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SCALABILITY OF G-FRAMES BY DIAGONAL OPERATORS

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Abstract. Tight frames are similar to orthogonal bases, except that the frame coefficients are not unique, but they are stable in calculations and numerical algorithms. Not all frames are tight frames, but some have the ability to become tight frames. These frames are called scalable frames. In this article, we extend this good property of frames to G-frames. For this purpose, we define the scalable G-frame based on the diagonal operators, and obtain a preconditioner for its analysis operator by block diagonal operator. We also provide the necessary and sufficient conditions for the scalability of the G-frames based on the frames induced by the G-frames.

1. Introduction

The frames are a generalization of orthogonal bases, with the difference that they are not linearly independent. That is, unlike the bases, the coefficients of the frames are not necessarily unique. Gabor [11] formulated a new method for signal decomposition and signal expansion based on preliminary signals. According to this study, Duffin and Schaeffer [8] provided frames for the Hilbert space to solve the non-harmonic series. Until 1980, the importance of frames was not known. After Daubechies, Grossmann, and Meyer [7] reintroduced the frames, a lot of research was done on this subject. Various extensions and generalizations of frames have been introduced so far, such as pseudo-frames [15] quasi projectors [10] and oblique frames [5]. These frames are a special feature of G-frames, which was first introduced by Sun [17].

The applications of frames can be divided into two categories. In the first applications, the frame is used for data analysis. In this case, the goal is to resist data deletion, data analysis, and compression. Other applications are data extensions. The approach is used in summarizing methods such as compressing sensors. In applications, there need to be stable numerical algorithms. Tight frames are one of the best subclasses of frames in this situation. One may ask how a tight frame can be obtained. One of the solutions that can be used to get a tight frame from a frame is the effect of the operator $S_F^{-\frac{1}{2}}$ on vectors of the frame $F = \{f_k\}$, where S_F is frame operator for F. But it's not easy to get this operator, and also make changes to the frame vectors. One of the most important

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ways to get a tight frame is to change the scale of each frame vector. For this purpose, Kutyniok and et al. [13] introduced a type of frames, which by scaling of its vectors, become Parseval frames. They called these type of frames scalable frames. They showed that for transforming a frame $F = \{f_k\}$ into a scalable frame, one should find a diagonal operator D such that the conditional number of the operator DT_F^*, where T_F^* is analysis operator of F, is one. In addition, they stated, with a geometric interpretation, that not all frames are necessarily scalable. After that, scalable frames have been investigated by several researchers [3, 13, 14].

In this study, we develop the concept of the scalability of frames to G-frames, and state the scalability of G-frames using non-negative diagonal operators instead of non-negative numbers for scale changes. Furthermore, we use block diagonal operator to precondition G-frames.

The present paper is organized as follows: In Section 2, we fix the notations of this paper and summarize some results needed for the rest of the paper. In Section 3, we present the concept of scalability of G-frames and prove some of the proper results for these G-frames. Finally, in the last section, we introduce some explicit constructions of scalable G-frames.

2. Notation and Preliminaries

Throughout this paper $\mathcal{K}$ and $\mathcal{H}$ are two Hilbert spaces and $\{\mathcal{H}_i, i \in I\}$ and $\{\mathcal{K}_i, i \in I\}$ are sequences of Hilbert spaces, where $I \subset \mathbb{Z}$. $\mathcal{L}(\mathcal{K}, \mathcal{H})$ is the collection of all bounded linear operators from $\mathcal{K}$ to $\mathcal{H}$ and for brevity, we denote $\mathcal{L}(\mathcal{H}, \mathcal{H})$ by $\mathcal{L}(\mathcal{H})$. We also recall that the space

$$\left(\bigoplus \mathcal{H}_i\right)_{l^2} = \left\{ \{x_i\} | x_i \in \mathcal{H}_i, i \in I : \quad \sum_{i\in I} \|x_i\|_2^2 < \infty \right\},$$

is a Hilbert space with pointwise operations and the inner product as

$$\langle \{x_i\}, \{y_i\} \rangle = \sum_{i\in I} \langle x_i, y_i \rangle.$$

Definition 2.1. An operator D defined on the (closed) linear span of a basis $\{e_i\}_{i\in I}$ in a normed space X is called diagonal operator whenever $De_k = \lambda_k e_k$, where $k \geq 1$ and λ_k's are complex numbers. If D is a continuous operator, one has

$$sup_{k\geq 1} \mid \lambda_k \mid < \infty.$$

$$\| D \| = sup_{k\geq 1} \mid \lambda_k \mid < \infty.$$

We denote the domain, the kernel, and the range of linear operator S by $domS$, $kerS$ and $rangS$, respectively.

Definition 2.2. Assume that D_i is an operator on $\mathcal{H}_i$; $\ i \in I$. We say that the operator D on $(\bigoplus \mathcal{H}_i)_{l^2}$ is a block diagonal operator with $\{D_i\}$ as its diagonal, whenever

$$D(\{f_i\}) = \{D_i f_i\}, \quad \{f_i\} \in domD,$$

where,

$$domD = \left\{ \{f_i\} \in \left(\bigoplus \mathcal{H}_i \right)_{l^2} : \{D_i f_i\} \in \left(\bigoplus \mathcal{H}_i \right)_{l^2} \right\}.$$

For the block diagonal operator D, we recall that $\|D\| = sup_{i\in I}\|D_i\|$.

Definition 2.3. An operator $S : \mathcal{H} \to \mathcal{K}$ will be called ICR (Injective Closed Range), whenever S has the following properties

i) S is closed linear operator,

ii) there exists $r > 0$ such that $\|Sf\| \geq r\|f\|, \quad \forall f \in \mathcal{H}$.

An operator P on the Hilbert space $\mathcal{H}$ is called non-negative operator if $\langle Pf, f\rangle \geq 0$. If P is non-negative, then its eigenvalues are non-negative real numbers [6].

2.1. Scalable frames. Here's a brief overview of topics on frames and scalable frames. For more information on frames see [2, 4] and for scalable frames see [1, 13, 14, 16, 18].

Definition 2.4. A sequence $\{f_n\}$ in a Hilbert space $\mathcal{H}$ is called a frame if there exist two constants $0 < A \leq B < \infty$ such that

$$A\|f\|^2 \leq \sum_{n\in\mathbb{N}} |\langle f, f_n\rangle|^2 \leq B\|f\|^2 \quad \forall f \in \mathcal{H}.$$

If $A = B$ the sequence $\{f_n\}$ is called tight frame and if $A = B = 1$, it is called Parseval frame.
Let $F = \{f_n\}$ be a frame, the map

$$T_F : l_2(\mathbb{N}) \to \mathcal{H}; \quad T_F(\{c_n\}) = \sum_{n\in\mathbb{N}} c_n f_n,$$

is called the synthesis operator of the frame F. This operator is bounded and onto. The adjoint T_F denoted by

$$T_F^* : \mathcal{H} \to l_2(\mathbb{N}); \quad T_F^*(f) = \{\langle f, f_n\rangle\},$$

is called analysis operator of F. The operator

$$S_F : \mathcal{H} \to \mathcal{H}; \quad S_F f = T_F T_F^* f = \sum_{n\in\mathbb{N}} \langle f, f_n\rangle f_n,$$

is called frame operator. For the frame F, the frame operator is bounded, self-adjoint, positive and invertible.

Definition 2.5. A frame $F = \{f_i\}_{i\in I}$ is called scalable frame whenever there exists a sequence of non-negative real numbers $\{\alpha_i\}_{i\in I}$ such that $\{\alpha_i f_i\}_{i\in I}$ is a Parseval frame. Also, if α_i's are positive numbers then the frame $F = \{f_i\}_{i\in I}$ is called strictly scalable.

Let $F = \{f_i\}_{i\in I}$ be a frame for Hilbert space $\mathcal{H}$. Then F is scalable if and only if there exists a non-negative diagonal operator D on $l^2(\mathbb{I})$ such that $\overline{T_F^* D} D T_F = I_{\mathcal{H}}$, [13].

Theorem 2.1. [13] *Let $F = \{f_i\}_{i=1}^{M} \subset \mathbb{R}^N \backslash \{0\}$ be a frame for $\mathbb{R}^N$. Then F is not scalable if and only if there exists a symmetric matrix $Y \in \mathbb{R}^{N\times N}$ with $tr(Y) < 0$ such that $f_j^T Y f_j \geq 0$ for all $j = 1, ..., M$, where f_j^T denotes the transpose of the matrix f_j.*

2.2. *G*-frames. In this section, we present the basic concepts of the G-frames. As we mentioned earlier, this notation has been introduced by Sun [17]. For more details see [5, 12, 17].

A sequence $\{G_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i)\}_{i\in I}$ is called G-frame for $\mathcal{H}$ with respect to $\{\mathcal{H}_i\}_{i\in I}$ whenever there exist two constants A, B such that $0 < A \leq B < \infty$ and

$$A \parallel f \parallel^2 \leq \sum_{i\in I} \parallel G_i f \parallel^2 \leq B \parallel f \parallel^2, \quad \forall f \in \mathcal{H}. \tag{2.1}$$

The sequence $\{G_i\}_{i\in I}$ is called tight G-frame whenever $A = B$, if $A = B = 1$, then $\{G_i\}_{i\in I}$ is called Parseval G-frame. Also, if the right hand (2.1) holds then the sequence $\{G_i\}_{i\in I}$ is called G-Bessel.

We recall that the synthesis operator for G-frame $\mathcal{G} = \{G_i\}_{i\in I}$ is

$$T_{\mathcal{G}} : \left(\bigoplus \mathcal{H}_i\right)_{l^2} \to \mathcal{H}; \quad T_{\mathcal{G}}(\{f_i\}_{i\in I}) = \sum_{i\in I} G_i^* f_i$$

and the analysis operator is

$$T_{\mathcal{G}}^* : \mathcal{H} \to \left(\bigoplus \mathcal{H}_i\right)_{l^2}; \quad T_{\mathcal{G}}^* f = \{G_i f\}_{i\in I}.$$

Now we are ready to recall the definition of G-frame operator

$$S_{\mathcal{G}} : \mathcal{H} \to \mathcal{H}; \quad S_{\mathcal{G}} f = \sum_{i\in I} G_i^* G_i f.$$

A G-frame operator is bounded, self-adjoint and invertible operator and the family $\{G_i S^{-1}\}_{i\in I}$ is called the canonical G-dual frame.

Suppose that $\{G_j\}_{j\in I}$ is a G-frame and $\{e_{j,k} : k \in \mathbb{K}_j\}$, where $\mathbb{K}_i \subseteq \mathbb{Z}$, is an orthonormal basis for $\mathcal{H}_j$. The sequence $\{u_{j,k} = G_j^* e_{j,k} : j \in I, k \in \mathbb{K}_j\}$ is called the frame induced by $\{G_j\}_{j\in I}$ with respect to $\{e_{j,k} : k \in \mathbb{K}_j\}$ [Theorem 3.1 [17]].

3. Scalibility of G-frames

In this section, we provide a new definition for the scalability of G-frames. For this definition, we use diagonal operators and block diagonal operators.
At first the question arises: if any sequence of operators which affects a G-frame creates a G-frame again. The answer to this question is certainly no. Below, we describe the conditions under which after the effect of a sequence of operators on a G-frame, a new G-frame is created.

A positive diagonal operator $D \in \mathcal{L}(\mathcal{H})$ is invertible, thus there exist two constants K, K' such that

$$K \parallel f \parallel \leq \parallel Df \parallel \leq K' \parallel f \parallel \quad \forall f \in \mathcal{H}.$$

Definition 3.1. A sequence of positive diagonal operators $\{D_i \in L(\mathcal{H}_i)\}_{i\in I}$ is called semi-normalized whenever

$$0 < inf_{i\in I} K_i \leq sup_{i\in I} K'_i < \infty,$$

where K_i and K'_i are the lower bound and the upper bound for D_i, $i \in I$, respectively.

Lemma 3.1. *Let $\{G_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i)\}_{i\in I}$ be a G-frame for the Hilbert space $\mathcal{H}$ with respect to $\{\mathcal{H}_i\}_{i\in I}$ with bounds A, B and $\{D_i \in L(\mathcal{H}_i)\}_{i\in I}$ be a sequence of semi-normalized diagonal operators. Then $\{D_iG_i\}_{i\in I}$ is a G-frame.*

Proof. Let $f \in \mathcal{H}$. Then

$$\sum_{i\in I} \| D_iG_if \|^2 \leq \sum_{i\in I} \| D_i \|^2 \| G_if \|^2 \leq$$

$$sup_{i\in I}K'_i \sum_{i\in I} \| G_if \|^2 \leq Bsup_{i\in I}K'_i \| f \|^2 .$$

On the other hand,

$$\sum_{i\in I} \| D_iG_if \|^2 \geq inf_{i\in I}K_i \sum_{i\in I} \| G_if \|^2$$

$$\geq inf_{i\in I}K_iA \| f \|^2 .$$

Therefore, the sequence $\{D_iG_i\}$ is a G-frame. □

Definition 3.2. A G-frame $\{G_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i)\}_{i\in I}$ is called scalable G-frame for $\mathcal{H}$ with respect to $\{\mathcal{H}_i\}$ whenever there exists a sequence of non-negative diagonal operators $\{D_i \in L(\mathcal{H}_i)\}_{i\in I}$ such that $\{D_iG_i\}$ is a Parseval G-frame. Also, if D_i's are positive operators then the G-frame $\{G_i\}_{i\in I}$ is called strictly scalable G-frame.

Here are some examples of the scalability of G-frames for better understanding of the definition. The following example shows that the new definition of scalability is a generalization of the definition of scalability.

Example 3.1. Let $\{f_i\}_{i\in\mathbb{I}}$ be a scalable frame for $\mathcal{H}$ with the sequence of positive scalars $\{\alpha_i\}_{i\in\mathbb{I}}$. For each $i \in I$, set $D_i = [\alpha_i]$, where $[\alpha_i]$ is an one-by-one matrix. Then the G-frame $\{G_i\}$ where $G_i : \mathcal{H} \to \mathbb{C}$ defined by $G_if = \langle f, f_i\rangle$ for each $f \in \mathcal{H}$, is a scalable G-frame for $\mathcal{H}$ with respect to $\mathbb{C}$ with diagonal operators $\{D_i\}$, because

$$\sum_{i\in I} \|D_iG_if\|^2 = \sum_{i\in I} |\alpha_i\langle f, f_i\rangle|^2 = \|f\|^2 \quad \forall f \in \mathcal{H}.$$

Example 3.2. Let $\mathcal{H}$ be a separable Hilbert space and $\{\mathcal{H}_i\}$ be a sequence of separable Hilbert subspaces such that $\mathcal{H} = \bigoplus_i \mathcal{H}_i$. Also, let $\{f_{ij}\}_{j\in\mathbb{K}_i}$ be a scalable frame for $\mathcal{H}_i$. Then $\{G_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i)\}_{i\in I}$ such that $G_if = \sum_{j\in\mathbb{K}_i} \langle f, f_{i,j}\rangle f_{i,j}$ for each $f \in \mathcal{H}$, is a scalable G- frame.

Example 3.3. Let $\mathcal{H} = \mathbb{C}^3$, and $\mathcal{H}_1 = \mathcal{H}_2 = \mathcal{H}_3 = \mathbb{C}^2$. Define

$$G_1 = \begin{bmatrix} -1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \qquad G_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -1 & 0 \end{bmatrix} \qquad G_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

and

$$D_1 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \qquad D_2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \qquad D_3 = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} \end{bmatrix}.$$

Now, we have

$$\sum_{i=1}^{3} \|D_iG_if\|^2 = |z_1|^2 + |z_2|^2 + |z_3|^2 = \|f\|^2, \quad \forall f = (z_1, z_2, z_3) \in \mathbb{C}^3.$$

To answer the question that was first raised in the section, we present a theorem that states the necessary and sufficient conditions for G-framing with the help of a block diagonal operator.

Theorem 3.1. *Let $\mathcal{G} = \{G_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i)\}_{i\in I}$ be a G-frame for the Hilbert space $\mathcal{H}$ with respect to $\{\mathcal{H}_i\}_{i\in I}$ with analysis operator $T^*_{\mathcal{G}}$. Also, assume that $\{D_i \in L(\mathcal{H}_i)\}_{i\in I}$ is a sequence of non-negative diagonal operators. Then $H = \{D_iG_i\}_{i\in I}$ is a G-frame if and only if $rangT^*_{\mathcal{G}} \subseteq domD$, $D|_{rangT^*_{\mathcal{G}}}$ is ICR, where D is a bounded block diagonal operator on $(\bigoplus \mathcal{H}_i)_{l^2}$ such that $\{D_i\}_{i\in I}$ is as its diagonal.*

Proof. Let $H = \{D_iG_i\}_{i\in I}$ be a G-frame and $T^*_{\mathcal{G}}$ be analysis operator of $\mathcal{G} = \{G_i\}_{i\in I}$.

$$(T^*_{\mathcal{G}}f)_j = D_jG_jf = (DG_jf)_j$$

Thus, $T^*_H = DT^*_{\mathcal{G}}$. Since $domT^*_H = \mathcal{H}$ then $rangT^*_{\mathcal{G}} \subseteq domD$. Also, since $\mathcal{G}$ is a G-frame, then $rangT^*_{\mathcal{G}}$ is closed. On the other hand, due to H is a G-frame, there exist $A_1, B_1 > 0$ such that for each $f \in \mathcal{H}$

$$A_1\|f\|^2 \leq \|T^*_Hf\|^2_{l^2} \leq B_1\|f\|^2.$$

Set $\nu = T^*_{\mathcal{G}}f \in rangT^*_{\mathcal{G}}$ so,

$$\|D\nu\|^2 = \|DT^*_{\mathcal{G}}f\|^2 \geq A_1\|f\|^2 \geq A_1\|T^*_{\mathcal{G}}\|^{-2}\|\nu\|^2.$$

Thus $D|_{rangT^*_{\mathcal{G}}}$ is ICR.

Conversely, let $D|_{rangT^*_{\mathcal{G}}}$ be ICR and $rangT^*_{\mathcal{G}} \subset domD$. Due to the closed graph theorem, the operator $D|_r angT^*_{\mathcal{G}}$ is a bounded operator and since $D|_{rangT^*_{\mathcal{G}}}$ is ICR, there exist two constants $a, b > 0$ such that

$$a\|y\|^2 \leq \|Dy\|^2 \leq b\|y\|^2, \quad \forall y \in rangT^*_{\mathcal{G}}. \tag{3.1}$$

On the other hand, we know that $T^*_{\mathcal{G}}$ is bounded and ICR, thus there exist two constants $a', b' > 0$ such that

$$a'\|f\|^2 \leq \|T^*_{\mathcal{G}}f\|^2 \leq b'\|f\|^2, \quad \forall f \in \mathcal{H}. \tag{3.2}$$

By using (3.1) and (3.2) we have

$$aa'\|f\|^2 \leq \|DT^*_{\mathcal{G}}f\|^2 \leq bb'\|f\|^2, \quad \forall f \in \mathcal{H}.$$

Therefore H is a G-frame.

Also, due to the boundedness of $DT^*_{\mathcal{G}}$, it is proved that $(DT^*_{\mathcal{G}})^* = \overline{T_{\mathcal{G}}D}$, which completes the proof. □

The block diagonal operator is not necessarily bounded. In the next proposition, we state the necessary and sufficient conditions for this operator to be bounded.

Proposition 3.1. *Let $\mathcal{G} = \{G_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i)\}_{i\in I}$ be a G-frame for the Hilbert space $\mathcal{H}$ with respect to $\{\mathcal{H}_i\}_{i\in I}$ such that G_i's be bounded below and $liminf\|G_i\| > 0$. Also, assume that on $\{D_i \in L(\mathcal{H}_i)\}_{i\in I}$ is a sequence of non-negative diagonal operators. Then $\mathcal{F} = \{D_iG_i\}_{i\in I}$ is a G-frame if and only if the block diagonal* *operator D with D_i as blocks is bounded and $D|_{rangT^*_{\mathcal{G}}}$ is ICR. In this case $S_{\mathcal{F}} = T_{\mathcal{G}} D^2 T^*_{\mathcal{G}}$.*

Proof. Let $\mathcal{F} = \{D_i G_i\}_{i\in I}$ be a G-frame. Then $\mathcal{F} = \{D_i G_i\}_{i\in I}$ is G-Bessel. So, there exists a positive constant B such that for each $f \in \mathcal{H}$,

$$\sum_{i\in I} \|D_i G_i f\|^2 \leq B\|f\|^2. \tag{3.3}$$

Now,

$$\|G_i f\|^2 \leq \|D_i\|^{-2}\|D_i G_i f\|^2 \leq \|D_i\|^{-2} B\|f\|^2. \tag{3.4}$$

Since G_i's are bounded below so, for each $i \in I$, there exists a $A_i > 0$ such that $A_i\|f\|^2 \leq \|G_i f\|^2$ for each $f \in \mathcal{H}$. Also, due to $liminf\|G_i\| > 0$ there exists a $\delta > 0$ and $j \in I$ such that $A_i \geq \delta$ for each $i \geq j$. So, by (3.4), $\|D_i\| \leq (B\delta)^{1/2}$ for each $i \geq j$. Therefore, there exists an $M > 0$ such that for each i, $\|D_i\| \leq M$. That is $\|D\| < \infty$. By Theorem 3.1, the converse is clear. □

Using the results of Proposition 3.1 and Theorem 3.1, the following proposition is obtained.

Proposition 3.2. *Let $\mathcal{G} = \{G_i\}_{i\in I}$ be a G-frame with the assumptions of Proposition 3.1 and analysis operator $T^*_{\mathcal{G}}$. Then $\mathcal{G}$ is scalable G-frame if and only if there exists a bounded block diagonal operator D on $(\bigoplus \mathcal{H}_i)_{l^2}$ such that*

$$T_{\mathcal{G}} D^2 T^*_{\mathcal{G}} = I_{\mathcal{H}}.$$

Proof. Let $\mathcal{G} = \{G_i\}_{i\in I}$ be a scalable G-frame, so there exists a sequence $\{D_i \in L(\mathcal{H}_i)\}_{i\in I}$ of non-negative diagonal operators such that $\mathcal{F} = \{D_i G_i\}_{i\in I}$ is a Parseval G-frame.
We define D as a block diagonal operator on $(\bigoplus \mathcal{H}_i)_{l^2}$ which $\{D_i\}_{i\in I}$ is as its diagonal blocks. By Theorem 3.1 we have

$$rangT^*_{\mathcal{G}} \subseteq domD, S_{\mathcal{F}} = T_{\mathcal{G}} D^2 T^*_{\mathcal{G}}$$

Since $\mathcal{F}$ is a Parseval frame then $S_{\mathcal{F}} = I_{\mathcal{H}}$.

Conversely, assume that there exists a block diagonal operator D on $(\bigoplus \mathcal{H}_i)_{l^2}$ with blocks $\{D_i \in L(\mathcal{H}_i)\}_{i\in I}$ such that $T_{\mathcal{G}} D^2 T^*_{\mathcal{G}} = I_{\mathcal{H}}$. This concludes that $rangT^*_{\mathcal{G}} \subseteq domD$. Also, $DT^*_{\mathcal{G}}$ is closed operator from $\mathcal{H}$ into $(\bigoplus \mathcal{H}_i)_{l^2}$ due to D and $T^*_{\mathcal{G}}$ are bounded operators. This implies that $D|_{rangT^*_{\mathcal{G}}}$ is ICR.
Besides, since $T_{\mathcal{G}} D^2 T^*_{\mathcal{G}} = I_{\mathcal{H}}$ then $DT^*_{\mathcal{G}}$ is isometric. Now using by Proposition 3.1, we conclude that $\mathcal{F}$ is a Parseval G-frame. □

As we have seen [17], a frame induced by a G-frame inherits the desired properties of the G-frame. In the following, we will describe another of these desirable properties that inherits the frame induced by G-frame.

Theorem 3.2. *Let $\{G_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i)\}_{i\in I}$ be a G-frame for the Hilbert space $\mathcal{H}$ with respect to $\{\mathcal{H}_i\}_{i\in I}$ and $\{u_{i,j}\}_{i\in I, j\in\mathbb{K}_i}$ be the frame induced by the G-frame $\{G_i\}_{i\in I}$, then $\{G_i\}_{i\in I}$ is scalable G-frame if and only if $\{u_{i,j}\}_{i\in I, j\in\mathbb{K}_i}$ is scalable frame.*

Proof. Let $\{G_i\}_{i\in I}$ be a scalable G-frame. Then there exists a sequence of non-negative diagonal operators $\{D_i\}_{i\in I}$ such that $\{D_i G_i\}_{i\in I}$ is a Parseval frame. By definition of diagonal operators there exists $\{\alpha_{i,j}\}_{i\in I, j\in\mathbb{K}_i}$ of non-negative scalars

such that $D_i e_{i,j} = \alpha_{i,j} e_{i,j}$ for all $i \in I$ and $j \in \mathbb{K}_i$, where, for each $i \in I$, the sequence $\{e_{i,j}\}_{j\in\mathbb{K}_i}$ is an orthonormal basis for $\mathcal{H}_i$.
Thus

$$\| f \|^2 = \sum_{i\in I} \| D_i G_i f \|^2 = \sum_{i\in I} \sum_{j\in\mathbb{K}_i} |\langle f, \alpha_{i,j} u_{i,j}\rangle|^2.$$

That is the desired result.

For the converse, let $\{u_{i,j}\}_{i\in I, j\in\mathbb{K}_i}$ be a scalable frame for the Hilbert space $\mathcal{H}$, then there exists a sequence of non-negative scalars $\{\alpha_{i,j}\}_{i\in I, j\in\mathbb{K}_i}$ such that $\{\alpha_{i,j} u_{i,j}\}_{i\in I, j\in\mathbb{K}_i}$ is a Parseval frame for Hilbert space $\mathcal{H}$. Set $D_i e_{i,j} = \alpha_{i,j} e_{i,j};\ i \in I,\ \ j \in \mathbb{K}_i$, thus

$$\| f \|^2 = \sum_{i\in I} \| D_i G_i f \|^2 = \sum_{i\in I} \sum_{j\in\mathbb{K}_i} |\langle f, \alpha_{i,j} u_{i,j}\rangle|^2.$$

□

In analogy to Theorem 2.1, we prove an equivalent condition to show that a G-frame is not scalable G-frame.

Proposition 3.3. *Let $\mathcal{G} = \{G_i \in \mathcal{L}(\mathbb{R}^N, \mathbb{R})\}_{i=1}^M$ be a G-frame for the Hilbert space $\mathbb{R}^N$ with respect to $\mathbb{R}$. Then $\mathcal{G}$ is not a scalable G-frame if and only if there exists a symmetric matrix Y on $\mathbb{R}^{N\times N}$ such that $tr(Y) < 0$ and $[G_i^* e_{ij}]^T Y [G_i^* e_{ij}] \geq 0$, where $[G_i^* e_{ij}]^T$ denotes the transpose of the matrix $[G_i^* e_{ij}]$.*

Proof. $\{G_i\}_{i=1}^M$ is not a scalable G-frame if and only if the frame $\{u_{ij}\}$ is not scalable frame for $\mathbb{R}^N$. By Theorem 2.1 there exists a symmetric matrix Y on $\mathbb{R}^{N\times N}$ such that $tr(Y) < 0$ and $[u_{ij}]^T Y [u_{ij}] \geq 0$. This means there exists a symmetric matrix Y on $\mathbb{R}^{N\times N}$ such that $tr(Y) < 0$ and $[G_i^* e_{ij}]^T Y [G_i^* e_{ij}] \geq 0$. □

4. New structures of scalable G-frames

In this section, we will describe new structures for scalable G-frames.

Proposition 4.1. *Let $\{G_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i)\}_{i\in I}$ be a scalable G-frame for the Hilbert space $\mathcal{H}$ with respect to $\{\mathcal{H}_i\}_{i\in I}$ and $U : \mathcal{H} \to \mathcal{H}$ be a bounded linear operator then $\{G_i U\}_{i\in I}$ is scalable G-frame if and only if $U : \mathcal{H} \to \mathcal{H}$ is a unitary operator.*

Proof. Since $\{G_i\}_{i\in I}$ is a scalable G-frame, then there exists a sequence $\{D_i \in L(\mathcal{H}_i)\}_{i\in I}$ of non-negative diagonal operators such that $\{D_i G_i\}_{i\in I}$ is Parseval G-frame.
Therefore,

$$\sum_{i\in I} \| D_i G_i U f \|^2 = \| U f \|^2 .$$

Thus, $\{G_i U\}_{i\in I}$ is scalable G-frame if and only if $\| Uf \| = \| f \|$. This completes the proof. □

Let $\mathcal{H}$ and $\mathcal{K}$ be two Hilbert spaces. The direct sum of Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$ is denoted by $\mathcal{H} \oplus \mathcal{K} = \{(h, k) : h \in \mathcal{H}, k \in \mathcal{K}\}$, which is a Hilbert space with pointwise operations and inner product

$$\langle (f, g), (f', g') \rangle = \langle f, f' \rangle_{\mathcal{H}} + \langle g, g' \rangle_{\mathcal{K}}, \quad \forall f, f' \in \mathcal{H}, \quad \forall g, g' \in \mathcal{K}.$$

If U and W are Hilbert spaces and $G \in \mathcal{L}(\mathcal{H}, U), T \in \mathcal{L}(\mathcal{K}, W)$, we recall that

$$G \oplus T \in \mathcal{L}(\mathcal{H} \oplus \mathcal{K}, U \oplus W) \ \ by \ \ G \oplus T(h,k) = (Gh, Tk) \ \ \forall h \in \mathcal{H}, \quad k \in \mathcal{K}.$$

Refer to [9, 12] for more information.

Theorem 4.1. *Let $\{G_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i)\}_{i\in I}$ be a scalable G-frame for $\mathcal{H}$ with respect to $\{\mathcal{H}_i\}_{i\in I}$ and $\{G'_i \in \mathcal{L}(\mathcal{K}, \mathcal{K}_i)\}_{i\in I}$ be a scalable G-frame for $\mathcal{K}$ with respect to $\{\mathcal{K}_i\}_{i\in I}$. Then $\{G_i \oplus G'_i \in \mathcal{L}(\mathcal{H} \oplus \mathcal{K}, \mathcal{H}_i \oplus \mathcal{K}_i)\}_{i\in I}$ is scalable G-frame.*

Proof. Since $\{G_i\}_{i\in I}$ is a scalable G-frame, there exists a sequence $\{D_i \in L(\mathcal{H}_i)\}_{i\in I}$ of non-negative diagonal operators such that $\{D_iG_i\}_{i\in I}$ is a Parseval G-frame. Also, there exists a sequence $\{D'_i \in L(\mathcal{K}_i)\}_{i\in I}$ of non-negative diagonal operators such that $\{D'_iG'_i\}_{i\in I}$ is a Parseval G-frame.
Now, set $S_i = \begin{bmatrix} D_i & 0 \\ 0 & D'_i \end{bmatrix}, i \in I$. The sequence $\{S_i \in L(\mathcal{H}_i \oplus \mathcal{K}_i)\}_{i\in I}$ is a sequence of non-negative diagonal operators on $\mathcal{H}_i \oplus \mathcal{K}_i$.
Therefore

$$\sum_{i\in I} \|S_iG_i \oplus G'_i(f,g)\|^2 = \sum_{i\in I} \|D_iG_i(f)\|^2 + \sum_{i\in I} \|D'_iG'_i(g)\|^2$$

$$= \|f\|^2 + \|g\|^2 = \|(f,g)\|^2 \quad \forall (f,g) \in \mathcal{H} \oplus \mathcal{K}.$$

□

Corollary 4.1. *Let $\{G_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i)\}_{i\in I}$ be scalable G-frame for $\mathcal{H}$ with respect to$\{\mathcal{H}_i\}_{i\in I}$ and $\{G'_i \in \mathcal{L}(\mathcal{K}, \mathcal{K}_i)\}_{i\in I}$ be A-tight G-frame for $\mathcal{K}$ with respect to $\{\mathcal{K}_i\}_{i\in I}$. Then $\{G_i \oplus G'_i \in \mathcal{L}(\mathcal{H} \oplus \mathcal{K}, \mathcal{H}_i \oplus \mathcal{K}_i)\}_{i\in I}$ is a scalable G-frame.*

Theorem 4.2. *Let $\{G_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i)\}_{i\in I}$ be a G-frame for $\mathcal{H}$ with respect to $\{\mathcal{H}_i\}_{i\in I}$ and for each $i \in I$, $\{T_{i,j} \in \mathcal{L}(\mathcal{H}_i, \mathcal{K}_{i,j})\}_{j\in J}$ be a scalable G-frame for $\mathcal{H}_i$ with respect to $\{\mathcal{K}_{i,j}\}_{j\in J}$ by a sequence of non-negative diagonal operators $\{D_{i,j} \in L(\mathcal{K}_{i,j})\}_{j\in J}$. Then $\{T_{i,j}G_i\}_{i\in I, j\in J}$ is a scalable G-frame for $\mathcal{H}$ with respect to $\{\mathcal{K}_{i,j}\}_{i\in I, j\in J}$ by the sequence $\{D_{i,j} \in L(\mathcal{K}_{i,j})\}_{i\in I, j\in J}$ if and only if $\{G_i\}_{i\in I}$ is a Parseval G-frame.*

Proof. Let $i \in I$. Under the above assumptions, $\{T_{i,j}\}_{j\in J}$ is a scalable G-frame for $\mathcal{H}_i$ with respect to $\{\mathcal{K}_{i,j}\}_{j\in J}$ by a sequence of non-negative diagonal operators $\{D_{i,j} \in L(\mathcal{K}_{i,j})\}_{j\in J}$. Then

$$\|f\|^2 = \sum_{j\in J} \|D_{i,j}T_{i,j}f\|^2 \quad \forall f \in \mathcal{K}_{i,j}. \tag{4.1}$$

Now, assume that $\{T_{ij}G_i\}_{i\in I, j\in J}$ is a scalable G-frame for $\mathcal{H}$ with respect to $\{\mathcal{K}_{ij}\}_{i\in I, j\in J}$ and the sequence $\{D_{i,j}\}_{i\in I, j\in J}$ of non-negative diagonal operators, for which the G-frame $\{D_{i,j}T_{i,j}G_i\}_{i\in I, j\in J}$ is Parseval G-frame. Hence, by (4.1)

$$\|f\|^2 = \sum_{i\in I, j\in J} \|D_{i,j}T_{i,j}G_if\|^2 = \sum_{i\in I} \|G_if\|^2 \quad \forall f \in \mathcal{H}.$$

Therefore $\{G_i\}_{i\in I}$ is Parseval G-frame.

Conversely, assume that $\{G_i\}_{i\in I}$ is Parseval G-frame. For each $i \in I$, $\{T_{i,j}G_i\}_{j\in J}$ is a scalable G-frame by a sequence of non-negative diagonal operators $\{D_{i,j} \in L(\mathcal{K}_{i,j})\}_{j\in J}$ for $\mathcal{H}_i$ with respect to $\{\mathcal{K}_{i,j}\}_{j\in J}$; thus for all $f \in \mathcal{H}$,

$$\sum_{i\in I, j\in J} \|D_{i,j}T_{i,j}G_i f\|^2 = \sum_{i\in I} \|G_i f\|^2 = \|f\|^2.$$

This completes the proof. □

Using the previous theorem, we can say that if any Hilbert space $\mathcal{H}_i$ has a scalable frame, then there exists a scalable G-frame for $\mathcal{H}$ with respect to $\{\mathcal{H}_i\}_{i\in I}$.

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