{-1}(0)\left(v\left(0\right)-qv\left(0-r\right)\right)\\
&=\int_0^t \Phi^{-1}(s)\left\{2A\left(s\right)qv\left(s-r\right)-b\left[f\left(v\left(s\right)+u^*\left(s\right)\right)-f\left(u^*\left(s\right)\right)\right]\right.\\
&\left.+bq\left[f\left(v\left(s-r\right)+u^*\left(s-r\right)\right)-f\left(u^*\left(s-r\right)\right)\right]\right\}ds. \qquad (3.7)
\end{aligned}$$

(3.7) can be expressed by

$$\begin{aligned}
v\left(t\right)&=\Phi(t,0)\left(v\left(0\right)-qv\left(0-r\right)\right)+qv\left(t-r\right)\\
&+\int_0^t \Phi(t,s)\left\{2A\left(s\right)qv\left(s-r\right)-b\left[f\left(v\left(s\right)+u^*\left(s\right)\right)-f\left(u^*\left(s\right)\right)\right]\right.\\
&\left.+bq\left[f\left(v\left(s-r\right)+u^*\left(s-r\right)\right)-f\left(u^*\left(s-r\right)\right)\right]\right\}ds,
\end{aligned}$$

then we have

$$\begin{aligned}
v\left(t\right)&=\Phi(t,0)\left(\psi\left(0\right)-q\psi\left(0-r\right)\right)+qv\left(t-r\right)\\
&+\int_0^t \Phi(t,s)\left\{2A\left(s\right)qv\left(s-r\right)-b\left[f\left(v\left(s\right)+u^*\left(s\right)\right)-f\left(u^*\left(s\right)\right)\right]\right.\\
&\left.+bq\left[f\left(v\left(s-r\right)+u^*\left(s-r\right)\right)-f\left(u^*\left(s-r\right)\right)\right]\right\}ds.
\end{aligned}$$

For all $\phi\in\mathbb{M}_\psi$, define the operators $\mathcal{A}$ and $\mathcal{B}$ by

$$\left(\mathcal{A}\phi\right)(t)=\begin{cases} 0,\ t\in\left[-r,0\right], \\ \int_0^t \Phi(t,s)\left\{2A\left(s\right)q\phi\left(s-r\right)-b\left[f\left(\phi\left(s\right)+u^*\left(s\right)\right)-f\left(u^*\left(s\right)\right)\right]\right. \\ \left.+bq\left[f\left(\phi\left(s-r\right)+u^*\left(s-r\right)\right)-f\left(u^*\left(s-r\right)\right)\right]\right\}ds,\ t\geq 0, \end{cases}$$

and

$$\left(\mathcal{B}\phi\right)(t)=\begin{cases} \psi\left(t\right),\ t\in\left[-r,0\right], \\ \Phi(t,0)\left(\psi\left(0\right)-q\psi\left(-r\right)\right)+q\phi\left(t-r\right),\ t\geq 0. \end{cases}$$

(i) For all $x,y\in\mathbb{M}_\psi$, $x\left(t\right)\to 0$ and $y\left(t\right)\to 0$ as $t\to\infty$, then $\left(\mathcal{B}y\right)(t)\to 0$ and

$$\begin{aligned}
&\lim_{t\to\infty}\left(\mathcal{A}x\right)(t)\\
&=\lim_{t\to\infty}\left\{\Phi\left(t\right)\int_0^t \Phi^{-1}(s)\left\{2A\left(s\right)qx\left(s-r\right)-b\left[f\left(x\left(s\right)+u^*\left(s\right)\right)-f\left(u^*\left(s\right)\right)\right]\right.\right.\\
&\left.\left.+bq\left[f\left(x\left(s-r\right)+u^*\left(s-r\right)\right)-f\left(u^*\left(s-r\right)\right)\right]\right\}ds\right\}\\
&=0,
\end{aligned}$$

therefore $\lim_{t\to\infty} (\mathcal{A}x + \mathcal{B}y)(t) = 0$. And

$$\begin{aligned}
\|\mathcal{A}x\| &= \sup_{t\geq 0} \left| \int_0^t \Phi(t,s) \left\{ 2A(s)\, qx(s-r) - b\left[f(x(s)+u^*(s)) - f(u^*(s)) \right] \right. \right. \\
&\quad \left. \left. + bq\left[f(x(s-r)+u^*(s-r)) - f(u^*(s-r)) \right] \right\} ds \right| \\
&\leq \left\{ 2\|A\|\,|q| \sup_{t\geq -r} |x(t)| + b\left[\sup_{|u|\leq H+Q} |f(u)| + \sup_{|u|\leq H} |f(u)| \right] \right. \\
&\quad \left. + b|q| \left[\sup_{|u|\leq H+Q} |f(u)| + \sup_{|u|\leq H} |f(u)| \right] \right\} \sup_{t\geq 0} \left| \int_0^t \Phi(t,s) ds \right| \\
&\leq \lambda \left[2\|A\|\,|q|\, Q + b(1+|q|) \left(\sup_{|u|\leq H+Q} |f(u)| + \sup_{|u|\leq H} |f(u)| \right) \right],
\end{aligned}$$

and

$$\begin{aligned}
\|\mathcal{B}y\| &= \sup_{t\geq -r} |(\mathcal{B}y)(t)| \\
&= \max\left\{ \|\psi\|, \sup_{t\geq 0} |\Phi(t,0)(\psi(0) - q\psi(-r)) + qy(t-r)| \right\} \\
&\leq \theta(1+|q|)\|\psi\| + \sup_{t\geq 0} |qy(t-r)| \\
&\leq \theta(1+|q|)\|\psi\| + |q|\, Q.
\end{aligned}$$

Thus,

$$\begin{aligned}
&\|\mathcal{A}x + \mathcal{B}y\| \\
&\leq \|\mathcal{A}x\| + \|\mathcal{B}y\| \\
&\leq \lambda \left[2\|A\|\,|q|\, Q + b(1+|q|) \left(\sup_{|u|\leq H+Q} |f(u)| + \sup_{|u|\leq H} |f(u)| \right) \right] \\
&+ \theta(1+|q|)\|\psi\| + |q|\, Q \\
&= (1 + 2\lambda\|A\|)\,|q|\, Q + \theta(1+|q|)\|\psi\| \\
&+ \lambda b(1+|q|) \left(\sup_{|u|\leq H+Q} |f(u)| + \sup_{|u|\leq H} |f(u)| \right).
\end{aligned}$$

According to the condition (3.6), $\|\mathcal{A}x + \mathcal{B}y\| \leq Q$. Thus, $\mathcal{A}x + \mathcal{B}y \in \mathbb{M}_\psi$.

(ii) For all $x \in \mathbb{M}_\psi$, $\|x\| \leq Q$. And

$$\left| (\mathcal{A}x)'(t) \right| = 0,\ t \in [-r, 0],$$

and

$$\begin{aligned}
&\left|(\mathcal{A}x)'(t)\right| \\
&= \left|A(t)\int_0^t \Phi(t,s)\{2A(s)qx(s-r) - b[f(x(s)+u^*(s)) - f(u^*(s))]\right. \\
&+bq[f(x(s-r)+u^*(s-r)) - f(u^*(s-r))]\}\,ds \\
&+\{2A(t)qx(t-r) - b[f(x(t)+u^*(t)) - f(u^*(t))] \\
&+bq[f(x(t-r)+u^*(t-r)) - f(u^*(t-r))]\}| \\
&\le (1+\lambda\|A\|)\left[2\|A\|\,|q|\,Q + b(1+|q|)\left(\sup_{|u|\le H+Q}|f(u)| + \sup_{|u|\le H}|f(u)|\right)\right],
\end{aligned}$$

here, the derivative of $(\mathcal{A}x)'(t)$ at zero means the left hand side derivative when $t\le 0$ and the right hand side derivative when $t\ge 0$. One can see $\left|(\mathcal{A}x)'(t)\right|$ is bounded for all $x\in\mathbb{M}_\psi$ and $\mathcal{A}\mathbb{M}_\psi$ is a precompact set of $\mathbb{S}$. Therefore $\mathcal{A}$ is compact.

Suppose $\{x_n\}\subset\mathbb{M}_\psi$, $x\in\mathbb{S}$, $x_n\to x$ as $n\to\infty$, then $|x_n(t)-x(t)|\to\infty$ uniformly for $t\ge -r$ as $n\to\infty$. Since

$$\begin{aligned}
&\|\mathcal{A}x_n - \mathcal{A}x\| \\
&= \sup_{t\ge 0}\left|\int_0^t \Phi(t,s)\{2A(s)q(x_n(s-r)-x(s-r))\right. \\
&- b[f(x_n(s)+u^*(s)) - f(x(s)+u^*(s))] \\
&+bq[f(x_n(s-r)+u^*(s-r)) - f(x(s-r)+u^*(s-r))]\}\,ds| \\
&\le \lambda[2\|A\|\,|q|\,\|x_n-x\| + b(1+|q|) \\
&\left.\times\sup_{t\ge -r}|f(x_n(t)+u^*(t)) - f(x(t)+u^*(t))|\right],
\end{aligned}$$

and f is continuous, therefore$\|\mathcal{A}x_n-\mathcal{A}x\|\to 0$ as $n\to\infty$ and $\mathcal{A}$ is continuous.

(iii) For all $x,y\in\mathbb{M}_\psi$,

$$\|\mathcal{B}x-\mathcal{B}y\| = \sup_{t\ge 0}|qx(t-r)-qy(t-r)| \le |q|\,\|x-y\|,$$

and $|q|<1$, therefore $\mathcal{B}$ is a contraction operator.

According to Krasnoselskii's fixed point theorem, there is a $\phi\in\mathbb{M}_\psi$ such that $(\mathcal{A}+\mathcal{B})\phi=\phi$ and ϕ is a solution for (3.1). Because the solution through ψ for the equation is unique, the solution $v(t)=\phi(t)\to 0$ as $t\to\infty$. □

When f satisfies the locally Lipschitz condition, H in Theorem 2.2 and Q in Theorem 3.1 exists, there is a constant $R>0$ such that

$$|f(v(t)+u^*(t)) - f(v(t))| < R\,|v(t)|.$$

Since ϕ satisfies

$$\begin{aligned}
\phi(t) &= \Phi(t,0)(\psi(0)-q\psi(-r)) + q\phi(t-r) \\
&\quad+\int_0^t \Phi(t,s)\{2A(s)q\phi(s-r) - b[f(\phi(s)+u^*(s)) - f(u^*(s))] \\
&\quad+bq[f(\phi(s-r)+u^*(s-r)) - f(u^*(s-r))]\}ds,
\end{aligned}$$

then

$$\|\phi\| \leq \theta(1+|q|)\|\psi\| + |q|\|\phi\| + \lambda[2\|A\||q|\|\phi\| + b(1+|q|)R\|\phi\|],$$

that is

$$[1-|q|-\lambda(2\|A\||q|+b(1+|q|)R)]\|\phi\| \leq \theta(1+|q|)\|\psi\|.$$

Then there clearly exists a $\delta > 0$ for each $\epsilon > 0$ such that $|\phi(t)| < \epsilon$ for all $t \geq -r$ if $\|\psi\| < \delta$. Thus we have the following theorem.

Theorem 3.2. *If R satisfies*

$$1-|q|-\lambda(2\|A\||q|+b(1+|q|))R > 0.$$

Then the zero solution for (3.1) is stable.

4. Conclusion

In this paper, we provided the existence and asymptotic stability of periodic solutions with sufficient conditions for neutral differential systems. The main tool of this paper is the method of fixed points. However, by introducing new fixed mappings, we get new existence and stability conditions. The obtained results have a contribution to the related literature, and they improve and extend the results in [8] from the case of neutral differential equations to that case with neutral differential systems.

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References

[1] A. Ardjouni and A. Djoudi, Fixed point and stability in neutral nonlinear differential equations with variable delays, *Opuscula Mathematica* **32(1)** (2012), 5–19.
[2] A. Ardjouni and A. Djoudi, Fixed points and stability in nonlinear neutral differential equations with variable delays, *Nonlinear Studies* **19(3)** (2012), 345–357.
[3] A. Ardjouni and A. Djoudi, Stability in nonlinear neutral differential equations with variable delays using fixed points theory, *Electronic Journal of Qualitative Theory of Differential Equations* **2011(43)** (2011), 1–11.
[4] A. Ardjouni, I. Derrardjia and A. Djoudi, Stability in totally nonlinear neutral differential equations with variable delay, *Acta Math. Univ. Comenianae* **LXXXIII(1)** (2014), 119–134.
[5] A. Ardjouni, A. Djoudi and I. Soualhia, Stability for linear neutral integro-differential equations with variable delays, *Electronic journal of Differential Equations* **2012(172)** (2012), 1–14.
[6] T. A. Burton, *Stability and periodic solutions of ordinary and functional differential equations*, Academic Press, Orlando, 1985.
[7] C. Chicone, *Ordinary differential equations with applications*, Springer, Berlin 1999.
[8] L. Ding and Z. Li, Periodicity and stability in neutral equations by Krasnoselskii's fixed point theorem, *Nonlinear Analysis: Real World Applications* **11** (2010) 1220–1228.
[9] J. K. Hale and S. M. Verduyn Lunel, *Introduction to functional differential equations*, Springer, New York, 1993.

[10] M. N. Islam and Y. N. Raffoul, Periodic solutions of neutral nonlinear system of differential equations with functional delay, *J. Math. Anal. Appl.* **331** (2007), 1175–1186.
[11] C. H. Jin and J. W. Luo, Stability in functional differential equations established using fixed point theory, *Nonlinear Anal.* **68** (2008), 3307–3315.
[12] C. H. Jin and J. W. Luo, Fixed points and stability in neutral differential equations with variable delays, *Proc. Am. Math. Soc.* **136(3)** (2008), 909–918.
[13] B. Mansouri, A. Ardjouni and A. Djoudi, Periodicity and stability in neutral nonlinear differential equations by Krasnoselskii's fixed point theorem, *CUBO A Mathematical Journal* **19(3)** (2017), 15–29.
[14] M. B. Mesmouli, A. Ardjouni and A. Djoudi, Periodic solutions and stability in a nonlinear neutral system of differential equations with infinite delay, *Boletin de la Sociedad Matematica Mexicana* **24(1)** (2018), 239–255.
[15] M. B. Mesmouli, A. Ardjouni and A. Djoudi, Stability solutions for a system of nonlinear neutral functional differential equations with functional delay, *Dynamic Systems and Applications* **25(1-2)** (2016), 253–262.
[16] M. B. Mesmouli, A. Ardjouni and A. Djoudi, Periodicity of solutions for a system of nonlinear integro-differential equations, *Sarajevo journal of mathematics* **11(1)** (2015), 49–63.
[17] M. B. Mesmouli, A. Ardjouni and A. Djoudi, Existence and uniqueness of periodic solutions for a system of nonlinear neutral functional differential equations with two functional delays, *Rend. Circ. Mat. Palermo* **63** (2014), 409–424.
[18] D. R. Smart, *Fixed point theorems*, Cambridge Univ. Press, Cambridge, 1980.

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