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GLOBAL BIFURCATION FROM INTERVALS IN NONLINEAR STURM-LIOUVILLE PROBLEM WITH INDEFINITE WEIGHT FUNCTION

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Abstract. We consider the nonlinear Sturm-Liouville problem with indefinite weight and not necessarily differentiable nonlinear term. This problem arise from a selection-migration model in population genetics. The existence of four families of unbounded continua of solutions of this problem having the usual nodal properties and emanating from the bifurcation intervals corresponding to the negative and positive eigenvalues of the linear problem, obtained from nonlinear problem by setting the nonlinear term equal to zero, is proved.

1. Introduction

We consider the nonlinear Sturm-Liouville problem

$$\ell(y) \equiv -(p\,(x)\,y')' + q(x)y = \lambda\rho(x)\,y + H(x, y, y', \lambda),\ x \in (0, 1), \tag{1.1}$$

$$\alpha_0 y(0) - \beta_0 y'(0) = 0, \tag{1.2}$$

$$\alpha_1 y(1) + \beta_1 y'(1) = 0, \tag{1.3}$$

where $\lambda \in \mathbb{R}$ is a spectral parameter, the functions $p \in C^1[0, 1]$, $q,\ \rho \in C[0, 1]$, p is strictly positive and q is nonnegative on $[0, 1]$, and $\rho(x)$ is real-valued on $[0, 1]$ that take on positive as well as negative values, α_i, β_i, $i = 0,\ 1$, are real constants such that $|\alpha_i| + |\beta_i| > 0$ and $\alpha_i\beta_i \geq 0$, $i = 0,\ 1$. The nonlinear term H is of the form $H = f + g$, where f and g are continuous functions on $[0, 1] \times \mathbb{R}^3$ that satisfy the following conditions:

$$uf(x, u, s, \lambda) \leq 0,\ ug(x, u, s, \lambda) \leq 0,\ (x, u, s, \lambda) \in [0, 1] \times \mathbb{R}^3; \tag{1.4}$$

there is a positive constant M such that

$$|f(x, u, s, \lambda)/u\,| \leq M,\ (x, u, s, \lambda) \in [0, 1] \times \mathbb{R}^3,\ 0 < |u| \leq 1,\ |s| \leq 1; \tag{1.5}$$

$$g(x, u, s, \lambda) = o\,(|u| + |s|) \text{ as } |u| + |s| \to 0, \tag{1.6}$$

uniformly with respect to $(x, \lambda) \in [0, 1] \times \Lambda$, for any bounded interval $\Lambda \subset \mathbb{R}$.

Nonlinear eigenvalue problems with indefinite weight arise from a selection-migration model in population genetics (see, e.g. [12-14, 17, 23]). Problem (1.1)-(1.3) for $p(x) \equiv 1$, $H(x, u, s, \lambda) = \lambda\rho(x)(F(u) - u)$, where $F(u) = u(1 -$

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$u)\,\{h(1-u)+(1-h)\,u\}$ for some constant h, $0<h<1$, describes the effects of population dispersal within a bounded habitat and selective advantages among the possible genotypes. In this model, weight function $\rho(x)$ that changes sign is a selection coefficient in the habitat, so that a selective advantage at some points of the habitat becomes a disadvantage at others [13, 14].

Global bifurcation in nonlinear Sturm-Liouville problem (1.1)-(1.3) with $\rho>0$ have been considered before in [7, 10, 19, 21, 22] without conditions $q(x)\geq 0$, $x\in[0,1]$, $\alpha_i\,\beta_i\geq 0$, $i=0,\,1$, and condition (1.4). These papers show the existence of unbounded continua of nontrivial solutions in $\mathbb{R}\times C^1[0,1]$ that having the usual nodal properties and emanating from bifurcation points and intervals of the line of trivial solutions corresponding to the eigenvalues of the linear problem which obtained from (1.1)-(1.3) by setting $H\equiv 0$. Note that similar results in nonlinear eigenvalue problems for fourth-order ordinary differential equations, elliptic partial differential equations, and Dirac systems were obtained in [1, 5, 7, 19, 20].

Problem (1.1)-(1.3) was considered in [6] for $f\equiv 0$, and in [4, 18] for $g\equiv 0$. These papers prove the existence of four families of global continua of nontrivial solutions in $\mathbb{R}\times C^1[0,1]$ which possesses usual nodal properties and bifurcating from points (for $f\equiv 0$) and intervals (for $g\equiv 0$) of the line of trivial solutions. It should be noted that in [4] the global bifurcation of solutions from intervals corresponding only to the principal eigenvalues was studied. In [10], it was possible to study the global bifurcation of solutions of (1.1)-(1.3) with $g\equiv 0$ from intervals corresponding to all eigenvalues, but it was not possible to accurately estimate the lengths of these intervals. Similar results in nonlinear eigenvalue problems for fourth-order ordinary differential equations and elliptic partial differential equations with indefinite weights were obtained in [2, 3, 8, 9, 11, 12, 14, 15, 17, 23, 24]. In these papers bifurcation of nontrivial solutions were studied from trivial solutions that correspond to the principal eigenvalues of the corresponding linear problems.

In this paper, we are able to estimate the length of the bifurcation intervals corresponding to all eigenvalues of the linear problem obtained from (1.1)-(1.3) by setting $H\equiv 0$ and completely study the unilateral global bifurcation of solutions of problem (1.1)-(1.3).

2. Preliminary

As is known [16, Ch. 10], the eigenvalues of the linear spectral problem

$$\begin{cases} -(p(x)y'(x))'+q(x)y(x)=\lambda\rho(x)y(x),\ x\in(0,\ 1), \\ y\in B.C., \end{cases} \tag{2.1}$$

where $B.C.$ is the set of functions that satisfy boundary conditions (1.2)-(1.3), are all real and simple, and consist of two unboundedly decreasing and unboundedly increasing sequences $\{\lambda_k^-\}_{k=1}^\infty$ and $\{\lambda_k^+\}_{k=1}^\infty$, respectively, such that $\lambda_1^-<0$ and $\lambda_1^+>0$. Moreover, for each $k\in\mathbb{N}$ the eigenfunction $y_k^-(x)$ ($y_k^+(x)$), corresponding to eigenvalue λ_k^- (λ_k^+), has exactly $k-1$ simple zeros in $(0,1)$.

Remark 2.1. Since the class of continuous functions $C[0,1]$ is dense in $L^\infty[0,1]$, the above statements for problem (2.1) also hold for $q\in L^\infty[0,1]$.

Multiplying both sides of (2.1) by $y(x)$ and integrating the result with respect to x from 0 to 1, using integration by parts, and taking boundary condition (1.2) and (1.3) into account we obtain

$$\int_0^1 \left\{p\,(x)y'^2(x) + q(x)y^2(x)\right\} dx + N[y] = \lambda \int_0^1 \rho(x)y^2(x)dx, \tag{2.2}$$

where

$$N[y] = -\,p(1)y'(1)y(1) + p(0)y'(0)y(0) \geq 0, \tag{2.3}$$

in view of conditions $\alpha_i\beta_i \geq 0$, $i = 0,\, 1$.

Setting $\lambda = \lambda_k^\sigma$ and $y = y_k^\sigma(x)$, $k \in \mathbb{N}$, $\sigma \in \{-,\, +\}$, in relation (2.2) we get

$$\int_0^1 \{p\,(x)\left((y_k^\sigma(x))'\right)^2 + q(x)\,(y_k^\sigma(x))^2\}dx + N[y] = \lambda_k^\sigma \int_0^1 \rho(x)\,(y_k^\sigma(x))^2\,dx.$$

Since $\sigma\lambda_k^\sigma > 0$ for $k \in \mathbb{N}$, $\sigma \in \{-,\, +\}$, it follows from last equality that

$$\sigma \int_0^1 \rho(x)(y_k^\sigma(x))^2(x)\,dx > 0.$$

Let E be the Banach space $E = C^1[0,1] \cap B.C.$ with the usual norm $||u||_1 = ||u||_\infty + ||u'||_\infty$, where $||u||_\infty = \max\limits_{x\in[0,1]} |u(x)|$. For each $k \in \mathbb{N}$, each $\sigma \in \{-\,,\,+\}$ and each $\nu \in \{-\,,\,+\}$ by $S_k^{\sigma,\,\nu}$ we denote be the set of functions $u \in E$ satisfying the following conditions:

1. $u(x)$ has exactly $k-1$ simple zeros in $(0,1)$;
2. $\sigma \int\limits_0^1 \rho(x)u^2(x)\,dx > 0$;
3. $\lim\limits_{x\to 0+} \nu\, \operatorname{sgn} u(x) \to 1$.

The sets $S_k^{-,\,-}$, $S_k^{-,\,+}$, $S_k^{+,\,-}$ and $S_k^{+,\,+}$ are open subsets of E, and $S_{k'}^{\sigma',\,\nu'} \cap S_k^{\sigma,\,\nu} = \emptyset$ for $(k', \sigma', \nu') \neq (k, \sigma, \nu)$. Moreover, if $y \in \partial S_k^{\sigma,\,\nu}$, then either there exists $\tau \in [0,1]$ such that $y(\tau) = y'(\tau) = 0$, or $\int\limits_0^1 \rho(x)\,y^2(x)\,dx = 0$.

Due to the presence of the nonlinear term f, the set of bifurcation points to problem (1.1)-(1.3) through the set $S_k^{\sigma,\,\nu}$, $k \in \mathbb{N}$, $\sigma,\, \nu \in \{-\,,\,+\}$, with respect to the line of trivial solutions should be contained in intervals surrounding trivial solutions $(\lambda_k^\sigma, 0)$ of this problem (see, e.g [1, 7, 21]).

To clarify the lengths of bifurcation intervals, we need to consider the following linear Sturm-Liouville problems

$$\begin{cases} \ell(y)(x) - \lambda\rho(x)y(x) = \mu y(x), \;\; x \in (0,1), \\ \quad y \in B.C. \end{cases} \tag{2.4}$$

$$\begin{cases} \ell(y)(x) + \psi(x)y(x) = \lambda\rho(x)y(x), \;\; x \in (0,1), \\ \; y \in B.C., \end{cases} \tag{2.5}$$

$$\begin{cases} \ell(y)(x) + \psi(x)y(x) - \lambda\rho(x)y(x) = \mu y(x), \;\; x \in (0,1), \\ \; y \in B.C., \end{cases} \tag{2.6}$$

where $\psi \in C[0,1]$ and $\psi(x) \geq 0$ for $x \in [0,1]$.

It is known that for each fixed $\lambda \in \mathbb{R}$ the spectrum of problem (2.4) consists of an infinitely increasing sequence of real simple eigenvalues

$$\mu_1(\lambda) < \mu_2(\lambda) < \ldots < \mu_k(\lambda) < \ldots ;$$

the eigenfunction $y_k(x, \lambda)$ that correspond to the eigenvalue $\mu_k(\lambda)$, $k \in \mathbb{N}$, has exactly $k-1$ simple zeros in the interval $(0,1)$. We also note that the eigenvalues λ_k^- and λ_k^+, $k \in \mathbb{N}$, of (2.1) are zeros of the function $\mu_k(\lambda)$.

Let $\lambda_{k,\psi}^-$ and $\lambda_{k,\psi}^+$ be respectively the k-th negative and positive eigenvalues of problem (2.5), and let $\mu_{k,\psi}(\lambda)$, $\lambda \in \mathbb{R}$ be the k-th eigenvalue of problem (2.6). It is obvious that the eigenvalues $\lambda_{k,\psi}^-$ and $\lambda_{k,\psi}^+$, $k \in \mathbb{N}$, of (2.5) are zeros of the function $\mu_{k,\psi}(\lambda)$.

Lemma 2.1 [4, Lemma 1]. *For each $k \in \mathbb{N}$ the following relations hold:*

$$\begin{aligned} &\mu_k(\lambda) \in C^\infty(\mathbb{R}), \quad \frac{d\mu_k(\lambda)}{d\lambda} = -\frac{\int_0^1 \rho(x) y_k^2(x,\lambda)\,dx}{\int_0^1 y_k^2(x,\lambda)\,dx}, \ \lambda \in \mathbb{R}; \\ &\mu_{k,\psi}(\lambda) \in C^\infty(\mathbb{R}), \quad \frac{d\mu_{k,\psi}(\lambda)}{d\lambda} = -\frac{\int_0^1 \rho(x) y_{k,\psi}^2(x,\lambda)\,dx}{\int_0^1 y_k^2(x,\lambda)\,dx}, \ \lambda \in \mathbb{R}, \end{aligned} \tag{2.7}$$

where $y_{k,\psi}(x,\lambda)$ is an eigenfunction corresponding to the eigenvalue $\mu_{k,\psi}(\lambda)$.

Lemma 2.2 [4, Lemma 2]. *The functions $\lambda \to \mu_1(\lambda)$ and $\lambda \to \mu_{1,\psi}(\lambda)$ are concave on $\mathbb{R}$.*

Remark 2.2. The question of the validity of the statements of Lemma 2.2 for the function $\mu_k(\lambda)$ and $\mu_{k,\psi}(\lambda)$ for $k \geq 2$ remains open.

Lemma 2.3 [4, formula (3.3)] *For each $k \in \mathbb{N}$ and each $\lambda \in \mathbb{R}$ the following relation holds:*

$$0 \leq \mu_{k,\psi}(\lambda) - \mu_k(\lambda) \leq K, \tag{2.8}$$

where $K = \sup\{\,\psi(x) : x \in [0,1]\,\}$.

Remark 2.3. By Lemma 2.3 it follows from (2.7) that

$$\lambda_{k,\psi}^- \leq \lambda_k^- < 0 \ \text{ and } \ 0 < \lambda_k^+ \leq \lambda_{k,\psi}^+.$$

Lemma 2.4 [4, Lemma 3] *For each $\sigma \in \{+,-\}$ the following formulas holds:*

$$|\lambda_{1,\psi}^\sigma - \lambda_1^\sigma| \leq \frac{\sigma K \int_0^1 (y_1^\sigma(x))^2\,dx}{\int_0^1 \rho(x)\,(y_1^\sigma(x))^2\,dx}. \tag{2.9}$$

Remark 2.4. By Remark 2.2 we cannot state that estimate (2.9) holds for the eigenvalues λ_k^σ and $\lambda_{k,\psi}^\sigma$ for $k \geq 2$.

3. Estimates of the distances between the corresponding eigenvalues of problems (2.1) and (2.4)

For brevity, we introduce the notations

$$\tilde{a}_k^\sigma = \left|\frac{d\mu_k(\lambda_k^\sigma)}{d\lambda}\right| = \frac{\sigma \int\limits_0^1 (y_k^\sigma(x))^2\, dx}{\int\limits_0^1 \rho(x) \left(y_k^\sigma(x)\right)^2\, dx}\,, \; a_k^\sigma = \tilde{a}_k^\sigma + \frac{1}{\tilde{a}_k^\sigma},\; k\in\mathbb{N},\; \sigma\in\{-\,,+\}.$$

Lemma 3.1 *For each $k\in\mathbb{N}$ and each $\sigma\in\{-\,,+\}$ the following estimate holds:*

$$\operatorname{dist}\{\lambda_{k,\psi}^\sigma, \lambda_k^\sigma\} \;\le K a_k^\sigma. \tag{3.1}$$

Proof. It follows from (2.8) that (3.1) holds for $k=1$. We will prove the theorem for $k\ge 2$ and $\sigma=+$ (the case of $\sigma=-$ is considered similarly).

It follows from Lemma 2.1 that $\mu_k(\lambda)$ and $\mu_{k,\psi}(\lambda)$ are decreasing functions in $(0,+\infty)$. Moreover, $\mu_k(\lambda_k^+)=0$ and $\mu_{k,\psi}(\lambda_{k,\psi}^+)=0$. By Remark 2.3 we have $0<\lambda_k^+\le\lambda_{k,\psi}^+$.

Let $A=(\lambda_k^+,0)$, $B=(\lambda_{k,\psi}^+,0)$, $C=(\lambda_{k,\psi}^+,\,\mu_k(\lambda_{k,\psi}^+))$, and let ℓ_k be the tangent to the graph of the function $\mu_k(\lambda)$ at the point A, γ_k be the angle formed by the line ℓ_k^+ and the positive direction of the axis $O\lambda$, and $\tilde{\ell}_{k,\psi}^+$ be the line parallel to the line ℓ_k^+ passing through the point B. It follows from the first relation of (2.7) that $\gamma_k^+\in\left(\frac{\pi}{2},\pi\right)$ and $\tilde{a}_k^+=-\tan\gamma_k$.

Consider two possible cases.

Case 1. Let $\ell_k^+\cap[BC]\ne\emptyset$, where $[BC]$ is a closed interval connecting points B and C (this case takes place for $k=1$ since the function $\mu_1(\lambda)$ is concave on $\mathbb{R}$). By virtue of (2.8) we have $|BC|\le K$, where $|BC|$ is the length of the closed interval $[BC]$.

Let $D=\ell_k^+\cap[BC]$. Then

$$|BD| = |AB|\tan(\pi-\gamma_k^+) = -|AB|\,\tan\gamma_k^+ = |AB|\,\tilde{a}_k^+,$$

which implies that

$$0\le\lambda_{k,\psi}^+-\lambda_k^+ = |AB| = \frac{|BD|}{\tilde{a}_k^+}\le\frac{|BC|}{\tilde{a}_k^+}\le\frac{K}{\tilde{a}_k^+}\le K\,a_k^+.$$

Case 2. Let $\ell_k\cap[BC]=\emptyset$. In this case let $\hat{\ell}_{k,\psi}^+$ be a line parallel to the axis $O\lambda$ passing through the point C. We denote: $E=\hat{\ell}_{k,\psi}^+\cap\ell_k^+$ and $F=\hat{\ell}_{k,\psi}^+\cap\tilde{\ell}_{k,\psi}^+$.

It is obvious that $\angle CBF=\gamma_k-\frac{\pi}{2}$, and consequently,

$$|\,CF| = |BC|\tan\angle CBF = |BC|\tan\left(\gamma_k^+-\frac{\pi}{2}\right) = -|BC|\,\tan^{-1}\gamma_k^+\le\frac{K}{\tilde{a}_k^+}. \tag{3.2}$$

Let $G\in[AE]$ such that $[BG]\perp[AE]$, and let $\delta_k^+=\angle EBC$. Note that

$$0<\delta_k^+\le\angle GBC=\angle GBF-\angle CBF=\frac{\pi}{2}-\gamma_k^+-\frac{\pi}{2}=\pi-\gamma_k^+\in\left(0,\frac{\pi}{2}\right).$$

Then from the right triangle $\triangle EBC$ we get

$$|EC| = |BC| \tan \delta_k^+ \leq |BC| \tan(\pi - \gamma_k^+) \leq -K \tan \gamma_k^+ = K\, \tilde{a}_k^+. \tag{3.3}$$

Therefore, it follows from (3.2) and (3.3) that

$$0 \leq \lambda_{k,\psi}^+ - \lambda_k^+ = |AB| = |EF| = |EC| + |\,CF| \leq K\, \tilde{a}_k^+ + \frac{K}{\tilde{a}_k^+} = K\, a_k^+.$$

The proof of this lemma is complete.

4. Existence and structure of bifurcation points of problem (1.1)-(1.3) through the set $S_k^{\sigma,\,\nu}$

We define the norm in $\mathbb{R} \times E$ by $||(\lambda, y)|| = \left\{|\lambda|^2 + ||y||_1^2\right\}^{\frac{1}{2}}$.

We say that $(\lambda, 0)$ is a bifurcation point of problem (1.1)-(1.3) with respect to the set $\mathbb{R} \times S_k^{\sigma,\,\nu}$, if in every small neighborhood of this point there is a solution to problem (1.1)-(1.3) which is contained in $\mathbb{R} \times S_k^{\sigma,\,\nu}$ (see [1]).

Remark 4.1. The nonlinear eigenvalue problem (1.1)-(1.3) cannot have a nontrivial solution of the form $(0, y) \in \mathbb{R} \times E$. Indeed, multiplying both sides of (1.1) for $\lambda = 0$ by $y(x)$ and integrating the result from 0 to 1, using the formula for the integration by parts, and taking boundary condition (1.2) and (1.3) into account we get

$$\begin{aligned} \int_0^1 \left\{p\,(x)\, y'^2(x) + q(x)\, y^2(x)\right\} dx + N[y] = \int_0^1 f(x, y(x), y'(x), \lambda)\, y(x)\, dx + \\ \int_0^1 g(x, y(x), y'(x), \lambda)\, y(x)\, dx, \end{aligned} \tag{4.1}$$

By (2.3) the left hand side of (4.1) is positive, and by (1.4) the right hand side of (4.1) is non-positive, a contradiction.

Remark 4.2. Let $(\lambda, y) \in \mathbb{R} \times E$ be a solution of (1.1)-(1.3) such that $y \in \partial S_k^{\sigma,\,\nu}$. Then by following the arguments in Lemma 4 of [4] with the use of conditions (1.4)-(1.6) we can show that $y \equiv 0$.

In the case of $f \equiv 0$ for (1.1)-(1.3) we have the following global bifurcation result.

Theorem 4.1. *For each $k \in \mathbb{N}$, each $\sigma \in \{-\,,+\}$ and each $\nu \in \{-\,,+\}$ there is a continuum $C_k^{\sigma,\,\nu}$ of solutions of problem* (1.1)-(1.3) *with $f \equiv 0$ that contain $(\lambda_k^\sigma, 0)$, lies in $(\mathbb{R}^\sigma \times S_k^{\sigma,\,\nu}) \cup \{(\lambda_k^\sigma, 0)\}$, and is unbounded in $\mathbb{R}^\sigma \times E$, where $\mathbb{R}^\sigma = \{z \in \mathbb{R} : 0 < \sigma z < +\infty\}$.*

The proof of this theorem is similar to that of [19, Theorem 2.3] with the use of Remarks 4.1 and 4.2.

We introduce the following notation:

$$b_k^\sigma = M a_k^\sigma,\ J_k^- = [\lambda_k^- - b_k^-, \lambda_k^-],\ J_k^+ = [\lambda_k^+, \lambda_k^+ + b_k^+],\, k \in \mathbb{N},\, \sigma \in \{-\,,+\}.$$

For each $k \in \mathbb{N}$, each $\sigma \in \{-\,,+\}$ and each $\nu \in \{-\,,+\}$ let $B_k^{\sigma,\,\nu}$ be the set of bifurcation points of problem (1.1)-(1.3) with respect to the set $\mathbb{R} \times S_k^{\sigma,\,\nu}$. The following theorems shows that these sets are not empty and in what intervals they are contained.

Theorem 4.2. *$B_k^{\sigma,\nu} \neq \emptyset$ for each $k \in \mathbb{N}$, each $\sigma \in \{-\,,+\}$ and each $\nu \in \{-\,,+\}$.*

Proof. For the proof of the theorem we will use the approximate problem

$$\begin{aligned} &\ell(y) = \lambda\rho(x)y + f(x, y|\,y|^{\varepsilon}, y', \lambda) + g(x, y, y', \lambda),\ \varepsilon \in (0,1),\ x \in (0,1),\\ &y \in B.C. \end{aligned} \tag{4.2}$$

By (1.5) we have

$$f(x, u|u|^{\varepsilon}, s, \lambda) = o(|u| + |s|) \text{ as } |u| + |s| \to 0$$

in the uniform sense of condition (1.6). Hence Theorem 4.1 imply that for each $k \in \mathbb{N}$, each $\sigma \in \{+\,,-\}$, each $\nu \in \{+\,,-\}$, and every $\varepsilon \in (0,1)$ there is an unbounded component $D_{k,\varepsilon}^{\sigma,\nu}$ of solutions of problem (4.2) such that

$$(\lambda_k^{\sigma}, 0) \in C_{k,\varepsilon}^{\sigma,\nu} \subset (\mathbb{R}^{\sigma} \times S_k^{\sigma,\nu}) \cup \{(\lambda_k^{\sigma}, 0)\}. \tag{4.3}$$

We now choose some fixed arbitrary $k_0 \in \mathbb{N}$, $\sigma_0 \in \{-\,,+\}$ and $\nu_0 \in \{-\,,+\}$, and we will prove the theorem for $k = k_0$, $\sigma = \sigma_0$ and $\nu = \nu_0$.

By (4.3) for every $\varepsilon \in (0,1)$ and each sufficiently small $\tau \in (0,1)$ there is a solution $(\lambda_{\varepsilon,\tau,0}, y_{\varepsilon,\tau,0}) = (\lambda_{k_0,\varepsilon,\tau}^{\sigma_0,\nu_0}, y_{k_0,\varepsilon,\tau}^{\sigma_0,\nu_0}) \in \mathbb{R}^{\sigma_0} \times E$ of problem (4.3) such that

$$y_{\varepsilon,\tau,0} \in S_{k_0}^{\sigma_0,\nu_0} \text{ and } ||\,y_{\varepsilon,\tau,0}||_1 = \tau. \tag{4.4}$$

It is obvious that $(\lambda_{\varepsilon,\tau,0}, y_{\varepsilon,\tau,0})$ solves the following linearizable Sturm-Liouville problem

$$\begin{aligned} &\ell y + \psi_{\varepsilon,\tau,0}(x)y = \lambda\rho(x)y + g(x, y, y, \lambda),\ x \in (0,1),\\ &y \in B.C., \end{aligned} \tag{4.5}$$

where

$$\psi_{\varepsilon,\tau,0}(x) = \begin{cases} -\dfrac{f(x, y_{\varepsilon,\tau,0}(x)\,|y_{\varepsilon,\tau,0}(x)|^{\varepsilon}, y'_{\varepsilon,\tau,0}(x), \lambda_{\varepsilon,\tau,0})}{y_{\varepsilon,\tau,0}(x)} & \text{if } y_{\varepsilon,\tau,0}(x) \neq 0,\\ 0, & \text{if } y_{\varepsilon,\tau,0}(x) = 0, \end{cases} \quad x \in [0,1]. \tag{4.6}$$

Since $\tau \in (0,1)$, by virtue of (1.4) and (1.5), it follows from (4.6) that

$$0 \le \psi_{\varepsilon,\tau,0}(x) \le M|\,y_{\varepsilon,\tau,0}(x)|^{\varepsilon} \le M,\ x \in [0,1]. \tag{4.7}$$

The linearization of (4.5) at $y = 0$ is given by

$$\begin{aligned} &\ell y + \psi_{\varepsilon,\tau,0}(x)y = \lambda\rho(x)y,\ x \in (0,1),\\ &y \in B.C. \end{aligned} \tag{4.8}$$

In view of (4.7), by Remarks 2.1, 2.3 and relation (3.1) we get

$$\lambda_{k,\psi_{\varepsilon,\tau,0}}^{\sigma} \in J_k^{\sigma}, \tag{4.9}$$

where $\lambda_{k,\psi_{\varepsilon,\tau,0}}^{-}$ and $\lambda_{k,\psi_{\varepsilon,\tau,0}}^{+}$ are the k-th negative and positive eigenvalues of the linear problem (4.8) respectively. By (4.4) it follows from Theorem 4.1 that we can choose a sufficiently small $\varrho_{\varepsilon,\tau} > 0$ such that

$$|\,\lambda_{\varepsilon,\tau,0} - \lambda_{k,\psi_{\varepsilon,\tau,0}}^{\sigma_0}| \le \varrho_{\varepsilon,\tau}. \tag{4.10}$$

Let $\sup\limits_{\varepsilon,\tau} \varrho_{\varepsilon\tau} = \varrho_0$. Then the relations (4.9) and (4.10) implies that

$$\lambda_{\varepsilon,\tau,0} \in J_{k_0}^{\sigma_0}(\varrho_0), \tag{4.11}$$

where

$$J_k^-(\varrho) = [\lambda_k^- - b_k^- - \varrho, \lambda_k^- + \varrho] \text{ and } J_k^+(\varrho) = [\lambda_k^+ - \varrho, \lambda_k^+ + b_k^+ + \varrho].$$

In view of (4.4) and (4.11), by (1.5), (1.6) it follows from equation in (4.2) that the set $\{y_{\varepsilon, \tau, 0} \in E : 0 < \varepsilon < 1\}$ is bounded in $C^2[0,1]$. Consequently, by the Arzelà-Ascoli theorem this set is precompact in E. Then from any sequence $\{\varepsilon_n\}_{n=1}^\infty \subset (0,1)$ converging to zero we can extract a subsequence $\{\varepsilon_{n_m}\}_{m=1}^\infty \subset (0,1)$ such that $(\lambda_{\varepsilon_{n_m}, \tau, 0}, y_{\varepsilon_{n_m}, \tau, 0}) \to (\lambda_{\tau, 0}, y_{\tau, 0})$ in $\mathbb{R}^{\sigma_0} \times E$ as $m \to \infty$. Equation (4.2) then shows that this subsequence is convergent also in $\mathbb{R} \times C^2[0,1]$. Hence setting $\varepsilon = \varepsilon_{n_m}$, $\lambda = \lambda_{\varepsilon_{n_m}, \tau, 0}$ and $y = y_{\varepsilon_{n_m}, \tau, 0}$ in (4.2), and passing to the limit as $m \to \infty$ we obtain that $(\lambda_{\tau, 0}, y_{\tau, 0})$ is a solution of (1.1)-(1.3) such that $||\, y_{\tau, 0}||_1 = \tau$. Then by Remark 4.2 it follows from (4.4) that $y_{\tau, 0} \in S_{k_0}^{\sigma_0, \tau_0}$.

Thus we have shown that for any sufficiently small $\tau > 0$ there exists a solution $(\lambda_{\tau, 0}, y_{\tau, 0}) \in \mathbb{R} \times E$ of problem (1.1)-(1.3) such that

$$\lambda_{\tau, 0} \in J_{k_0}^{\sigma_0}(\varrho_0), \; ||\, y_{\tau, 0}||_1 = \tau, \text{ and } y_{\tau, 0} \in S_{k_0}^{\sigma_0, \nu_0}. \tag{4.12}$$

Then any sequence $\{(\lambda_{\tau_n, 0}, y_{\tau_n, 0})\}_{n=1}^\infty \subset \mathbb{R} \times E$, where $\tau_n > 0$, $n \in \mathbb{N}$, and $\tau_n \to 0$ as $n \to \infty$, of solutions to problem (1.1)-(1.3), which satisfies (4.12), contains a subsequence $\{(\lambda_{\tau_{n_l}, 0}, y_{\tau_{n_l}, 0})\}_{l=1}^\infty$ converging to $(\lambda_0, 0)$ for some $\lambda_0 \in J_{k_0}^{\sigma_0}(\varrho_0)$. It means that $(\lambda_0, 0) \in J_{k_0}^{\sigma_0}(\varrho_0) \times \{0\}$ is a bifurcation point of problem (1.1)-(1.3) with respect to the set $\mathbb{R} \times S_{k_0}^{\sigma_0, \nu_0}$. The proof of this theorem is complete.

Theorem 4.3. *$B_k^{\sigma, \nu} \subseteq J_k^\sigma \times \{0\}$ for each $k \in \mathbb{N}$, each $\sigma \in \{-\,, +\}$ and $\nu \in \{-\,, +\}$.*

Proof. By virtue of Remark 4.1 and Theorem 4.2, to prove the theorem it suffices to show that for each $k \in \mathbb{N}$, each $\sigma \in \{-\,, +\}$ and each $\nu \in \{-\,, +\}$ the following relation holds:

$$B_k^{\sigma, \nu} \cap (\mathbb{R}^\sigma \setminus J_k^\sigma) = \emptyset. \tag{4.13}$$

We again choose arbitrarily fixed $k_0 \in \mathbb{N}$, $\sigma_0 \in \{-\,, +\}$ and $\nu_0 \in \{-\,, +\}$. If (4.13) does not hold for $k = k_0$, $\sigma = \sigma_0$ and $\nu = \nu_0$, then there exists a sequence $\{(\lambda_{n,0}^*, y_{n,0}^*)\}_{n=1}^\infty \subset \mathbb{R} \times E$ of solutions of problem (1.1)-(1.3) such that

$$(\lambda_{n,0}^*, y_{n,0}^*) \to (\lambda_0^*, 0) \text{ as } n \to \infty, \; y_{n,0}^* \in S_{k_0}^{\sigma_0, \nu_0}, \; \lambda_0^* \notin J_{k_0}^{\sigma_0} \subset \mathbb{R}^{\sigma_0}. \tag{4.14}$$

Then we can choose a sufficiently small $\epsilon_0 > 0$ such that

$$\lambda_{n,0}^* \notin J_{k_0}^{\sigma_0}(\epsilon_0) \text{ for a sufficiently large } n \in \mathbb{N}. \tag{4.15}$$

It is obvious that $(\lambda_{n,0}^*, y_{n,0}^*)$, $n \in \mathbb{N}$, solves the nonlinear eigenvalue problem

$$\begin{aligned} &\ell y + \psi_{n,0}^*(x)\, y = \lambda \rho(x)\, y + g(x, y, y', \lambda), \; x \in (0,1), \\ &y \in B.C., \end{aligned} \tag{4.16}$$

where

$$\psi_{n,0}^*(x) = \begin{cases} -\frac{f(x, y_{n,0}^*(x), (y_{n,0}^*)'(x), \lambda_{n,0}^*)}{y_{n,0}^*(x)} & \text{if } y_{n,0}^*(x) \neq 0, \\ 0, & \text{if } y_{n,0}^*(x) = 0, \end{cases} \quad x \in [0,1]. \tag{4.17}$$

In view of (1.6), we see from Theorem 4.1 that

$$|\lambda_{n,0}^* - \lambda_{k, \psi_{n,0}^*}^{\sigma_0}| < \epsilon_0 \text{ for a sufficiently large } n \in \mathbb{N}, \tag{4.18}$$

where $\lambda^{+}_{k,\,\psi^*_{n,\,0}}$ and $\lambda^{-}_{k,\,\psi^*_{n,\,0}}$ are the k-th negative and positive eigenvalues, respectively, of the linear spectral problem

$$\begin{aligned} &\ell y + \psi^*_{n,\,0}(x)\, y = \lambda \rho(x)\, y,\ x \in (0,1), \\ &y \in B.C. \end{aligned} \tag{4.19}$$

By (1.4) and (1.5) it follows from (4.17) that

$$0 \le \psi^*_{n,\,0}(x) \le M \text{ for } x \in [0,1] \text{ and } n \in \mathbb{N}.$$

Consequently, Remark 2.1 and relation (3.1) implies that

$$\lambda^{\sigma_0}_{k,\,\psi^*_{n,\,0}} \in J^{\sigma_0}_{k_0}. \tag{4.20}$$

In view of (4.18), by (4.20) for a sufficiently large $n \in \mathbb{N}$ we get $\lambda^*_{n,\,0} \in J^{\sigma_0}_{k_0}(\epsilon_0)$ in contradiction with the relation (4.15). The proof of this theorem is complete.

5. Unilateral global bifurcation of solutions of problem (1.1)-(1.3)

Let $\mathcal{D}$ be the closure in $\mathbb{R} \times E$ of the set of nontrivial solutions to the problem (1.1)-(1.3). For each $k \in \mathbb{N}$, each $\sigma \in \{-\,,+\}$ and each $\nu \in \{-\,,+\}$ by $\hat{D}^{\sigma,\,\nu}_k$ we denote the union of all the connected components $D^{\sigma,\,\nu}_{k,\,\lambda}$ of $\mathcal{D}$ bifurcating from points $(\lambda, 0) \in J^\sigma_k \times \{0\}$ with respect to the set $S^{\sigma,\nu}_k$. By [21, Theorem 3.1] and Theorems 4.2, 4.3 we have $\hat{D}^{\sigma,\,\nu}_k \neq \emptyset$. Note that $D^{\sigma,\,\nu}_k = \hat{D}^{\sigma,\,\nu}_k \cup (J^\sigma_k \times \{0\})$ is a connected subset of $\mathbb{R} \times E$, but $\hat{D}^{\sigma,\,\nu}_k$ may not be connected in $\mathbb{R} \times E$.

The main result of this paper is the following theorem.

Theorem 5.1. *For each $k \in \mathbb{N}$, each $\sigma \in \{-\,,+\}$ and each $\nu \in \{-\,,+\}$ the set $D^{\sigma,\,\nu}_k$ lies in $(\mathbb{R}^\sigma \times S^{\sigma,\,\nu}_k) \cup (J^\sigma_k \times \{0\})$ and is unbounded $\mathbb{R} \times E$.*

Proof. Since $D^{\sigma,\,\nu}_k$ for each $\nu \in \{-\,,+\}$ is a connected component of $\mathcal{D}$ emanating from $J^\sigma_k \times \{0\} \subset \mathbb{R}^\sigma \times \{0\}$ it follows from Remark 4.1 that $D^{\sigma,\,-}_k,\, D^{\sigma,\,+}_k \subset \mathbb{R}^\sigma \times E$. Moereover, in view of Remark 4.2 and Theorem 4.3 we obtain $(D^{\sigma,\,\nu}_k \backslash (J^\sigma_k \times \{0\})) \cap (\mathbb{R}^\sigma \times \partial S^{\sigma,\,\nu}_k) = \emptyset$, which implies that $D^{\sigma,\,\nu}_k \subset (\mathbb{R}^\sigma \times S^{\sigma,\,\nu}_k) \cup (J^\sigma_k \times \{0\})$.

Next we choose some fixed arbitrary $k_0 \in \mathbb{N}$, $\sigma_0 \in \{-\,,+\}$ and $\nu_0 \in \{-\,,+\}$, and we will prove that $D^{\sigma,\,\nu}_k$ is unbounded in $\mathbb{R} \times E$ for $k = k_0$, $\sigma = \sigma_0$ and $\nu = \nu_0$.

If $D^{\sigma_0,\,\nu_0}_{k_0}$ is bounded in $\mathbb{R} \times E$, then $D^{\sigma_0,\,\nu_0}_{k_0}$ is precompact in $\mathbb{R} \times E$ in view of (1.1)-(1.3). Hence by following the arguments in Lemma 1.2 of [19] we can find a neighborhood $Q_0 \subset \mathbb{R}^{\sigma_0} \times E$ of this set such that $D^{\sigma_0,\,\nu_0}_{k_0} \cap \partial Q_0 = \emptyset$ (see also [7]). Since $D^{\sigma_0,\,\nu_0}_{k_0,\,\varepsilon}$ is an unbounded connected component of solutions of (4.2) satisfying (4.3) it follows that for any $\varepsilon_n \in (0,1)$ there exists $(\lambda_{\varepsilon,\,0}, y_{\varepsilon,\,0}) \in D^{\sigma_0,\,\nu_0}_{k_0,\,\varepsilon} \cap \partial Q_0$. Then, as in the proof of Theorem 4.1, we can show that there is a sequence $\{\varepsilon_n\}_{n=1}^\infty \subset (0,1)$ converging to 0 as $n \to \infty$ such that $(\lambda_{\varepsilon_n,\,0}, y_{\varepsilon_n,\,0})$ converges to a solution (λ_0, y_0) of problem (1.1)-(1.3), and consequently, $y_0 \in S^{\sigma_0,\,\nu_0}_{k_0} \cup \partial S^{\sigma_0,\,\nu_0}_{k_0}$. If $y_0 \in \partial S^{\sigma_0,\,\nu_0}_{k_0}$, then it follows from Remark 4.2 that $y_0 \equiv 0$. By following the arguments in the proof of Theorem 4.2 we get $\lambda_0 \in J^{\sigma_0}_{k_0}$ which contradicts the condition $J^{\sigma_0}_{k_0} \cap \partial Q_0 = \emptyset$. Hence we have $y_0 \in S^{\sigma_0,\,\nu_0}_{k_0}$ in contradiction with the condition $D^{\sigma_0,\,\nu_0}_{k_0} \cap \partial Q_0 = \emptyset$. The proof of this theorem is complete.

If $g \equiv 0$ and f satisfies condition (1.5) for all $(x, u, s, \lambda) \in [0,1] \times \mathbb{R}^3$, then for problem (1.1)-(1.3) we get the following global bifurcation result.

Theorem 5.2. *Let $g \equiv 0$ and condition (1.5) is satisfied for all $(x, u, s, \lambda) \in [0,1] \times \mathbb{R}^3$. Then for each $k \in \mathbb{N}$, each $\sigma \in \{+\,, -\}$ and each $\nu \in \{+\,, -\}$ the set $D_k^{\sigma,\,\nu}$ lies in the strip $(J_k^\sigma \times S_k^{\sigma,\,\nu}) \cup (J_k^\sigma \times \{0\})$ and is unbounded in $\mathbb{R} \times E$.*

The proof of this theorem follows from Theorem 5.1, taking into account the following statement.

Lemma 5.1. *Let the conditions of Theorem 5.2 be satisfied and let $(\lambda^*, y^*) \in \mathbb{R} \times E$ be a solution to problem (1.1)-(1.3) such that $y^* \in S_k^{\sigma,\,\nu}$, $k \in \mathbb{N}$ and $\sigma,\, \nu \in \{+\,, -\}$. Then $\lambda^* \in J_k^\sigma$.*

Proof. Let $(\lambda^*, y^*) \in \mathbb{R} \times E$ be a solution to problem (1.1)-(1.3) such that $y^* \in S_{k_0}^{\sigma_0,\,\nu_0}$ for some fixed arbitrary $k_0 \in \mathbb{N}$, $\sigma_0 \in \{-\,, -+\}$ and $\nu_0 \in \{-\,, +\}$. Then (λ^*, y^*) is a solution of the linear problem (4.19) with $\psi^*_{n,\,0}(x)$ replaced by $\psi^*_0(x)$, where $\psi^*_0(x)$ is defined as the right-hand side of (4.17) with $(\lambda^*_{n,\,0}, y^*_{n,\,0})$ replaced by (λ^*, y^*). Since $y^* \in S_{k_0}^{\sigma_0,\,\nu_0}$ it follows that λ^* is a k-th eigenvalue $\lambda_k^{*,\,\sigma_0}$ of this problem. Then by (3.1) and Remark 2.3 we get $\lambda^* \in J_{k_0}^{\sigma_0}$. The proof of this lemma is complete.

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