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CONVERSE THEOREM OF THE APPROXIMATION THEORY OF FUNCTIONS IN MORREY-SMIRNOV CLASSES RELATED TO THE DERIVATIVES OF FUNCTIONS

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Abstract. Let $G \subset \mathbb{C}$ be a finite Jordan domain with a Dini-smooth boundary. In this study an inverse theorem for polynomial approximation in the Morrey-Smirnov spaces $E^{p,\lambda}(G)$, $0 < \lambda \leq 1$ and $1 < p < \infty$, is proved.

1. Introduction and the main result

Let $\mathbb{T}$ denote the interval $[0, 2\pi]$. Let $L^p(\mathbb{T})$, $1 \leq p < \infty$, be the Lebesgue space of all measurable $2\pi-$periodic functions defined on $\mathbb{T}$ such that

$$\|f\|_p := \left(\int_{\mathbb{T}} |f(x)|^p \, dx \right)^{\frac{1}{p}} < \infty.$$

The *Morrey space* $L_0^{p,\lambda}(\mathbb{T})$ for a given $0 \leq \lambda \leq 1$ and $p \geq 1$, we define as the set of functions $f \in L_{loc}^p(\mathbb{T})$ such that

$$\|f\|_{L_0^{p,\lambda}(\mathbb{T})} := \left\{ \sup_I \frac{1}{|I|^{1-\lambda}} \int_I |f(t)|^p \, dt \right\}^{\frac{1}{p}} < \infty,$$

where the supremum is taken over all intervals $I \subset [0, 2\pi]$. Note that $L_0^{p,\lambda}(\mathbb{T})$ becomes a Banach space, for $\lambda = 1$ coincides with $L^p(\mathbb{T})$ and for $\lambda = 0$ with $L^\infty(\mathbb{T})$. If $0 \leq \lambda_1 \leq \lambda_2 \leq 1$, then $L_0^{p,\lambda_1}(\mathbb{T}) \subset L_0^{p,\lambda_2}(\mathbb{T})$. Also, if $f \in L_0^{p,\lambda}(\mathbb{T})$, then $f \in L^p(\mathbb{T})$ and hence $f \in L^1(\mathbb{T})$. The Morrey spaces were introduced by C. B. Morrey in 1938. The properties of these spaces have been investigated intensively by various authors and together with weighted Lebesgue spaces L_ω^p play an important role in the theory of partial equations, especially in the study of local behavior of the solutions of elliptic differential equations and describe local regularity more precisely than Lebesgue spaces L^p. The detailed

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information about properties of the Morrey spaces can be found in [12], [14], [15], [16], [24], [30], [31], [33], [35], [36], [39] and [41].

Denote by $C^{\infty}(\mathbb{T})$ the set of all functions that are realized as the restriction to $\mathbb{T}$ of elements in $C^{\infty}(\mathbb{R})$. Also, we define $L^{p,\lambda}(\mathbb{T})$ to be closure of $C^{\infty}(\mathbb{T})$ in $L_0^{p,\lambda}(\mathbb{T})$.

Let Γ be a rectifiable Jordan curve in the complex plane $\mathbb{C}$. This curve separates the plane into two domains $G := \operatorname{int}\Gamma$, $G^- := \operatorname{ext}\Gamma$. Without loss of generality we may assume $0 \in G$. Let D be the unit disc, $\mathbb{T} := \partial D$, $D^- := \operatorname{ext}\mathbb{T}$. We denote by ϕ the conformal mapping of G^- onto D^- normalized by $\phi(\infty) = \infty$ and $\lim\limits_{z\to\infty} \phi(z)/z > 0$. Let $\psi(w)$ be the inverse to $\phi(z)$. Also, let $w = \varphi_1(z)$ indicate a function that maps the domain G conformally onto the disk D. The inverse mapping of φ_1 will be shown by ψ_1. Let Γ_r be the image of the circle $|\varphi_1(z)| = r$, $0 < r < 1$ under the mapping $z = \psi_1(w)$.

Definition 1.1. Let us denote by $E^p(G)$, where $p > 0$, the class of all functions $f(z) \neq 0$ that are analytic in B and have the property that the integral

$$\int\limits_{\Gamma_r} |f(z)|^p \, |dz|$$

is bounded for $0 < r < 1$. We shall call the $E^p(G)$-class the *Smirnov class.*

If the function $f(z)$ belongs to E^p, then $f(z)$ has definite limiting values $f(z')$ almost everywhere (a. e.) on Γ, over all nontangential paths; $|f(z')|$ is summable on L; and

$$\lim_{r\to 1} \int\limits_{\Gamma_r} |f(z)|^p \, |dz| = \int\limits_{\Gamma} \left|f(z')\right|^p |dz| .$$

For $p > 1$ $E^p(G)$ is a Banach space with respect to the norm

$$\|f\|_{E^p(G)} := \|f\|_{L^p(\Gamma)} := \left(\int\limits_{\Gamma} |f(z)|^p \, |dz| \right)^{\frac{1}{p}} .$$

It is known that $\varphi' = E^1(G^-)$ and $\psi' \in E^1(D^-)$. Note that the general information about Smirnov classes can be found in the books [11, pp. 168-185]. [13, pp. 438-453].

We define Morrey spaces $L^{p,\lambda}(\Gamma)$ for a given $0 \leq \lambda \leq 1$ and $p \geq 1$, as the set of functions $f \in L^p_{loc}(\Gamma)$ such that

$$\|f\|_{L^{p,\lambda}(\Gamma)} := \left\{ \sup_B \frac{1}{|B|^{1-\lambda}} \int\limits_{B\cap\Gamma} |f(z)|^p \, |dz| \right\}^{\frac{1}{p}} < \infty,$$

where the supremum is taken over all balls of $\mathbb{C}$. Note that $L^{p,\lambda}(\Gamma)$ becomes a Banach space. If $\lambda = 1$, then $L^{p,\lambda}(\Gamma)$ coincides with $L^p(\Gamma)$ and if $\lambda = 0$, then we obtain the spaces $L^{\infty}(\Gamma)$.

We define also the *Morrey– Smirnov classes* $E^{p,\lambda}(G)$, $0 \leq \lambda \leq 1$ and $p \geq 1$, of analytic functions in G as

$$E^{p,\lambda}(G,\omega) := \left\{ f \in E^p(G) : f \in L^{p,\lambda}(\Gamma) \right\}.$$

If we define the norm of $f\in E^{p,\lambda}(G)$, $0\le\lambda\le 1$ and $p\ge 1$ by

$$\|f\|_{E^{p,\lambda}(G)}:=\|f\|_{L^{p,\lambda}(\Gamma)},$$

then the class $E^{p,\lambda}(G)$, $0\le\lambda\le 1$ and $p\ge 1$ will be a Banach space. Note that for $\lambda=1$ the class $E^{p,\lambda}(G)$ coincides with the classical Smirnov class $E^{p}(G)$ and for $\lambda=0$ with $E^{\infty}(G)$. If $G=D:=\{z:|z|<1\}$, then we have Morrey-Hardy space $H^{p,\lambda}(D):=E^{p,\lambda}(D)$ on the unit disk D.

Let $f\in L^{p,\lambda}(\mathbb{T})$, $0\le\lambda\le 1$ and $p\ge 1$. The function

$$\omega_{p,\lambda}(f,\delta)=\sup_{|h|\le\delta}\left\|f\left(e^{i(\cdot+h)}\right)-f\left(e^{i\cdot}\right)\right\|_{L^{p,\lambda}(\mathbb{T})}$$

is called modulus of continuity of f.

The modulus of continuity $\omega_{p,\lambda}(f,\delta)$ has the following properties [22], [23] :

1) $\omega_{p,\lambda}(f,\delta)$ is an increasing function,
2) $\lim\limits_{\delta\to 0}\omega_{p,\lambda}(f,\delta)=0$ for every $f\in L^{p,\lambda}(\mathbb{T})$, $0\le\lambda\le 1$ and $p\ge 1$,
3) $\omega_{p,\lambda}(f+g,\delta)\le\omega_{p,\lambda}(f,\delta)+\omega_{p,\lambda}(g,\delta)$ for $f,\ g\in L^{p,\lambda}(\mathbb{T})$
4) $\omega_{p,\lambda}(f,n\delta)\le n\omega_{p,\lambda}(f,\delta),\quad n\in\mathbb{N}$,
5) $\omega_{p,\lambda}(f,s\delta)\le(s+1)\,\omega_{p,\lambda}(f,\delta),\ s>0$,
6)$\omega_{p,\lambda}(f,\delta)\le[(n+1)\,\delta+1]\,\omega_{p,\lambda}(f,\frac{1}{n+1}),\ n\in\mathbb{N}$.

We denote $f_r(w)=f^{(r)}[\psi(w)]$. We define the modulus of continuity for $f^{(r)}(z)\in L^{p,\lambda}(\Gamma)$ as

$$\Omega_{\Gamma,p,\lambda}\left(\delta,f^{(r)}\right):=\omega_{p,\lambda}(\delta,f_r=\sup_{|h|\le\delta}\left\|f_r\left(e^{i(\cdot+h)}\right)-f_r\left(e^{i\cdot}\right)\right\|_{L^{p,\lambda}(\mathbb{T})}.$$

Let h be a continuous function on $[0,2\pi]$. Its modulus of continuity is defined by

$$\omega(t,h):=\sup\{|h(t_1)-h(t_2)|:t_1,t_2\in[0,2\pi],\quad |t_1-t_2|\le t\},\quad t\ge 0.$$

The curve Γ is called *Dini-smooth* if it has a parametrization

$$\Gamma:\varphi_0(s),\ 0\le s\le 2\pi$$

such that $\varphi_0'(s)$ is Dini-continuous, i.e.

$$\int\limits_0^{\pi}\frac{\omega(t,\varphi_0')}{t}dt<\infty$$

and $\varphi_0'(s)\ne 0$ [34, p. 48].

If Γ Dini-smooth curve, then there exist [40] the constants c_1, c_2, c_3 and c_4 such that

$$0\le c_1\le\left|\psi'(w)\right|\le c_2<\infty,\ |w|>1.$$

$$0\le c_3\le\left|\phi'(z)\right|\le c_4<\infty,\ z\in\overline{G}^{-}\tag{1.1}$$

We denote by $E_n(f)_{E^{p,\lambda}(G)}$ the best approximation of $f\in L^{p,\lambda}(\Gamma)$ by polynomials of degree not exceeding n, i.e.,

$$E_n(f)_{E^{p,\lambda}(G)}=\inf\left\{\|f-P_n\|_{L^{p,\lambda}(\Gamma)}:P_n\in\Pi_n\right\},$$

where Π_n denotes the class of algebraic polynomials of degree at most n.

We use the constants $c_1, c_2, c_3, ...$ (in general, different in different relations) which depend only on the quantities that are not important for the questions of interest.

The problems of approximation theory in the weighted and non-weighted Morrey spaces have been investigated by several authors (see, for example, [5-9], [17], [22], [23], [27-29] and [32]).

When $\Gamma = \partial G$ is a Dini-smooth curve, in this study the inverse theorem of approximation theory in Morey-Smirnov classes $E^{p,\lambda}(G)$, $0 \leq \lambda \leq 1$ and $p \geq 1$ has been proved. The result obtained in this work is generalization of the theorem proved for the Smirnov classes $E^p(G)$, $p > 1$, in the study [1], to more general Morrey- Smirnov classes $E^{p,\lambda}(G)$, $0 \leq \lambda \leq 1$ and $p > 1$. Simillar results in different classes were investigated in [2-4], [10], [18-21], [25], [26]. [37] and [38].

Our main result is the following theorem.

Theorem 1.1. *Let G be a bounded simply connected domain with a Dini-smoth boundary Γ and let $L^{p,\lambda}(\Gamma)$ be a Morrey space with $0 \leq \lambda \leq 1$ and $1 < p < \infty$. For each natural number n there exists a polynomial $P_n(z)$ of degree n, such that*

$$\|f - P_n\|_{L^{p,\lambda}(\Gamma)} \leq \frac{c_5}{n^{r+\alpha}}, \tag{1.2}$$

where $0 < \alpha \leq 1$ and r is a nonegative integer. Then $f \in E^{p,\lambda}(G)$, $0 \leq \lambda \leq 1$ and $p > 1$ and for the modulus of continuity $\Omega^{\alpha}_{\Gamma,p,\lambda}\left(\delta, f^{(r)}\right)$ the following inequalities hold:

$$\Omega^{\alpha}_{\Gamma,p,\lambda}\left(\delta, f^{(r)}\right) \leq c_6 \delta^{\alpha}, \; 0 < \alpha < 1, \tag{1.3}$$

$$\Omega^{\alpha}_{\Gamma,p,\lambda}\left(\delta, f^{(r)}\right) \leq c_7 \delta \left(1 + |\ln \delta|\right), \; \alpha = 1. \tag{1.4}$$

2. Auxiliary results

Let $f \in L_1(\Gamma)$. Then the functions f^+ and f^- defined by

$$f^+(z) = \frac{1}{2\pi i} \int\limits_{\Gamma} \frac{f(\zeta)}{\zeta - z} d\zeta, \; z \in B$$

and

$$f^-(z) = \frac{1}{2\pi i} \int\limits_{\Gamma} \frac{f(\zeta)}{\zeta - z} d\zeta, \; z \in B^-$$

are analytic in G and G^- respectively, and $f^-(\infty) = 0$. Thus the limit

$$S_\Gamma(f)(z) := \lim_{\varepsilon \to 0} \frac{1}{2\pi i} \int\limits_{\Gamma \cap \{\zeta : \, |\zeta - z| > \varepsilon\}} \frac{f(\zeta)}{\zeta - z} d\zeta$$

exists and is finite for almost all $z \in \Gamma$.

The quantity $S_\Gamma(f)(z)$ is called the *Cauchy singular integral* of f at $z \in \Gamma$.

According to the Privalov's theorem [13, p. 431] if one of the functions f^+ or f^- has the non-tangential limits a. e. on Γ, then $S_\Gamma(f)(z)$ exists a. e. on

L and also the other one has the non-tangential limits a. e. on Γ. Conversely, if $S_\Gamma(f)(z)$ exists a. e. on Γ, then the functions $f^+(z)$ and $f^-(z)$ have non-tangential limits a. e. on Γ. In both cases, the formulae

$$f^+(z) = S_\Gamma(f)(z) + \frac{1}{2}f(z),\ f^-(z) = S_\Gamma(f)(z) - \frac{1}{2}f(z)$$

and hence

$$f = f^+ - f^-$$

holds a. e. on Γ.

The following lemmas for the Morrey spaces $f \in L^{p,\lambda}(\mathbb{T})$, $0 < \lambda \leq 1$ and $1 < p < \infty$ play an important role in the proof of the main result.

Lemma 2.1.[23]. *Let* $L^{p,\lambda}(\mathbb{T})$, $0 < \lambda \leq 1$ *and* $p > 1$, *be a Morrey space. Then for each trigonometric polynomial* T_n *of degree* n, *and* $k \in \mathbb{N} := \{1, 2, ...\}$ *the inequality*

$$\left\| T_n^{(k)} \right\|_{L^{p,\lambda}(\mathbb{T})} \leq c_8 n^k \left\| T_n \right\|_{L^{p,\lambda}(\mathbb{T})},\ n \in \mathbb{N} := \{0, 1, 2, ...\} \tag{2.1}$$

holds with a constant c_8 *independent of* n.

Lemma 2.2. *Let* Γ *be a Dini-smooth curve and let* $L^{p,\lambda}(\Gamma)$ *be a Morrey space with* $0 < \lambda \leq 1$ *and* $1 < p < \infty$. *Then for a polynomial* $P_n(z)$ *of degree* n *the inequality*

$$\left\| P_n' \right\|_{L^{p,\lambda}(\Gamma)} \leq c_9 n \left\| P_n \right\|_{L^{p,\lambda}(\Gamma)} \tag{2.2}$$

holds with a constant c_9 *independent of* n.

Proof. For the trigonometric polynomial T_n this inequality was obtained in the study [23]. For $z = e^{i\theta}$ we get

$$P_n(z) = T_n(\theta) \quad P_n'(z)\, ie^{i\theta} = T_n'(\theta).$$

For polynomial $P_n(z)$ with respect to Faber polynomials the expansion

$$\begin{aligned} P_n(z) &= \sum_{k=0}^{n} a_k \phi_k(z) \\ &= \sum_{k=0}^{n} a_k [\phi(z)]^k + \frac{1}{2\pi i} \int_\Gamma \frac{d\zeta}{\varsigma - z} \sum_{k=0}^{n} a_k [\phi(\varsigma)]^k, \quad z \in ext\Gamma. \end{aligned} \tag{2.3}$$

holds. Then if $z \in ext\Gamma$ we obtain

$$P_n'(z) = \sum_{k=1}^{n} k a_k [\phi(z)]^{k-1} \phi'(z) + \frac{1}{2\pi i} \int_\Gamma \frac{d\zeta}{\varsigma - z} \sum_{k=1}^{n} k a_k [\phi(\varsigma)]^{k-1} \phi'(\varsigma). \tag{2.4}$$

Consider the function $P_{n,0}(\tau) = P_n[\psi(\tau)] \in L^{p,\lambda}(\mathbb{T})$. Then the Cauchy type integral

$$\frac{1}{2\pi i} \int_\mathbb{T} \frac{P_{n,0}(\tau)}{\tau - w} d\tau$$

represents analytic functions $P_{n,0}^+$ and $P_{n,0}^-$ in D and D^-, respectively. For $w \in \mathbb{T}$ we get

$$P_{n,0}^+(w) = \sum_{k=0}^{n} a_k w^k$$

Using (1.1) we have

$$\|P_n[\psi(\cdot)]\|_{L^{p,\lambda}(\mathbb{T})} \le c_{10}\|P_n(\cdot)\|_{L^{p,\lambda}(\Gamma)}. \tag{2.5}$$

Using (2.5) and the boundedness of the singular integral we have [23]

$$\|S_{\mathbb{T}}(P_{n,0})(\cdot)\|_{L^{p,\lambda}(\mathbb{T})} \le c_{11}\|P_n(\cdot)\|_{L^{p,\lambda}(\Gamma)}. \tag{2.6}$$

Applying the inequalities (2.5), (2.6) and Minkowski inequality we obtain

$$\left\|P^+_{n,0}(\cdot)\right\|_{L^{p,\lambda}(\mathbb{T})} = \left\|S_T(P_{n,0})(\cdot) - \frac{1}{2}P_{n,0}(\cdot)\right\|_{L^{p,\lambda}(\mathbb{T})}$$

$$\le \|S_{\mathbb{T}}(P_{n,0})(\cdot)\|_{L^{p,\lambda}(\mathbb{T})} + \frac{1}{2}\|P_{n,0}(\cdot)\|_{L^{p,\lambda}(\mathbb{T})} \le c_{12}\|P_n(\cdot)\|_{L^{p,\lambda}(\Gamma)}. \tag{2.7}$$

Consideration of (2.1), (2.7) for the nontangential limits on $\mathbb{T}$ gives us

$$\left\|\left(P^+_{n,0}(\cdot)\right)'\right\|_{L^{p,\lambda}(\mathbb{T})} \le c_{13}n\|P_n(\cdot)\|_{L^{p,\lambda}(\Gamma)}.$$

Taking (1.1) into account, we have

$$\left\|\sum_{k=1}^{n} ka_k[\phi(\cdot)]^{k-1}\phi'(\cdot)\right\|_{L^{p,\lambda}(\Gamma)}$$

$$\le \quad c_{14}\left\|\sum_{k=1}^{n} ka_k w^{k-1}\right\|_{L^{p,\lambda}(\mathbb{T})} \le c_{15}n\|P_n(\cdot)\|_{L^{p,\lambda}(\Gamma)}. \tag{2.8}$$

Using (2.8), Minkowski's inequality and the boundedness of the singular operator S_Γ in the Morrey spaces [23] for the non-tangential limits on Γ in (2.4) of the integral, we have

$$\left\|\frac{1}{2}\left(\sum_{k=1}^{n} ka_k[\phi(\cdot)]^{k-1}\phi'(\cdot)\right) + S_\Gamma\left[\sum_{k=1}^{n} ka_k[\phi(\cdot)]^{k-1}\phi'(\cdot)\right]\right\|_{L^{p,\lambda}(\Gamma)}$$

$$\le c_{16}\left\|\sum_{k=1}^{n} ka_k[\phi(\cdot)]^{k-1}\phi'(\cdot)\right\|_{L^{p,\lambda}(\Gamma)} \le c_{17}n\|P_n(\cdot)\|_{L^{p,\lambda}(\Gamma)}. \tag{2.9}$$

Using (2.4), (2.8) and (2.9) we reach inequality (2.2).

3. Proof of the main result

Proof of Theorem 1.1. The inequality

$$\|P_n\|_{L^{p,\lambda}(\Gamma)} \le \|P_n - f\|_{L^{p,\lambda}(\Gamma)} + \|f\|_{L^{p,\lambda}(\Gamma)} \le c_{18}, \tag{3.1}$$

holds, where constant c_{18} independent of n.

The sequence $\{P_n(\varsigma)\}$ converges in $L^{p,\lambda}(\Gamma)$. Therefore, the sequence $\{P_n(\varsigma)\}$ converges with respect to a measure. Since, condition (3.1) is satisfied, according to [22] and [23] sequence $\{P_n(\varsigma)\}$ converges uniformly within the domain to the function $f(z) \in E^{p,\lambda}(G)$ and nontangential boundary values of the function $f(z)$ (from inside Γ) coincides with $f(\varsigma)$ a.e. on Γ.

We define the following polynomials sequence:

$$T_0(z) = P_1(z), \qquad T_k(z) = P_{2^k}(z) - P_{2^{k-1}}(z) \qquad (k = 1, 2, ...).$$

The series $\sum\limits_{k=0}^{\infty} T_k(z)$ converges uniformly to the function $f(z)$ into G. Then the series $\sum\limits_{k=0}^{\infty} T_k^{(r)}$ converges uniformly to the function $f^{(r)}(z)$ into G. We define the following sequence:

$$K_n(s) = \sum_{k=0}^{n} T_k^{(r)}(s).$$

Now show that the sequence $K_n(\varsigma)$ converges in $L^{p,\lambda}(\Gamma)$. Taking into account (1.2), we have

$$\begin{aligned} & \|T_k\|_{L^{p,\lambda}(\Gamma)} \\ \leq \; & \|f - P_{2^k}\|_{L^{p,\lambda}(\Gamma)} + \left\|f - P_{2^{k-1}}\right\|_{L^{p,\lambda}(\Gamma)} \leq \frac{c_{19}}{2^{k(r+\alpha)}}. \end{aligned} \tag{3.2}$$

Use of (3.2) and (2.2) gives us

$$\left\|T_k^{(r)}\right\|_{L^{p,\lambda}(\Gamma)} \leq \frac{c_{20}}{2^{k\alpha}}. \tag{3.3}$$

From (3.3) we conclude that

$$\|K_m - K_n\|_{L^{p,\lambda}(\Gamma)} \leq \sum_{k=n+1}^{m} \left\|T_k^{(r)}\right\|_{L^{p,\lambda}(\Gamma)} \leq \frac{c_{21}}{2^{n\alpha}}, \qquad (m > n). \tag{3.4}$$

Then according to inequality (3.4) the sequence $K_n(s)$ is a Cauchy sequence. Since $L^{p,\lambda}(\Gamma)$ is a Banach space, the sequence $\{K_n(\varsigma)\}$ converges in $L^{p,\lambda}(\Gamma)$. Therefore, the sequence $K_n(\varsigma)$ converges with respect to a measure.

Since $\|K_n(\varsigma)\|_{L^{p,\lambda}(\Gamma)} \leq c_{22}$, the sequence $K_n(\varsigma)$ converges with respect to a measure to $f^{(r)}(\varsigma)$ nontangential boundary values of the function $f^{(r)}(z)$. There exists subsequence $K_{n_i}(s)$ of the sequence $K_n(s)$ such that

$$K_{n_i}(s) \to f^{(r)}(s)$$

a.e on Γ. Then we have a.e. on Γ

$$|K_{n_i}(\varsigma) - K_n(\varsigma)| \to \left|f^{(r)}(\varsigma) - K_n(\varsigma)\right|.$$

According to Fatou's Lemma and (3.4) we obtain

$$\left\|f^{(r)} - K_n\right\|_{L^{p,\lambda}(\Gamma)} \leq \frac{c_{23}}{2^{n\alpha}}, \tag{3.5}$$

where $K_n(\varsigma)$ polynomial of degree 2^n. We fix δ, satisfying the condition $0 < \delta \leq \frac{1}{2}$ and choose $m \in N$, such that $2^{m-1} \leq \frac{1}{\delta} < 2^m$. If we pass on to the complex plane (w), then from the inequality (3.5) we get

$$\begin{aligned} & \left\|f_r\left(\cdot e^{ih}\right) - f_r(\cdot)\right\|_{L^{p,\lambda}(T)} \\ \leq \; & \frac{c_{24}}{2^{m\alpha}} + \left\|K_{m-1}\left[\psi\left(\cdot e^{ih}\right)\right] - K_{m-1}\left[\psi(\cdot)\right]\right\|_{L^{p,\lambda}(T)}. \end{aligned} \tag{3.6}$$

We define the sequence of the polynomials in the following form:

$$Q_1(z) = K_1(z), \qquad Q_k(z) = K_k(z) - K_{k-1}(z),$$

where the polynomial $Q_k(z)$ is of degree 2^k.

By (3.5) we have

$$\|Q_k\|_{L^{p,\lambda}(\Gamma)} = \|K_k - K_{k-1}\|_{L^{p,\lambda}(\Gamma)} \le \frac{c_{25}}{2^{k\alpha}}, \qquad k \ge 2. \tag{3.7}$$

Putting $w = e^{ix}$ we define $Q_k\left[\psi\left(e^{ix}\right)\right] = \nu_k(x)$. Then we have

$$\|\nu_k(\cdot + h) - \nu_k(\cdot)\|_{L^{p,\lambda}(\mathbb{T})}$$

$$= \left\{ \sup_I \frac{1}{|I|^{1-\lambda}} \int_I |\nu_k(x+h) - \nu_k(x)|^p \, dx \right\}^{\frac{1}{p}}$$

$$= \left\{ \sup_I \frac{1}{|I|^{1-\lambda}} \int_I \left| \int_0^h \nu_k'(x+h) \right|^p dx \right\}^{\frac{1}{p}}$$

$$\le h \|\nu_k'\|_{L^{p,\lambda}(\mathbb{T})} \tag{3.8}$$

Since the curve Γ is Dini-smooth, taking into account (1.1) we obtain

$$\|\nu_k'\|_{L^{p,\lambda}(\mathbb{T})} \le c_{26} \|Q_k'\|_{L^{p,\lambda}(\Gamma)} . \tag{3.9}$$

Use of (3.8) and (3.9) gives us

$$\left\| Q_k\left[\psi\left(\cdot e^{ih}\right)\right] - Q_k\left[\psi(\cdot)\right] \right\|_{L_M(\mathbb{T})} \le c_{27} h \|Q_k'(\cdot)\|_{L_M(\Gamma)} . \tag{3.10}$$

Taking into account the relations (2.2), (3.9) and (3.10) we have

$$\left\| K_{m-1}\left[\psi\left(\cdot e^{ih}\right)\right] - K_{m-1}\left[\psi(\cdot)\right] \right\|_{L^{p,\lambda}(\mathbb{T})}$$

$$\le c_{28} h \sum_{k=1}^{m-1} \|Q_k'(\cdot)\|_{L^{p,\lambda}(\Gamma)} \le c_{29} h \sum_{k=1}^{m-1} 2^k \|Q_k(\cdot)\|_{L^{p,\lambda}(\Gamma)} . \tag{3.11}$$

Use of (3.6), (3.11) and (3.7) for $|h| \le \delta$ gives us

$$\Omega_{\Gamma,p,\lambda}\left(\delta, f^{(r)}\right) \le \frac{c_{30}}{2^{m\alpha}} + c_{31}\delta \sum_{k=2}^{m-1} 2^{k(1-\alpha)} + c_{30}\delta. \tag{3.12}$$

Taking into account (3.12), we obtain (1.3) and (1.4). The proof of Theorem 1.1 is completed.

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