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PARA-COMPLEX STRUCTURES ON LINEAR COFRAME BUNDLE WITH SASAKIAN METRIC

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Abstract. By using a Riemannian metric on a differentiable manifold, the Sasakian metric is introduced on the linear coframe bundle of the Riemannian manifold. Geometric properties of Levi-Civita connection of Sasakian metric are investigated. Also, para-complex structures on the linear coframe bundle with Sasakian metric are constructed and some interesting properties of those structures are studied.

1. Introduction

Let M be an $n-$dimensional manifold of class C^{∞}. The problem of extending differential-geometrical structures on M to its fiber bundles has been the subject of a number of papers. An account of these can be found in Yano and Ishihara [22] (see, also [2]). In [21], Sasaki by using a Riemannian metric g on a differentiable manifold M, constructed a Riemannian metric $\tilde{g}$ on the tangent bundle $T(M)$ of M. Then, some geometers such as Kowalski, Aso, Musso and Tricerri studied interesting geometric properties of this metric, that is called Sasaki metric (see [1], [8], [11]). Some properties and applications for the Riemannian metrics of the cotangent, linear frame, linear coframe and tensor bundles are given in [3, 5, 9, 10, 16-18]. Also noteworthy are the papers devoted to the study of various differential geometric structures, including submanifolds of Sasakian manifolds. V.A.Khan and M.A.Khan studied pseudo-slant submanifolds of Sasakian manifolds (see [7]). The main results related to the projective curvature tensor in Sasakian manifolds are due to U.K.Gautam, A.Haseeb and R.Prasad (see [6]).

Let M_{2k} be a $2k$-dimensional differentiable manifold endowed with an almost (para) complex structure φ and a pseudo-Riemannian metric g of signature (k,k) such that $g(\varphi X,Y)=g(X,\varphi Y)$ for arbitrary vector fields X and Y on M_{2k}, i.e. g is pure with respect to φ. The metric g is called Nordenian metric. Nordenian metrics are also referred to as anti-Hermitian metrics or B-metrics. They present extensive application in mathematics as well as in theoretical physics. Many authors considered almost (para) complex Nordenian structures on the tangent, cotangent and tensor bundles [4, 12-15].

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This paper is devoted to the investigation of para-Nordenian structures in the linear coframe bundle with Sasakian metric. In 2 we briefly describe the definitions and results that are needed later, after which the Sasakian metric ${}^{S}g$ and the Levi-Civita connection ${}^{S}\nabla$ of this metric in the linear coframe bundle $F^{*}(M)$ over a Riemannian manifold (M,g) are studied in 3. The para-Nordenian structures on the linear coframe bundle $F^{*}(M)$ with Sasakian metric are introduced in 4. In 5 we study the almost para-holomorphic vector fields on the linear coframe bundle $F^{*}(M)$ with Sasakian metric. The integrabilities of almost para-Nordenian structures on the $F^{*}(M)$ with Sasakian metric are investigated in 6.

2. Preliminaries

In this section we shall summarize briefly the basic definitions and results which be used later. Let (M,g) be an $n-$dimensional Riemannian manifold and $F^{*}(M)$ its coframe bundle (see, [19,20]). The coframe bundle $F^{*}(M)$ over M consists of all pairs (x,u^{*}), where x is a point of M and u^{*} is a basis (coframe) for the cotangent space $T_{x}^{*}M$. We denote by π the natural projection of $F^{*}(M)$ to M defined by $\pi(x,u^{*})=x$. If $(U;x^{1},x^{2},...,x^{n})$ is a system of local coordinates in M, then a coframe $u^{*}=(X^{\alpha})=(X^{1},X^{2},...,X^{n})$ for $T_{x}^{*}M$ can be expressed uniquely in the form $X^{\alpha}=X_{i}^{\alpha}(dx^{i})_{x}$ and hence

$$\left(\pi^{-1}(U);x^{1},x^{2},...,x^{n},X_{1}^{1},X_{2}^{1},...,X_{n}^{n}\right)$$

is a system of local coordinates in $F^{*}(M)$ (see, [19]). Indices $i,j,k,...,\alpha,\beta,\gamma,...$ have range in $\{1,2,...,n\}$, while indices $A,B,C,...$ have range in

$$\left\{1,...,n,n+1,...,n+n^{2}\right\}.$$

We put $h_{\alpha}=\alpha\cdot n+h$. Summation over repeated indices is always implied.

We denote by $\Im_{s}^{r}(M)$ the set of all differentiable tensor fields of type (r,s) on M. Let $V=V^{i}\partial_{i}$ and $\omega=\omega_{i}dx^{i}$ be the local expressions in $U\subset M$ of a vector and a covector (1-form) fields $V\in\Im_{0}^{1}(M)$ and $\omega\in\Im_{1}^{0}(M)$, respectively. Then the complete and horizontal lifts ${}^{C}V,{}^{H}V\in\Im_{0}^{1}(F^{*}(M))$ of V and the $\beta-$th vertical lifts ${}^{V_{\beta}}\omega\in\Im_{0}^{1}(F^{*}(M))$ $(\beta=1,2,...,n)$ of ω are given, respectively, by

$$ {}^{C}V=\begin{pmatrix} V^{i} \\ -X_{j}^{\alpha}\partial_{i}V^{j}\end{pmatrix},\qquad {}^{H}V=\begin{pmatrix} V^{i} \\ X_{j}^{\alpha}\Gamma_{ik}^{j}V^{k}\end{pmatrix},\qquad {}^{V_{\beta}}\omega=\begin{pmatrix} 0 \\ \delta_{\beta}^{\alpha}\omega_{i}\end{pmatrix} \tag{2.1}$$

with respect to the natural frame $\{\partial_{i},\partial_{i_{\alpha}}\}=\left\{\frac{\partial}{\partial x^{i}},\frac{\partial}{\partial X_{i}^{\alpha}}\right\}$, (see [10] for more details).

The vertical lift of a smooth function f on M is a function ${}^{V}f$ on $F^{*}(M)$ defined by ${}^{V}f=f\circ\pi$.

Let (U,x^{i}) be a coordinate system in M. In $U\in M$, we put

$$X_{(i)}=\frac{\partial}{\partial x^{i}},\theta^{(i)}=dx^{i},i=1,2,...,n.$$

Taking account of (2.1), we easily see that the components of ${}^{H}X_{(i)}$ and ${}^{V_{\alpha}}\theta^{(i)}$ are respectively, given by

$$D_{i}={}^{H}X_{(i)}=\left(A_{i}^{H}\right)=\begin{pmatrix}\delta_{i}^{h} \\ X_{j}^{\alpha}\Gamma_{ih}^{j}\end{pmatrix}, \tag{2.2}$$

$$D_{i_\alpha} = {}^{V_\alpha}\theta^{(i)} = \left(A^H_{i_\alpha}\right) = \begin{pmatrix} 0 \\ \delta^\alpha_\beta \delta^i_h \end{pmatrix} \tag{2.3}$$

with respect to the natural frame $\{\partial_i, \partial_{i_\alpha}\}$. We call the set $\left\{{}^H X_{(i)}, {}^{V_\alpha}\theta^{(i)}\right\}$ the frame adapted to the Levi-Civita connection ∇_g. On putting

$$D_i = {}^H X_{(i)}, \quad D_{i_\alpha} = {}^{V_\alpha}\theta^{(i)},$$

we write the adapted frame as $\{D_I\} = \{D_i, D_{i_\alpha}\}$. From equations (2.2), (2.3), and (2.1) we see that ${}^H V$ and ${}^{V_\alpha}\omega$ have respectively, components

$${}^H V = V^i D_i, {}^H V = \left({}^H V^I\right) = \begin{pmatrix} V^i \\ 0 \end{pmatrix}, \tag{2.4}$$

$${}^{V_\alpha}\omega = \sum_i \omega_i \delta^\beta_\alpha D_{i_\alpha}, {}^{V_\alpha}\omega = \left({}^{V_\alpha}\omega^I\right) = \begin{pmatrix} 0 \\ \delta^\beta_\alpha \omega_i \end{pmatrix} \tag{2.5}$$

with respect to the adapted frame $\{D_I\}$, where V^i and ω_i being local components of $V \in \Im^1_0(M)$ and $\omega \in \Im^0_1(M)$, respectively.
For each $x \in M$, the scalar product $g^{-1} = (g^{ij})$ is defined on the cotangent space $T^*_x M$ by $g^{-1}(\omega, \theta) = g^{ij}\omega_i\theta_j$ for all $\omega, \theta \in \Im^0_1(M)$.

The bracket operation of vertical and horizontal vector fields is given by the formulas

$$[{}^{V_\beta}\omega, {}^{V_\gamma}\theta] = 0, [{}^H X, {}^{V_\beta}\omega] = {}^{V_\beta}(\nabla_X \omega),$$

$$[{}^H X, {}^H Y] = {}^H[X, Y] + \sum_{\sigma=1}^n {}^{V_\sigma}(X^\sigma \circ R(X, Y)) \tag{2.6}$$

for all $X, Y \in \Im^1_0(M)$ and $\omega, \theta \in \Im^0_1(M)$, where R denotes the curvature tensor field of the linear connection ∇.

3. The Sasakian metric on the coframe bundle

We define a Riemannian metric ${}^S g$ on the coframe bundle $F^*(M)$ by the following three equations

$${}^S g({}^H X, {}^H Y) = {}^V(g(X, Y)) = g(X, Y) \circ \pi, \tag{3.1}$$

$${}^S g({}^H X, {}^{V_\beta}\omega) = 0, \tag{3.2}$$

$${}^S g({}^{V_\beta}\omega, {}^{V_\gamma}\theta) = \delta^{\beta\gamma V}(g^{-1}(\omega, \theta)) = \delta^{\beta\gamma}(g^{-1}(\omega, \theta) \circ \pi) \tag{3.3}$$

for any $X, Y \in \Im^1_0(M)$ and $\omega, \theta \in \Im^0_1(M)$ (see, [5]). We call the metric ${}^S g$ the Sasakian metric on the coframe bundle $F^*(M)$ over the Riemannian manifold (M, g). Since any tensor field of type $(0, 2)$ on the $F^*(M)$ is completely determined by its action on vector fields of type ${}^H X$ and ${}^{V_\beta}\omega$ it follows that ${}^S g$ is completely determined by its eqs (3.1), (3.2) and (3.3). The metric ${}^S g$ is a Riemannian metric on $F^*(M)$ uniquely determined by the metric g.

From equations (3.1), (3.2) and (3.3) it follows that

$${}^S g_{ij} = {}^S g(D_i, D_j) = {}^V(g(\partial_i, \partial_j)) = g_{ij},$$

$${}^S g_{i_\alpha j} = {}^S g(D_{i_\alpha}, D_j) = 0,$$

$${}^S g_{i_\alpha j_\beta} = {}^S g(D_{i_\alpha}, D_{j_\beta}) = \delta^{\alpha\beta V}(g^{-1}(dx^i, dx^j)) = \delta^{\alpha\beta} g^{ij},$$

i.e. ${}^{S}g$ has components in the form

$$ {}^{S}g = \begin{pmatrix} g_{ij} & 0 \\ 0 & \delta^{\alpha\beta} g^{ij} \end{pmatrix} \tag{3.4}$$

with respect to the adapted frame $\{D_I\}$.

Let us consider local 1−forms $\tilde{\eta}^I$ in $\pi^{-1}(U)$ defined by

$$\tilde{\eta}^I = \bar{A}^I{}_J dx^J,$$

where

$$A^{-1} = (\bar{A}^I{}_J) = \begin{pmatrix} \bar{A}^i{}_j & \bar{A}^i{}_{j_\beta} \\ \bar{A}^{i_\alpha}_j & \bar{A}^{i_\alpha}_{j_\beta} \end{pmatrix} = \begin{pmatrix} \delta^i_j & 0 \\ -X^\alpha_m \Gamma^m_{ij} & \delta^\alpha_\beta \delta^j_i \end{pmatrix}. \tag{3.5}$$

The matrix (3.8) is the inverse of the matrix

$$A = (A_K{}^J) = \begin{pmatrix} A_k^{\ j} & A_{k_\gamma}^{\ j} \\ A_k^{\ j_\beta} & A_{k_\gamma}^{\ j_\beta} \end{pmatrix} = \begin{pmatrix} \delta^j_k & 0 \\ X^\beta_m \Gamma^m_{jk} & \delta^\beta_\gamma \delta^k_j \end{pmatrix} \tag{3.6}$$

of the transformation $D_K = A_K{}^J \partial_J$ (see (2.2) and (2.3)). It is easy to establish that the set $\{\tilde{\eta}^I\}$ is the coframe dual to the adapted frame $\{D_K\}$, i.e.

$$\tilde{\eta}^I(D_K) = \bar{A}^I{}_J A_K{}^J = \delta^I_K.$$

Since the adapted frame is non-holonomic, we put

$$[D_I, D_J] = \Omega_{IJ}{}^K D_K$$

from which we have

$$\Omega_{IJ}{}^K = \left(D_I A^L_J - D_J A^L_I\right) \bar{A}^K_L.$$

According to (2.2), (2.3), (3.5) and (3.6), the components of non-holonomic object $\Omega_{IJ}{}^K$ are given by

$$\begin{cases} \Omega_{ij_\beta}{}^{k_\gamma} = -\Omega_{j_\beta i}{}^{k_\gamma} = -\delta^\gamma_\beta \Gamma^j_{ik}, \\ \Omega_{ij}{}^{k_\gamma} = X^\gamma_m R^m_{ijk}, \end{cases} \tag{3.7}$$

all the others being zero, where R^m_{ijk} local components of the curvature tensor field R of ∇_g.

Let ${}^{S}\nabla$ be the Levi-Civita connection determined by the Sasakian metric ${}^{S}g$ on the coframe bundle $F^*(M)$. We put

$${}^{S}\nabla_{D_I} D_J = {}^{S}\Gamma^K_{IJ} D_K.$$

From the equation

$${}^{S}\nabla_X Y - {}^{S}\nabla_Y X = [X, Y], \forall X, Y \in \Im^1_0(F^*(M))$$

we have

$${}^{S}\Gamma^K_{IJ} - {}^{S}\Gamma^K_{JI} = \Omega_{IJ}{}^K. \tag{3.8}$$

The equation

$$\left({}^{S}\nabla_X {}^{S}g\right)(Y, Z) = 0$$

has form

$$D_L {}^{S}g_{IJ} - {}^{S}\Gamma^K_{LI} {}^{S}g_{KJ} - {}^{S}\Gamma^K_{LJ} {}^{S}g_{IK} = 0 \tag{3.9}$$

with respect to the adapted frame $\{D_K\}$. By using (3.8) and (3.9), we obtain:

$$\begin{aligned} {}^{S}\Gamma^K_{IJ} = \tfrac{1}{2} {}^{S}g^{KL} \left(D_I {}^{S}g_{LJ} + D_J {}^{S}g_{IL} - D_L {}^{S}g_{IJ}\right) + \\ + \tfrac{1}{2}\left(\Omega_{IJ}{}^K + \Omega^K_{\cdot\ IJ} + \Omega^K_{\cdot\ JI}\right), \end{aligned} \tag{3.10}$$

where $\Omega^{K}_{\cdot\ \ IJ} = {}^{S}g^{KL\,S}g_{PJ}\Omega_{LI}{}^{P}$ and

$$\left({}^{S}g\right)^{-1} = \left({}^{S}g^{KJ}\right) = \begin{pmatrix} g^{kj} & 0 \\ 0 & \delta_{\gamma\beta}g_{kj} \end{pmatrix}. \tag{3.11}$$

Taking account (3.4), (3.7) and (3.11), we obtain from (3.10)

$$\begin{cases} {}^{S}\Gamma^{k}_{ij} = \Gamma^{k}_{ij}, & {}^{S}\Gamma^{k}_{i_\alpha j_\beta} = {}^{S}\Gamma^{k_\gamma}_{i_\alpha j} = {}^{S}\Gamma^{k_\gamma}_{i_\alpha j_\beta} = 0, \\ {}^{S}\Gamma^{k}_{ij_\beta} = \frac{1}{2}X^{\beta}_{m}R^{k\ jm}_{\cdot\ i\ \cdot}, & {}^{S}\Gamma^{k}_{i_\alpha j} = \frac{1}{2}X^{\alpha}_{m}R^{k\ \ \ im}_{\cdot\ j\ \cdot}, \\ {}^{S}\Gamma^{k_\gamma}_{ij} = \frac{1}{2}X^{\gamma}_{m}R^{m}_{ijk}, & {}^{S}\Gamma^{k_\gamma}_{ij_\beta} = -\delta^{\beta}_{\gamma}\Gamma^{j}_{ik}. \end{cases} \tag{3.12}$$

Let $\tilde{X}, \tilde{Y} \in \Im^{1}_{0}(F^{*}(M))$ and $\tilde{X} = \tilde{X}^{I}D_{I}, \tilde{Y} = \tilde{Y}^{J}D_{J}$. Then the covariant derivative ${}^{S}\nabla_{\tilde{Y}}\tilde{X}$ along $\tilde{Y}$ has components in the form

$${}^{S}\nabla_{\tilde{Y}}\tilde{X}^{I} = \tilde{Y}^{J}D_{J}\tilde{X}^{I} + {}^{S}\Gamma^{I}_{JK}\tilde{X}^{K}\tilde{Y}^{J} \tag{3.13}$$

with respect to the adapted frame $\{D_{I}\}$.

Using (2.4)-(2.6), (3.1)-(3.3), (3.12) and (3.13), we have

Theorem 3.1. *Let M be a Riemannian manifold with metric g and ${}^{S}\nabla$ be the Levi-Civita connection of the coframe bundle $F^{*}(M)$ equipped with Sasakian metric ${}^{S}g$. Then ${}^{S}\nabla$ satisfies*

$$i)\,{}^{S}\nabla_{{}^{V_\alpha}\omega}{}^{V_\beta}\theta = 0,$$
$$ii)\,{}^{S}\nabla_{{}^{V_\alpha}\omega}{}^{H}Y = \tfrac{1}{2}{}^{H}\left(R(\tilde{X}^{\alpha}, \tilde{\omega})Y\right),$$
$$iii)\,{}^{S}\nabla_{{}^{H}X}{}^{V_\beta}\theta = {}^{V_\beta}\left(\nabla_{X}\theta\right) + \tfrac{1}{2}{}^{H}\left(R(\tilde{X}^{\beta}, \tilde{\theta})X\right),$$
$$iv)\,{}^{S}\nabla_{{}^{H}X}{}^{H}Y = {}^{H}\left(\nabla_{X}Y\right) + \tfrac{1}{2}\sum_{\sigma=1}^{n}{}^{V_\sigma}\left(X^{\sigma}\circ R(X,Y)\right)$$

for all $X, Y \in \Im^{1}_{0}(M)$ and $\omega, \theta \in \Im^{0}_{1}(M)$, where $\tilde{\omega} = g^{-1}\circ\omega \in \Im^{1}_{0}(M)$, $\tilde{\theta} = g^{-1}\circ\theta \in \Im^{1}_{0}(M)$, $\tilde{X}^{\alpha} = g^{-1}\circ X^{\alpha} \in \Im^{1}_{0}(M)$.

We note that the analogue of Theorem 3.1 in the case of cotangent bundle is proved in [18].

4. Para-Nordenian structures on $\left(F^{*}(M), {}^{S}g)\right)$

An almost para-complex manifold is an almost product manifold (M_{n}, φ), $\varphi^{2} = I$, such that the two eigenbundles $T^{+}(M_{n})$ and $T^{-}(M_{n})$ associated to the two eigenvalues $+1$ and -1 of φ, respectively, have the same rank. The dimension of an almost paracomplex manifold is necessarily even.

A tensor field $t \in \Im^{0}_{q}\left(M_{2n}\right)$ is said to be a pure with respect to the para-complex structure φ, if

$$t(\varphi X_{1}, X_{2}, ..., X_{q}) = t(X_{1}, \varphi X_{2}, ..., X_{q}) = t(X_{1}, X_{2}, ..., \varphi X_{q})$$

for any $X_{1}, X_{2}, ..., X_{q} \in \Im^{1}_{0}(M_{2n})$.

We define the following operator ϕ_{φ} associated with φ and apply to the pure tensor field t :

$$(\phi_{\varphi}t)(Y, X_{1}, X_{2}, ..., X_{q}) = (\varphi Y)(t(X_{1}, X_{2}, ..., X_{q}))$$

$$-Y(t(\varphi X_1, X_2, ..., X_q)) + t((L_{X_1}\varphi)Y, X_2, ..., X_q))$$
$$+ \cdots + t(X_1, X_2, ..., (L_{X_q}\varphi)Y).$$

We note that $\phi_\varphi t \in \Im^0_q(M_{2n})$. If $\phi_\varphi t = 0$ then t is said to be an almost para-holomorphic (see [5, 15, 16, 18]).

Definition 4.1. In a manifold with almost para-complex structure φ, a vector field $X$is called an almost para-holomorphic vector field if $L_X\varphi = 0$.

A Riemannian manifold (M_{2n}, g) with an almost para-complex structure φ is said to be almost para-Nordenian if the Riemannian metric g is pure with respect to φ. It is well known that the almost para-Nordenian manifold is para-Kahler $(\nabla_g\varphi = 0)$ if and only if g is para-holomorphic $(\phi_\varphi g = 0)$ (see [18]).

Let $(F^*(M), {}^Sg)$ be the linear coframe bundle with the Sasakian metric ${}^Sg$. Define a tensor field F_α of type $(1,1)$ on $F^*(M)$ for each $\alpha = 12, ..., n$, by

$$F_\alpha({}^HX) = {}^{V_\alpha}\tilde{X}, \quad F_\alpha({}^{V_\beta}\omega) = \delta^{\beta H}_\alpha\tilde{\omega} \tag{4.1}$$

for any $X \in \Im^1_0(M)$ and $\omega \in \Im^0_1(M)$, where $\tilde{X} = g \circ X \in \Im^0_1(M)$, $\tilde{\omega} = g^{-1} \circ \omega \in \Im^1_0(M)$ and the horizontal lifts are considered with respect to the Levi-Civita connection of g. Each F_α satisfies the condition

$$F^2_\alpha = I.$$

Indeed, by virtue of (4.1), we have

$$F^2_\alpha({}^HX) = F_\alpha(F_\alpha({}^HX)) = F_\alpha({}^{V_\alpha}\tilde{X}) = \delta^{\alpha H}_\alpha\tilde{\tilde{X}} = {}^HX,$$
$$F^2_\alpha({}^{V_\beta}\omega) = F_\alpha(F_\alpha({}^{V_\beta}\omega)) = F_\alpha(\delta^{\beta H}_\alpha\tilde{\omega}) = \delta^{\beta V_\alpha}_\alpha\tilde{\tilde{\omega}} = {}^{V_\beta}\omega$$

for any $X \in \Im^1_0(M)$ and $\omega \in \Im^0_1(M)$, which implies $F^2_\alpha = I$ for each $\alpha = 12, ..., n$.

The following theorem holds.

Theorem 4.1. *The triple $(F^*(M), {}^Sg, F_\alpha)$, for each $\alpha = 12, ..., n$, is an almost para-Nordenian manifold.*

Proof. If we put

$$A(\tilde{X}, \tilde{Y}) = {}^Sg(F_\alpha\tilde{X}, \tilde{Y}) - {}^Sg(\tilde{X}, F_\alpha\tilde{Y})$$

for any $\tilde{X}, \tilde{Y} \in \Im^1_0(F^*(M))$. Then from (3.1)-(3.3) and (4.1), we have

$$A({}^HX, {}^HY) = {}^Sg(F_\alpha({}^HX), {}^HY) - {}^Sg({}^HX, F_\alpha({}^HY))$$
$$= {}^Sg({}^{V_\alpha}\tilde{X}, {}^HY) - {}^Sg({}^HX, {}^{V_\alpha}\tilde{Y}) = 0,$$
$$A({}^HX, {}^{V_\beta}\omega) = {}^Sg(F_\alpha({}^HX), {}^{V_\beta}\omega) - {}^Sg({}^HX, F_\alpha({}^{V_\beta}\omega))$$
$$= {}^Sg({}^{V_\alpha}\tilde{X}, {}^{V_\beta}\omega) - {}^Sg({}^HX, \delta^{\beta H}_\alpha\tilde{\omega})$$
$$\delta^{\alpha\beta}g^{-1}(gX, \omega) - g(X, g^{-1}\omega)\delta^\beta_\alpha = 0,$$
$$A({}^{V_\beta}\omega, {}^HY) = -A({}^HY, {}^{V_\beta}\omega) = 0,$$
$$A({}^{V_\beta}\omega, {}^{V_\gamma}\theta) = {}^Sg(F_\alpha({}^{V_\beta}\omega), {}^{V_\gamma}\theta) - {}^Sg({}^{V_\beta}\omega, F_\alpha({}^{V_\gamma}\theta))$$
$$= {}^Sg(\delta^{\beta H}_\alpha\tilde{\omega}, {}^{V_\gamma}\theta) - {}^Sg({}^{V_\beta}\omega, \delta^{\gamma H}_\alpha\tilde{\theta})$$
$$= \delta^{\beta S}_\alpha g({}^H\tilde{\omega}, {}^{V_\gamma}\theta) - \delta^{\gamma S}_\alpha g({}^{V_\beta}\omega, {}^H\tilde{\theta}) = 0,$$

i.e. ${}^Sg$ is pure with respect to F_α, for each $\alpha = 12, ..., n$. Thus Theorem 4.1 is proved.

Let us consider the covariant derivative of F_α, for each $\alpha = 12, ..., n$, with respect to the Levi-Civita connection ${}^S\nabla$ of metric ${}^Sg$. Taking into account $(i) - (iv)$ of Theorem 3.1, we obtain

$$({}^S\nabla_{{}^HX}F_\alpha)({}^HY) = {}^H\nabla_{{}^HX}F_\alpha({}^HY) - F_\alpha({}^S\nabla_{{}^HX}{}^HY)$$

$$= {}^S\nabla_{{}^HX}{}^{V_\alpha}\tilde{Y} - F_\alpha({}^H(\nabla_XY) + \frac{1}{2}\sum_{\beta=1}^{n}{}^{V_\beta}(X^\beta \circ R(X,Y)))$$

$$= {}^{V_\alpha}(\nabla_X\tilde{Y}) + \frac{1}{2}(R(\tilde{X}^\alpha, \tilde{\tilde{Y}})X) - F_\alpha({}^H(\nabla_XY) + \frac{1}{2}\sum_{\beta=1}^{n}{}^{V_\beta}(X^\beta \circ R(X,Y)))$$

$$= \frac{1}{2}{}^H(R(\tilde{X}^\alpha, Y)X) - \frac{1}{2}\delta_\alpha^\beta\sum_{\beta=1}^{n}{}^H(g^{-1}X^\beta \circ R(X,Y)) \tag{4.2}$$

$$= \frac{1}{2}{}^H(g^{-1}X^\alpha(R(\quad ,Y)X - R(X,Y))),$$

$$({}^S\nabla_{{}^{V_\beta}\omega}F_\alpha)({}^HY) = {}^H\nabla_{{}^{V_\beta}\omega}F_\alpha({}^HY) - F_\alpha({}^S\nabla_{{}^{V_\beta}\omega}{}^HY) \tag{4.3}$$

$$= {}^S\nabla_{{}^{V_\beta}\omega}{}^{V_\alpha}\tilde{Y} - \frac{1}{2}F_\alpha{}^H(R(\tilde{X}^\beta, \tilde{\omega})Y) = -\frac{1}{2}F_\alpha{}^H(X^\beta g^{-1} \circ R(\quad ,\tilde{\omega})Y)$$

$$= -\frac{1}{2}{}^{V_\alpha}(X^\beta \circ R(\quad ,\tilde{\omega})Y),$$

$$({}^S\nabla_{{}^HX}F_\alpha)({}^{V_\gamma}\theta) = {}^H\nabla_{{}^HX}F_\alpha({}^{V_\gamma}\theta) - F_\alpha({}^S\nabla_{{}^HX}{}^{V_\gamma}\theta)$$

$$= {}^S\nabla_{{}^HX}\delta_\alpha^\gamma{}^H\tilde{\theta} - F_\alpha({}^{V_\gamma}(\nabla_X\theta) + \frac{1}{2}{}^H(R(\tilde{X}^\gamma, \tilde{\theta})X))$$

$$= \delta_\alpha^\gamma{}^H(\nabla_X\tilde{\theta}) + \delta_\alpha^\gamma\frac{1}{2}\sum_{\beta=1}^{n}{}^{V_\beta}(X^\beta \circ R(X,\tilde{\theta})) - \delta_\alpha^\gamma{}^H(g^{-1} \circ \nabla_X\theta)$$

$$-\frac{1}{2}{}^{V_\alpha}(g \circ R(\tilde{X}^\gamma, \tilde{\theta})X) = \delta_\alpha^\gamma\frac{1}{2}\sum_{\beta=1}^{n}{}^{V_\beta}(X^\beta \circ R(X,\tilde{\theta})) \tag{4.4}$$

$$-\frac{1}{2}{}^{V_\alpha}(g \circ R(\tilde{X}^\gamma, \tilde{\theta})X),$$

$$({}^S\nabla_{{}^{V_\beta}\omega}F_\alpha)({}^{V_\gamma}\theta) = {}^H\nabla_{{}^{V_\beta}\omega}F_\alpha({}^{V_\gamma}\theta) - F_\alpha({}^S\nabla_{{}^{V_\beta}\omega}{}^{V_\gamma}\theta)$$

$$= {}^S\nabla_{{}^{V_\beta}\omega}(\delta_\alpha^\gamma{}^H\tilde{\theta}) = \delta_\alpha^\gamma{}^S\nabla_{{}^{V_\beta}\omega}{}^H\tilde{\theta} = \delta_\alpha^\gamma\frac{1}{2}{}^H(R(\tilde{X}^\beta, \tilde{\omega})\tilde{\theta}) \tag{4.5}$$

$$= \delta_\alpha^\gamma\frac{1}{2}{}^H(X^\beta g^{-1} \circ R(\quad ,\tilde{\omega})\tilde{\theta}).$$

From (4.2)-(4.5), we get

Theorem 4.2. *The linear coframe bundle $F^*(M)$ of a Riemannian manifold (M, g) is para-Kahlerian (para-holomorphic Nordenian) with respect to the metric ${}^Sg$ and almost para-complex structure F_α, for each $\alpha = 12, ..., n$, if and only if the Riemannian manifold (M, g) is flat.*

5. Para-holomorphic vector-fields on $\left(F^*(M), {}^S g\right)$

Let (M, g) be a Riemannian manifold, and let $F^*(M)$ be its linear coframe bundle with Sasakian metric ${}^S g$ and with the almost para-Nordenian structures F_α, $\alpha = 1, 2, ..., n$. A vector field $\tilde{X} \in \Im_0^1(F^*(M))$ with respect to which the almost para-Nordenian structure F_α has a vanishing Lie derivative ($L_{\tilde{X}} F_\alpha = 0$) is said to be almost para-holomorphic (see, [18]). It is well known that [5]

$$\left[{}^C X, {}^H Y\right] = {}^H[X, Y] + \sum_{\alpha=1}^{n} {}^{V_\alpha}(X^\alpha \circ (L_X \nabla) Y), \tag{5.1}$$

$$\left[{}^C V, {}^{V_\gamma}\omega\right] = {}^{V_\gamma}(L_V \omega) \tag{5.2}$$

for any $X, Y \in \Im_0^1(M)$ and $\omega \in \Im_1^0(M)$, where

$$(L_X \nabla) Y = \nabla_Y \nabla X + R(X, Y)$$

and

$$(L_X \nabla)(Y, Z) = L_X(\nabla_Y X) - \nabla_Y(L_X Z) - \nabla_{[X,Y]} Z.$$

A vector field $X \in \Im_0^1(M)$ is called a Killing vector field (or infinitesimal isometry) if $L_X g = 0$ and X called an infinitesimal affine transformation if $L_V \nabla g = 0$. A Killing vector field is necessarily an infinitesimal affine transformation, i.e. we have $L_V \nabla g = 0$ as a consequence of $L_X g = 0$. Now we consider the Lie derivative of F_α ($\alpha = 1, 2, ..., n$) with respect to the complete lift ${}^C X$. Taking account of (4.1), (5.1) and (5.2), we find

$$(L_{{}^C X} F_\alpha)({}^{V_\gamma}\theta) = L_{{}^C X}(F_\alpha({}^{V_\gamma}\theta) - F_\alpha(L_X {}^{V_\gamma}\theta)$$

$$= \delta_\alpha^\gamma L_{{}^C X} {}^H\tilde{\theta} - \delta_\alpha^\gamma(g^{-1} \circ (L_X \theta)) = \delta_\alpha^\gamma \left[{}^H[X, \tilde{\theta}] + \sum_{\beta=1}^{n} {}^{V_\beta}(X^\beta \circ L_X \nabla\tilde{\theta})\right.$$

$$\left. - {}^H(g^{-1} \circ (L_X \theta))\right] = \delta_\alpha^\gamma \left[{}^H(L_X(g^{-1} \circ \theta) - g^{-1} \circ (L_X \theta)) \right. \tag{5.3}$$

$$\left. + \sum_{\beta=1}^{n} {}^{V_\beta}(X^\beta \circ L_X \nabla\tilde{\theta})\right],$$

$$(L_{{}^C X} F_\alpha)({}^H Y) = L_{{}^C X}(F_\alpha({}^H Y) - F_\alpha(L_X {}^H Y)$$

$$= L_{{}^C X} {}^{V_\alpha}\tilde{Y} - F_\alpha({}^H[X, Y] + \sum_{\beta=1}^{n} {}^{V_\beta}(X^\beta \circ L_X \nabla Y)$$

$$= L_{{}^C X} {}^{V_\alpha}\tilde{Y} - {}^{V_\alpha}(g \circ [X, Y]) + \sum_{\beta=1}^{n} \delta_\alpha^\beta {}^{V_\beta}(g^{-1} X^\beta \circ L_X \nabla Y) \tag{5.4}$$

$$= {}^{V_\alpha}(L_X(g \circ Y) - g \circ L_X Y) + \sum_{\beta=1}^{n} \delta_\alpha^\beta {}^{V_\beta}(g^{-1} X^\beta \circ L_X \nabla Y).$$

Let now X be a Killing vector field ($L_X g = 0$). Then by virtue of $L_X \nabla = 0$, from (5.3) and (5.4), we have $L_{{}^C X} F_\alpha = 0$, $\alpha = 1, 2, ..., n$, i.e. ${}^C X$ is para-holomorphic with respect to each F_α. Hence, we have

Theorem 5.1. *Let (M,g) be a Riemannian manifold and let $(F^*(M), {}^Sg, F_\alpha)$ be para-Nordenian manifold for each $\alpha = 1,2,...,n$. Then the complete lift ${}^CX$ of vector field $X \in \Im^1_0(M)$ to $F^*(M)$ is almost para-holomorphic vector field with respect to the each almost para-Nordenian structure $(F_\alpha, {}^Sg)$, if X is a Killing vector field on a Riemannian manifold (M,g).*

6. Integrability of para-Nordenian structures on $\big(F^*(M), {}^Sg)\big)$

In this section, we study the integrability of almost para-Nordenian structure F_α for each $\alpha = 1,2,...,n$, on $\big(F^*(M), {}^Sg)\big)$. We assume that ∇ is a Levi-Civita connection of a Riemannian metric g. Denoting by N_{F_α} the Nijenhuis tensor of F_α, we have

$$N_{F_\alpha}(\tilde{X},\tilde{Y}) = [F_\alpha\tilde{X}, F_\alpha\tilde{Y}] - F_\alpha[F_\alpha\tilde{X},\tilde{Y}] - F_\alpha[\tilde{X},F_\alpha\tilde{Y}] + [\tilde{X},\tilde{Y}] \tag{6.1}$$

for all $\tilde{X},\tilde{Y} \in \Im^1_0(F^*(M))$. Then taking account of (2.6) and (6.1), we obtain

$$\begin{aligned}
N_{F_\alpha}({}^HX, {}^HY) &= [F_\alpha{}^HX, F_\alpha{}^HY] - F_\alpha[F_\alpha{}^HX, {}^HY] \\
-F_\alpha[{}^HX, F_\alpha{}^HY] + [{}^HX, {}^HY] &= [{}^{V_\alpha}\tilde{X}, {}^{V_\alpha}\tilde{Y}] - F_\alpha[{}^{V_\alpha}\tilde{X}, {}^HY] \\
-F_\alpha[{}^HX, {}^{V_\alpha}\tilde{Y}] &+ {}^H[X,Y] + \sum_{\beta=1}^n {}^{V_\beta}(X^\beta \circ R(X,Y)) \\
&= F_\alpha{}^{V_\alpha}(\nabla_Y\tilde{X}) - F_\alpha{}^{V_\alpha}(\nabla_X\tilde{Y}) + {}^H[X,Y] + \sum_{\beta=1}^n {}^{V_\beta}(X^\beta \circ R(X,Y)) \\
&= \delta_\alpha^{\alpha}{}^H(\nabla_Y X - \nabla_X Y) + {}^H[X,Y] + \sum_{\beta=1}^n {}^{V_\beta}(X^\beta \circ R(X,Y)) \\
&= \sum_{\beta=1}^n {}^{V_\beta}(X^\beta \circ R(X,Y)),
\end{aligned}$$

$$\begin{aligned}
N_{F_\alpha}({}^HX, {}^{V_\beta}\omega) &= [F_\alpha{}^HX, F_\alpha{}^{V_\beta}\omega] - F_\alpha[F_\alpha{}^HX, {}^{V_\beta}\omega] \\
-F_\alpha[{}^HX, F_\alpha{}^{V_\beta}\omega] + [{}^HX, {}^{V_\beta}\omega] &= [{}^{V_\alpha}\tilde{X}, \delta_\alpha^\beta{}^H\tilde{\omega}] - F_\alpha[{}^{V_\alpha}\tilde{X}, {}^{V_\beta}\omega] \\
-F_\alpha[{}^HX, \delta_\alpha^\beta{}^H\tilde{\omega}] + {}^{V_\beta}(\nabla_X\omega) &= \delta_\alpha^\beta[{}^{V_\alpha}\tilde{X}, {}^H\tilde{\omega}] + {}^{V_\beta}(\nabla_X\omega) \\
-\delta_\alpha^\beta F_\alpha({}^H[X,\tilde{\omega}] + \sum_{\gamma=1}^n {}^{V_\gamma}(X^\gamma \circ R(X,\tilde{\omega}))) &= -\delta_\alpha^\beta{}^{V_\alpha}(\nabla_{\tilde{\omega}}\tilde{X}) \\
+{}^{V_\beta}(\nabla_X\omega) - \delta_\alpha^\beta{}^{V_\alpha}(g\circ[X,\tilde{\omega}]) &- \delta_\alpha^\beta \sum_{\gamma=1}^n \delta_\alpha^\gamma{}^H(g^{-1}X^\gamma \circ R(X,\tilde{\omega})) \\
&= -{}^{V_\beta}(g\circ\nabla_{\tilde{\omega}}X) + {}^{V_\beta}(g\circ\nabla_X\tilde{\omega}) - {}^{V_\beta}(g\circ[X,\tilde{\omega}]) \\
-\delta_\alpha^\beta \sum_{\gamma=1}^n \delta_\alpha^\gamma{}^H(g^{-1}X^\gamma \circ R(X,\tilde{\omega})) &= {}^{V_\beta}(g\circ[X,\tilde{\omega}]) - {}^{V_\beta}(g\circ[X,\tilde{\omega}]) \\
&= -\delta_\alpha^\beta \sum_{\gamma=1}^n \delta_\alpha^\gamma{}^H(g^{-1}X^\gamma \circ R(X,\tilde{\omega})),
\end{aligned}$$

$$N_{F_\alpha}({}^{V_\beta}\omega, {}^{V_\gamma}\theta) = [F_\alpha{}^{V_\beta}\omega, F_\alpha{}^{V_\gamma}\theta] - F_\alpha[F_\alpha{}^{V_\beta}\omega, {}^{V_\gamma}\theta]$$

$$-F_\alpha[{}^{V_\beta}\omega, F_\alpha{}^{V_\gamma}\theta] + [{}^{V_\beta}\omega, {}^{V_\gamma}\theta] = [\delta_\alpha^\beta{}^H\tilde{\omega}, \delta_\alpha^\gamma{}^H\tilde{\theta}]$$
$$-F_\alpha[\delta_\alpha^\beta{}^H\tilde{\omega}, {}^{V_\gamma}\theta] - F_\alpha[{}^{V_\beta}\omega, \delta_\alpha^\gamma{}^H\tilde{\theta}] = \delta_\alpha^\beta\delta_\alpha^\gamma[{}^H\tilde{\omega}, {}^H\tilde{\theta}]$$
$$-\delta_\alpha^\beta F_\alpha[{}^H\tilde{\omega}, {}^{V_\gamma}\theta] - \delta_\alpha^\gamma F_\alpha[{}^{V_\beta}\omega, {}^H\tilde{\theta}] = \delta_\alpha^\beta\delta_\alpha^\gamma{}^H[\tilde{\omega}, \tilde{\theta}]$$
$$+\delta_\alpha^\beta\delta_\alpha^\gamma\sum_{\sigma=1}^{n}{}^{V_\sigma}(X^\sigma \circ R(\tilde{\omega}, \tilde{\theta})) - \delta_\alpha^\beta F_\alpha{}^{V_\gamma}(\nabla_{\tilde{\omega}}\theta)$$
$$+\delta_\alpha^\gamma F_\alpha{}^{V_\beta}(\nabla_{\tilde{\theta}}\omega) = \delta_\alpha^\beta\delta_\alpha^\gamma{}^H[\tilde{\omega}, \tilde{\theta}] + \delta_\alpha^\beta\delta_\alpha^\gamma\sum_{\sigma=1}^{n}{}^{V_\sigma}(X^\sigma \circ R(\tilde{\omega}, \tilde{\theta}))$$
$$-\delta_\alpha^\beta\delta_\alpha^\gamma{}^H(g^{-1} \circ \nabla_{\tilde{\omega}}\theta) + \delta_\alpha^\gamma\delta_\alpha^\beta{}^H(g^{-1} \circ \nabla_{\tilde{\theta}}\omega) = \delta_\alpha^\beta\delta_\alpha^\gamma{}^H[\tilde{\omega}, \tilde{\theta}]$$
$$+\delta_\alpha^\beta\delta_\alpha^\gamma\sum_{\sigma=1}^{n}{}^{V_\sigma}(X^\sigma \circ R(\tilde{\omega}, \tilde{\theta})) - \delta_\alpha^\beta\delta_\alpha^\gamma{}^H(\nabla_{\tilde{\omega}}\tilde{\theta})$$
$$+\delta_\alpha^\gamma\delta_\alpha^\beta{}^H(\nabla_{\tilde{\theta}}\tilde{\omega}) = \delta_\alpha^\beta\delta_\alpha^\gamma\sum_{\sigma=1}^{n}{}^{V_\sigma}(X^\sigma \circ R(\tilde{\omega}, \tilde{\theta}))$$

for all $X, Y \in \Im_0^1(M)$, $\omega, \theta \in \Im_1^0(M)$ and $1 \le \alpha \le n$.
From the above equations we conclude that $N_{F_\alpha} = 0$ if and only if $R(X, Y) = 0$ for all $X, Y \in \Im_0^1(M)$. Therefore, we have

Theorem 6.1. *Let (M, g) be a Riemannian manifold and $F^*(M)$ be its linear coframe bundle equipped with Sasakian metric ${}^Sg$. Then the almost para-Nordenian manifold $(F^*(M), {}^Sg, F_\alpha)$, for each $\alpha = 1, 2, ..., n$, is para-Nordenian if and only if $R = 0$.*

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