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SYMMETRY SOLUTIONS AND CONSERVED VECTORS FOR A GENERALIZED SHORT PULSE EQUATION

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Abstract. This work investigates a generalized short pulse equation. The short pulse equation governs the generation of ultra short optical pulses in nonlinear media. Firstly, we find its Lie symmetries and later utilize them to secure an optimal system of one-dimensional subalgebras (OSOSs). Thereafter, invariant solutions are determined under each element of the OSOSs. Three different cases for the constants a and b in the equation are discussed, viz., a and b not zero simultaneously; $a = 0$ but $b \neq 0$; $a \neq 0$ but $b = 0$. We also depict graphically the 3D and 2D representations of some of the gained solutions for the underlying equation. Secondly, by invoking the general multiplier method we acquire conserved vectors for the generalized short pulse equation.

1. Introduction

Most natural phenomena, for example in plasma physics, applied physics, fluid dynamics, oceanography, nonlinear optics, are modelled by partial differential equations (PDEs). To better understand these natural phenomena one needs to find explicit solutions of the PDEs that describe them. One of the key problems in the study of PDEs in the eighteenth and nineteenth century was seeking solutions to PDEs in closed form, that is finding their explicit solutions. Possibly the first explicit (special) solution to a PDE was the travelling wave solution that appeared in the work of d'Alembert as a solution to the linear wave equation. In the study of heat conduction equations, Fourier developed the method of separation of variables, which was generalized later by Sturm and Liouville in 1830s. Similarity solutions materialized in the works of Weierstrass around 1870, and later in 1890 of Bolzman.

Many of the fascinating phenomena of the world are administered by nonlinear partial differential equations (NPDEs). For example, many semi-conductor appliance models are described by a nonlinear Poisson equation for the electrostatic potential, together with convection-diffusion-reaction equations [2], the extended 2D Boussinesq equation that governs shallow water waves [14], the two important physical models, the Hirota equation and the Hirota-Maccari System [10],

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the Schrödinger equation that has applications in optical fibers [23], the (2+1)-dimensional ZK modified equal width equation and the Chafee-Infante equation [15], the PLMP equation that describes incompressible fluid [13], the Hamiltonian amplitude equation which describes instabilities of modulated wave trains [41], and the Boussinesq-Burgers-type equations that characterizes shallow water waves in lakes and ocean beaches [22]. For more NPDE models, see for instance [4, 65, 63, 31, 16, 21, 20, 24, 32, 46, 47, 33].

Finding explicit solutions to NPDEs is a dreadful exercise as there are no structured techniques that can be invoked to determine their solutions. Nevertheless, over the years researchers have made valuable efforts in developing certain techniques that could be used to get special solutions for NPDEs. Some of these special techniques are $(G'/G)-$expansion method [61], trial equation method [30], bifurcation technique [64], modified hyperbolic tangent function method [17], inverse scattering transform method [1], Hirota's technique [25], simplest equation technique [34], Kudryashov's technique [35], homogeneous balance technique [62], the power series method [39], Lie group theoretic method [49, 60, 28, 27] and so on.

Among the techniques mentioned above, Lie group analysis technique is completely algorithmic and efficient in finding explicit solutions of NPDEs. This technique was introduced by the eminent Norwegian scholar Sophus Lie (1842-1899) in the late nineteenth century. S. Lie took great inspiration from the works of Galois (1811-1832), who pioneered Galois theory and group theory [37, 38].

In investigating differential equations (DEs), conservation laws are very advantageous. As an example, conservation laws can be utilized to reduce the order of PDEs and to find their solutions, acquire numerical schemes, discover the scope of integrability of DEs and so on. In classical physics, conservation laws administer energy, momentum, angular momentum, electric charge, etc. For further information, see for instance [48, 6, 36, 29, 9, 45] and the references therein. DEs that come from variational principle, Noether's theorem [48] provides an effective way of establishing conservation laws as it offers a technique for finding conservation laws. However, for DEs that may or may not have Lagrangian formulation, the multiplier method can be employed to derive conservation laws [49].

In this work, we study a NPDE called the generalized short pulse (gSP) equation, which was first introduced and studied in [55] and is given by

$$u_{tx} - au^2u_{xx} - buu_x^2 = 0 \tag{1.1}$$

with a, b constants, not equal to zero at the same time. It was shown in [55] that equation (1.1) is integrable in two unconnected cases; the first one being when $a/b = 1/2$ and the second $a/b = 1$. These cases conform to the nonlinear equations

$$u_{tx} = \frac{1}{6}(u^3)_{xx} \tag{1.2}$$

and

$$u_{tx} = \frac{1}{2}u(u^2)_{xx}, \tag{1.3}$$

respectively. It was further shown in [55] that equations (1.2) and (1.3) are transformable to linear Klein-Gordon equations. Recently, the integrability of the SP equation

$$u_{tx} = u + au^2u_{xx} + buu_x^2 \tag{1.4}$$

was studied in [54] and it was shown that for $a/b = 1/2$ and $a/b = 1$ equation (1.4) is integrable. These two cases correspond to

$$u_{tx} = u + \frac{1}{6}(u^3)_{xx} \tag{1.5}$$

and

$$u_{tx} = u + \frac{1}{2}u(u^2)_{xx}, \tag{1.6}$$

via scale transformations of variables. Equation (1.5) first appeared in differential geometry [3, 53] and later came across in optics [59, 11]. Equations (1.5) and (1.6) have been widely studied by researchers, see for example [56, 7, 8, 57, 58, 42, 43, 50, 40, 52, 51, 19, 44, 18]. The SP equation (1.5) narrates the generation of ultra short optical pulses in non-linear media. In [56] the authors showed that (1.5) is integrable. Recently, Hone et al. [26] studied certain generalized short pulse equations from the standpoint of integrability.

In this work, we investigate the gSP equation (1.1) from the Lie group analysis viewpoint. In Section 2, we find Lie symmetries of (1.1) and using these symmetries we build up an optimal system of one-dimensional subalgebras (OSOSs). Thereafter, invariant solutions of (1.1) are derived under each of the elements of OSOSs. Various cases of the constants a and b are considered. In Section 3, conservation laws are established by invoking the general multiplier method for various a and b values. Lastly, concluding remarks are delivered in Section 4.

2. Solutions of the gSP equation (1.1)

We first consider the gSP equation (1.1) for the case when $a, b \neq 0$.

2.1. Solutions of the gSP equation (1.1) when $a, b \neq 0$. Here we find Lie symmetries, optimal system and symmetry reductions of equation (1.1). Exact solutions of the reduced ODEs are presented in certain cases.

2.1.1. *Lie symmetries of (1.1).* We work out Lie symmetries for the gSP equation (1.1) when $a, b \neq 0$. Equation (1.1) is unchanged under symmetry group which has the generator

$$X = \tau(t,x,u)\frac{\partial}{\partial t} + \xi(t,x,u)\frac{\partial}{\partial x} + \eta(t,x,u)\frac{\partial}{\partial u} \tag{2.1}$$

on condition that

$$X^{[2]}F|_{F=0} = 0, \tag{2.2}$$

where $F \equiv u_{tx} - au^2u_{xx} - buu_x^2$ and $X^{[2]}$ is the 2nd prolongation [49] of X that is defined by

$$X^{[2]} = \tau\frac{\partial}{\partial t} + \xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial u} + \zeta_2\frac{\partial}{\partial u_x} + \zeta_{12}\frac{\partial}{\partial u_{tx}} + \zeta_{22}\frac{\partial}{\partial u_{xx}} \tag{2.3}$$

with ζ_2, ζ_{12} and ζ_{22} determined as follows:

$$
\begin{aligned}
\zeta_1 &= D_t(\eta) - u_t D_t(\tau) - u_x D_t(\xi),\\
\zeta_2 &= D_x(\eta) - u_t D_x(\tau) - u_x D_x(\xi),\\
\zeta_{12} &= D_x(\zeta_1) - u_{tt} D_x(\tau) - u_{tx} D_x(\xi),\\
\zeta_{22} &= D_x(\zeta_2) - u_{tx} D_x(\tau) - u_{xx} D_x(\xi),
\end{aligned}
\tag{2.4}
$$

where D_t and D_x are the total derivatives given by

$$
D_t = \frac{\partial}{\partial t} + u_t \frac{\partial}{\partial u} + u_{tt} \frac{\partial}{\partial u_t} + u_{tx} \frac{\partial}{\partial u_x} + \cdots ,
$$

$$
D_x = \frac{\partial}{\partial x} + u_x \frac{\partial}{\partial u} + u_{xt} \frac{\partial}{\partial u_t} + u_{xx} \frac{\partial}{\partial u_x} + \cdots .
$$

From equation (2.2) we obtain

$$
\left[\tau \frac{\partial}{\partial t} + \xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial u} + \zeta_2 \frac{\partial}{\partial u_x} + \zeta_{12} \frac{\partial}{\partial u_{tx}} + \zeta_{22} \frac{\partial}{\partial u_{xx}}\right] \left(u_{tx} - au^2 u_{xx} - buu_x^2\right) = 0
$$

on (1.1), which on expansion gives

$$
\eta(-2auu_{xx} - bu_x^2) + \zeta_2(-2buu_x) + \zeta_{12} + \zeta_{22}(-au^2)\Big|_{(1.1)} = 0.
$$

Substituting the values of ζ_2, ζ_{12} and ζ_{22} in the above equation, replacing u_{tx} by $au^2 u_{xx} + buu_x^2$ and splitting on appropriate derivatives of u gives

$$
\begin{aligned}
&\tau_u = 0,\ \tau_x = 0,\ \xi_u = 0,\ \eta_{uu} = 0,\ \eta_{xu} = 0,\ \eta_{tx} - au^2 \eta_{xx} = 0,\\
&\eta + u(\eta_u + \tau_t - \xi_x) = 0,\ 2au\eta + au^2 \tau_t - au^2 \xi_x + \xi_t = 0,\\
&\eta_{tu} + au^2 \xi_{xx} - \xi_{tx} - 2bu\eta_x = 0,
\end{aligned}
$$

whose solution is

$$
\tau = C_1 - 2C_3 t + C_4,\ \xi = C_1 x + C_2,\ \eta = C_3 u.
$$

Thus, the Lie symmetries of (1.1) are

$$
\begin{aligned}
X_1 &= \frac{\partial}{\partial t}, \quad \text{time translation}\\
X_2 &= \frac{\partial}{\partial x}, \quad \text{space translation}\\
X_3 &= t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x}, \quad \text{dilation}\\
X_4 &= 2t\frac{\partial}{\partial t} - u\frac{\partial}{\partial u}, \quad \text{dilation,}
\end{aligned}
$$

which give rise to a four dimensional Lie algebra L_4.

2.1.2. *One-parameter transformation groups of Lie algebra of (1.1).* Using the Lie equations together with the initial conditions, viz.,

$$
\frac{d\bar{t}}{da} = \tau(t,x,u),\ \bar{t}|_{a=0} = t,\ \frac{d\bar{x}}{da} = \xi(t,x,u),\ \bar{x}|_{a=0} = x,\ \frac{d\bar{u}}{da} = \eta(t,x,u),\ \bar{u}|_{a=0} = u
$$

the one-parameter transformation groups G_i generated by the symmetries X_i are given by

$$G_1 : (\bar{t}, \bar{x}, \bar{u}) \longrightarrow (t + a_1, x, u),\ G_2 : (\bar{t}, \bar{x}, \bar{u}) \longrightarrow (t, x + a_2, u),$$
$$G_3 : (\bar{t}, \bar{x}, \bar{u}) \longrightarrow (e^{a_3} t, e^{a_3} x, u),\ G_4 : (\bar{t}, \bar{x}, \bar{u}) \longrightarrow (e^{2a_4} t, x, e^{-a_4} u).$$

Furthermore, if $u = f(t, x)$ is a solution of the gSP equation (1.1), then so are the functions

$$u_1 = f(t - a_1, x),\ u_2 = f(t, x - a_2),$$
$$u_3 = f(e^{-a_3} t, e^{-a_3} x),\ u_4 = e^{-a_4} f(e^{-2a_4} t, x),$$

where a_i are any real numbers.

2.1.3. *Optimal system of one-dimensional subalgebras (OSOSs) of (1.1).* In this section we construct an OSOSs. Commutators of the four symmetries of (1.1) are designated in Table 1 and the adjoint representation of subalgebras is provided in Table 2.

TABLE 1. Commutators of symmetries of (1.1)

$[X_i, X_j]$	X_1	X_2	X_3	X_4
X_1	0	0	X_1	$2X_1$
X_2	0	0	X_2	0
X_3	$-X_1$	$-X_2$	0	0
X_4	$-2X_1$	0	0	0

TABLE 2. Adjoint representation of subalgebras

Ad	X_1	X_2	X_3	X_4
X_1	X_1	X_2	$-\varepsilon X_1 + X_3$	$-2\varepsilon X_1 + X_4$
X_2	X_1	X_2	$-\varepsilon X_2 + X_3$	X_4
X_3	$e^{\varepsilon} X_1$	$e^{\varepsilon} X_2$	X_3	X_4
X_4	$e^{2\varepsilon} X_1$	X_2	X_3	X_4

Subsequently, invoking the method in [49] we can conclude that an OSOSs is given by

$$\{X_1, X_2, X_3, X_4, X_1 + cX_2, X_2 + \nu X_4, X_3 + X_4\}, \tag{2.5}$$

where $c, \nu = \pm 1$.

2.1.4. *Symmetry reductions and invariant solutions.* We utilize OSOSs obtained above to carry out symmetry reductions and construct invariant solutions for (1.1).

Case 1.1. Subalgebra: $X_1 = \partial/\partial t$

The characteristic equations associated with the symmetry X_1 are

$$\frac{dt}{1} = \frac{dx}{0} = \frac{du}{0}$$

and these yield two invariants $J_1 = x$, $J_2 = u$. Hence, invariant solution is $u = h(x)$. Substituting this value of u into (1.1), we get

$$ah^2h'' + bhh'^2 = 0.$$

Since $h(x) \neq 0$, we obtain $ahh'' + bh'^2 = 0$, whose solution is

$$h(x) = C_1(ax + bx - aC_2)^{a/(a+b)}$$

with C_1, C_2 constants. If we take $b = a$, we get $hh'' + h'^2 = 0$, or $(hh')' = 0$, which gives $hh' = C_1$, where C_1 is constant of integration. Solving this equation, we get $h(x) = \pm\sqrt{2C_1x + 2C_2}$, where C_2 is an integration constant. Hence

$$u(t, x) = \pm\sqrt{2C_1x + 2C_2}$$

is the invariant solution of (1.1) with $b = a$ corresponding to the symmetry X_1.

Case 1.2. Subalgebra: $X_2 = \partial/\partial x$

The characteristic equations of X_2 give the invariants $J_1 = t$, $J_2 = u$ and so the solution is $u = \phi(t)$, with ϕ being arbitrary function of t. Proceeding as in Case 1.1 we conclude that the invariant solution of (1.1) corresponding to X_2 is $u = \phi(t)$.

Case 1.3. Subalgebra: $X_3 = t\partial/\partial t + x\partial/\partial x$

The characteristic equations for X_3 provide two invariants $J_1 = t/x$, $J_2 = u$ and the solution $u = \phi(\zeta)$, $\zeta = t/x$ with ϕ being arbitrary function of ζ. Inserting this u in (1.1) we gain the nonlinear ordinary differential equation (NODE)

$$\phi' + \zeta\phi'' + 2a\zeta\phi^2\phi' + a\zeta^2\phi^2\phi'' + b\zeta^2\phi\phi'^2 = 0.$$

If we let $b = 2a$ the above equation becomes

$$\phi' + \zeta\phi'' + 2a\zeta\phi^2\phi' + a\zeta^2\phi^2\phi'' + 2a\zeta^2\phi\phi'^2 = 0,$$

which on integration gives

$$\zeta\phi' + a\zeta^2\phi^2\phi' = C_1, \tag{2.6}$$

where C_1 is an integration constant. The solution to equation (2.6), using Maple is

$$C_1\phi(\zeta) + \ln\left\{(aC_1^2\zeta\phi^2 - 2aC_1\zeta\phi + 2a\zeta + C_1^2)/\zeta\right\} = C_1^2C_2/(2a)$$

and hence solution of equation (1.1) with $b = 2a$ under symmetry X_3 in implicit form is

$$C_1u(\zeta) + \ln\left\{(aC_1^2\zeta u^2 - 2aC_1\zeta u + 2a\zeta + C_1^2)/\zeta\right\} = C_1^2C_2/(2a),$$

where $\zeta = t/x$ and C_1, C_2 are constants.

When $C_1 = 0$ in equation (2.6), we get $\phi'(1 + a\zeta\phi^2) = 0$. Hence, $\phi' = 0$ or $1 + a\zeta\phi^2 = 0$, which on integration yields $\phi = C_2$ or $\phi = \pm\sqrt{-1/(a\zeta)}$. Thus, we conclude that

$$u(t, x) = \pm\sqrt{\frac{-x}{at}},$$

is a solution of (1.1) with $b = 2a$ under X_3 provided $a < 0$.

Case 1.4. Subalgebra: $X_4 = 2t\partial/\partial t - u\partial/\partial u$

The symmetry X_4 provides two invariants $J_1 = x$, $J_2 = \sqrt{t}u$ and the solution $u = \phi(x)/\sqrt{t}$ with ϕ an arbitrary function of x. Inserting this expression of u in (1.1), we get the NODE $\phi' + 2a\phi^2\phi'' + 2b\phi\phi'^2 = 0$.

For $b = 2a$ the above equation integrates to $\phi + 2a\phi^2\phi' = C_1$ with C_1 a constant. When $C_1 = 0$, the integration of the above equation yields $\phi = \pm\sqrt{(C_2 - x)/a}$. Thus,

$$u(t,x) = \pm\sqrt{\frac{C_2 - x}{at}} \tag{2.7}$$

is the solution of (1.1) with $b = 2a$ under X_4, where C_2 is an arbitrary constant.

Case 1.5. Subalgebra: $X_1 + cX_2 = \partial/\partial t + c\partial/\partial x$, $c = \pm 1$

The symmetry $X_5 = X_1 + cX_2$ provides the solution

$$u = f(\xi), \ \ \xi = x - ct$$

with f arbitrary function of ξ. Insertion of u in (1.1) leads to $cf'' + af^2f'' + bff'^2 = 0$. By letting $b = 2a$, we get

$$cf'' + af^2f'' + 2aff'^2 = 0,$$

which on integration gives

$$cf' + af^2f' + C_1 = 0 \tag{2.8}$$

with C_1 an arbitrary constant. The solution of (2.8) via Maple is

$$f(\xi) = \\ \frac{1}{2a}\left\{4a^2\sqrt{9C_1^2C_2^2 + 18C_1^2C_2\xi + 9C_1^2\xi^2 + 4c^3/a} - 12a^2C_1C_2 - 12a^2C_1\xi\right\}^{1/3} \\ - \frac{2c}{\left\{4a^2\sqrt{9C_1^2C_2^2 + 18C_1^2C_2\xi + 9C_1^2\xi^2 + 4c^3/a} - 12a^2C_1C_2 - 12a^2C_1\xi\right\}^{1/3}}.$$

Hence

$$u(t,x) = \\ \frac{1}{2a}\left\{4a^2\sqrt{9C_1^2C_2^2 + 18C_1^2C_2\xi + 9C_1^2\xi^2 + 4c^3/a} - 12a^2C_1C_2 - 12a^2C_1\xi\right\}^{1/3} \\ - \frac{2c}{\left\{4a^2\sqrt{9C_1^2C_2^2 + 18C_1^2C_2\xi + 9C_1^2\xi^2 + 4c^3/a} - 12a^2C_1C_2 - 12a^2C_1\xi\right\}^{1/3}}, \tag{2.9}$$

where $\xi = x - ct$, is the solution of (1.1) with $b = 2a$ under X_5, with C_1 and C_2 being arbitrary constants. The wave profile depicting the travelling wave solution (2.9) is presented in Figure 1 using parametric values $a = 100$, $c = 1$, $C_1 = 10$, $C_2 = 0.4$ with $-10 \le t, x \le 10$.

Case 1.6. Subalgebra: $X_2 + \nu X_4 = \partial/\partial x + 2\nu t\partial/\partial t - \nu u\partial/\partial u$, $\nu = \pm 1$

The symmetry $X_6 = X_2 + \nu X_4$ gives the invariant solution

$$u = \frac{\phi(\zeta)}{\sqrt{t}}, \ \ \zeta = \frac{e^{\nu x}}{\sqrt{t}}.$$

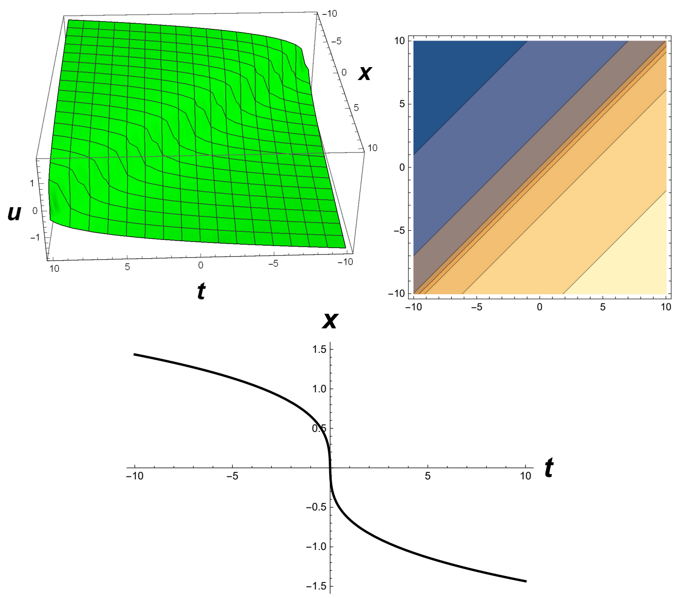

FIGURE 1. Wave profile of travelling wave solution (2.9) for $-10 \leq t, x \leq 10$.

Inserting this u into (1.1) gives

$$\phi' + \frac{1}{2}\zeta\phi'' + a\nu\phi^2\phi' + a\nu\zeta\phi^2\phi'' + b\nu\zeta\phi\phi'^2 = 0. \tag{2.10}$$

For $b = 2a$, the above equation can be written as

$$\phi' + \frac{1}{2}\zeta\phi'' + a\nu(\zeta\phi^2\phi')' = 0,$$

whose solution in the implicit form is

$$2aC_1\nu\phi(\zeta) + \left(2aC_1{}^2\nu + 1\right)\ln\{\phi(\zeta) - C_1\} + a\nu\phi(\zeta)^2 + \ln(\zeta) + C_2 = 0$$

for C_1, C_2 arbitrary constants. Consequently, solution of (1.1) with $b = 2a$ under X_6 is $u = \phi(\zeta)/\sqrt{t}$, where $\zeta = e^{\nu x}/\sqrt{t}$, and ϕ solves

$$2aC_1\nu\phi(\zeta) + \left(2aC_1{}^2\nu + 1\right)\ln\{\phi(\zeta) - C_1\} + a\nu\phi(\zeta)^2 + \ln(\zeta) + C_2 = 0.$$

Case 1.7. Subalgebra: $X_3 + X_4 = 3t\partial/\partial x + x\partial/\partial x - u\partial/\partial u$
The symmetry $X_7 = X_3 + X_4$ leads to the solution

$$u = t^{-1/3}\phi(\zeta),\ \zeta = t^{-1/3}x$$

with ϕ being a function of ζ. Inserting this expression of u in (1.1), we get

$$\frac{2}{3}\phi' + \frac{1}{3}\zeta\phi'' + a\phi^2\phi'' + b\phi\phi'^2 = 0.$$

When $b = 2a$, the above equation becomes

$$\frac{2}{3}\phi' + \frac{1}{3}\zeta\phi'' + a(\phi^2\phi')' = 0,$$

which on integration gives

$$\frac{1}{3}\phi + \frac{1}{3}\zeta\phi' + a\phi^2\phi' + C_1 = 0 \tag{2.11}$$

with C_1 being a constant. The solution of (2.11) via Maple is

$$\phi(\zeta) = \frac{1}{6a}\left\{12a^2\sqrt{729\,C_1^2\zeta^2 - 486\,C_1C_2\,\zeta + 81\,C_2^2 + 12\zeta^3/a} - 324C_1a^2\zeta + 108C_2a^2\right\}^{\frac{1}{3}} - \frac{2\zeta}{\left\{12a^2\sqrt{729C_1^2\zeta^2 - 486\,C_1C_2\,\zeta + 81\,C_2^2 + 12\zeta^3/a} - 324C_1a^2\zeta + 108C_2a^2\right\}^{\frac{1}{3}}},$$

where C_2 is a constant. Thus, the solution of (1.1) with $b = 2a$ under X_7 is

$$u(t,x) = \frac{t^{-\frac{1}{3}}}{6a}\left\{12a^2\sqrt{729C_1^2\zeta^2 - 486C_1C_2\zeta + 81C_2^2 + 12\zeta^3/a} - 324C_1a^2\zeta + 108C_2a^2\right\}^{\frac{1}{3}} - \frac{2t^{-1/3}\zeta}{\left\{12a^2\sqrt{729C_1^2\zeta^2 - 486C_1C_2\zeta + 81C_2^2 + 12\zeta^3/a} - 324C_1a^2\zeta + 108C_2a^2\right\}^{\frac{1}{3}}},$$

where $\zeta = t^{-1/3}x$.

2.2. Solutions of the gSP equation (1.1) when $a = 0$, $b \neq 0$. In this case, the equation (1.1) becomes

$$u_{tx} - buu_x^2 = 0. \tag{2.12}$$

Using the Lie algorithm, we find that Lie symmetries of (2.12) are

$$X_1 = \frac{\partial}{\partial t},\ X_2 = \frac{\partial}{\partial x},\ X_3 = t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x},\ X_4 = 2t\frac{\partial}{\partial t} - u\frac{\partial}{\partial u},$$

which are the same as for the case when $a, b \neq 0$. Consequently, the OSOSs is also the same, that is

$$\{X_1, X_2, X_3, X_4, X_1 + cX_2, X_2 + \nu X_4, X_3 + X_4\}, \tag{2.13}$$

where $c, \nu = \pm 1$.

2.2.1. *Symmetry reductions and invariant solutions of (2.12).* **Case 2.1. Subalgebra:** $X_1 = \partial/\partial t$

The symmetry X_1 provides the invariant solution as $u(t,x) = C_1$, with C_1 being a constant.

Case 2.2. Subalgebra: $X_2 = \partial/\partial x$

The symmetry X_2 gives $u = \phi(t)$ as a solution of (2.12), where ϕ is a function of t.

Case 2.3. Subalgebra: $X_3 = t\partial/\partial t + x\partial/\partial x$

The symmetry X_3 yields the invariant solution $u = \phi(\zeta)$, $\zeta = t/x$, where ϕ satisfies the NODE $\phi' + \zeta\phi'' + b\zeta^2\phi\phi'^2 = 0$.

Case 2.4. Subalgebra: $X_4 = 2t\partial/\partial t - u\partial/\partial u$

The symmetry X_4 gives the invariant solution $u = \phi(x)/\sqrt{t}$. Inserting the value of u in (2.12), we get $\phi' + 2b\phi\phi'^2 = 0$, which on integration gives $\phi(x) =$

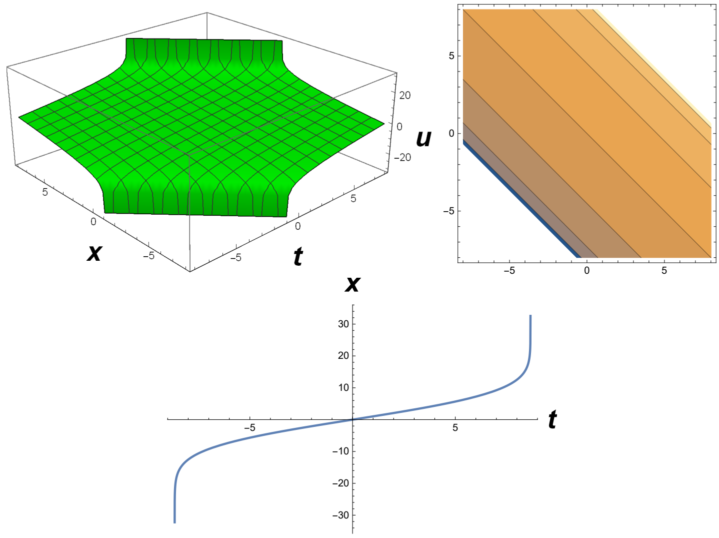

FIGURE 2. Wave profile of travelling wave solution (2.14) for $-8 \leq t, x \leq 8$.

C_1 or $\phi(x) = \pm\sqrt{(C_2 - x)/b}$, with C_1, C_2 constants and consequently, solution of (2.12) based on symmetry X_4 is

$$u(t,x) = \frac{C_1}{\sqrt{t}} \text{ or } u(t,x) = \pm\sqrt{\frac{C_2 - x}{bt}}.$$

Case 2.5. Subalgebra: $X_1 + cX_2 = \partial/\partial t + c\partial/\partial x$, $c = \pm 1$
The symmetry $X_5 = X_1 + cX_2$ provides the invariant solution

$$u = f(\xi), \ \xi = x - ct$$

and after inserting u in (2.12), we get $cf'' + bff'^2 = 0$, whose solution via Mathematica is

$$f(\xi) = -\sqrt{-\frac{2c}{b}}\,\mathrm{erf}^{-1}\left(\sqrt{-\frac{2b}{\pi c}}C_1(\xi + C_2)\right), \ c/b < 0,$$

and accordingly, the solution of (2.12) corresponding to symmetry X_5 is

$$u(t,x) = -\sqrt{-\frac{2c}{b}}\,\mathrm{erf}^{-1}\left(\sqrt{-\frac{2b}{\pi c}}C_1(x - ct + C_2)\right), \ c/b < 0 \tag{2.14}$$

with C_1, C_2 constants and erf^{-1} being the inverse error function [5]. The wave profile representing the travelling wave solution (2.14) is given in Figure 2 using constant values $b = 0.02$, $c = -1$, $C_1 = -1$, $C_2 = 0$, $\pi = 3$, where $-8 \leq t, x \leq 8$.

Case 2.6. Subalgebra: $X_2 + \nu X_4 = \partial/\partial x + 2\nu t\partial/\partial t - \nu u\partial/\partial u$, $\nu = \pm 1$
The symmetry $X_6 = X_2 + \nu X_4$ gives the invariant solution

$$u = \frac{\phi(\zeta)}{\sqrt{t}}, \ \zeta = \frac{e^{\nu x}}{\sqrt{t}}.$$

Utilizing this value of u in (2.12), we get

$$\phi' + \frac{1}{2}\zeta\phi'' + b\nu\zeta\phi\phi'^2 = 0.$$

Using Mathematica the solution of the above NODE is

$$\phi(\zeta) = -\frac{1}{\sqrt{-b\nu}}\mathrm{erf}^{-1}\left(\sqrt{-\frac{4b\nu}{\pi}}\left(\frac{C_1\zeta + C_2}{\zeta}\right)\right), \quad b\nu < 0 \tag{2.15}$$

and hence the invariant solution of (2.12) based on symmetry X_6 is

$$u(t,x) = -\frac{1}{\sqrt{-b\nu t}}\mathrm{erf}^{-1}\left(\sqrt{-\frac{4b\nu}{\pi}}\left(\frac{C_1\zeta + C_2}{\zeta}\right)\right), \quad b\nu < 0, \tag{2.16}$$

where $\zeta = e^{\nu x}/\sqrt{t}$, C_1, C_2 are constants and erf^{-1} is the inverse error function [5].

Case 2.7. Subalgebra: $X_3 + X_4 = 3t\partial/\partial x + x\partial/\partial x - u\partial/\partial u$

The symmetry $X_7 = X_3 + X_4$ provides the solution $u = t^{-1/3}\phi(t^{-1/3}x)$, where ϕ satisfies the NODE $2\phi' + \zeta\phi'' + 3b\phi\phi'^2 = 0$.

2.3. Solutions of the gSP equation (1.1) when $a \neq 0$, $b = 0$**.** In this case, equation (1.1) becomes

$$u_{xt} - au^2 u_{xx} = 0. \tag{2.17}$$

It has the same symmetries as for the previous two cases, namely

$$X_1 = \frac{\partial}{\partial t}, \ X_2 = \frac{\partial}{\partial x}, \ X_3 = t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x}, \ X_4 = 2t\frac{\partial}{\partial t} - u\frac{\partial}{\partial u}$$

and consequently has the same OSOSs

$$\{X_1, X_2, X_3, X_4, X_1 + cX_2, X_2 + \nu X_4, X_3 + X_4\}, \tag{2.18}$$

where $c, \nu = \pm 1$.

2.3.1. *Symmetry reductions and invariant solutions of (2.17).* **Case 3.1. Subalgebra:** $X_1 = \partial/\partial t$

The symmetry X_1 gives the group invariant solution $u(t,x) = C_1 x + C_2$ with C_1, C_2 arbitrary constants.

Case 3.2. Subalgebra: $X_2 = \partial/\partial x$

The symmetry X_2 provides $u(t,x) = \phi(t)$, with ϕ being a function of t, as the solution of (2.17).

Case 3.3. Subalgebra: $X_3 = t\partial/\partial t + x\partial/\partial x$

The symmetry X_3 gives the invariant solution $u = \phi(t/x)$, where ϕ solves the NODE

$$\phi' + \zeta\phi'' + a\phi^2\left(2\zeta\phi' + \zeta^2\phi''\right) = 0.$$

Case 3.4. Subalgebra: $X_4 = 2t\partial/\partial t - u\partial/\partial u$

The subalgebra X_4 gives the invariant solution $u = \phi(x)/\sqrt{t}$. Inserting u in (2.17), we get $\phi' + 2a\phi^2\phi'' = 0$, whose solution via Mathematica is

$$\phi(x) = -\frac{1}{2aC_1}\left\{W\left(-e^{-2aC_1{}^2(C_2+x)-1}\right) + 1\right\}$$

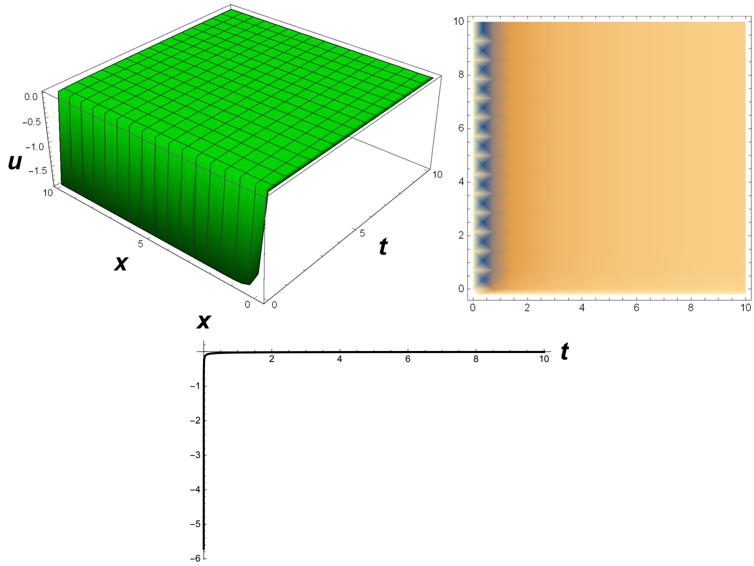

FIGURE 3. Wave profile of Lambert function solution (2.19) for $-10 \leq t, x \leq 10$.

with C_1, C_2 constants and W being the Lambert W function (also known as Product Log function) [12]. Hence, the invariant solution of (2.17) based on symmetry X_4 is

$$u(t,x) = -\frac{1}{2aC_1\sqrt{t}}\left\{W\left(-e^{-2aC_1{}^2(C_2+x)-1}\right)+1\right\}. \tag{2.19}$$

The wave profile of Lambert function solution (2.19) is given in Figure 3 with parameter values $a = 200$, $C_1 = 0.1$, $C_2 = 0.2$ for $-10 \leq t, x \leq 10$.

Case 3.5. Subalgebra: $X_1 + cX_2 = \partial/\partial t + c\partial/\partial x$, $c = \pm 1$

The symmetry $X_5 = X_1 + cX_2$ provides the invariant solution $u = f(x - ct)$. Substituting u in (2.17), we obtain

$$cf''(\xi) + af^2(\xi)f''(\xi) = 0, \;\; \xi = x - ct,$$

which on integration gives $f(\xi) = K_1\xi + K_2$, where K_1 and K_2 are arbitrary constants. Consequently, we have $u(t,x) = K_1(x - ct) + K_2$ as a group invariant solution of (2.17) under X_5.

Case 3.6. Subalgebra: $X_2 + \nu X_4 = \partial/\partial x + 2\nu t\partial/\partial t - \nu u\partial/\partial u$, $\nu = \pm 1$

Now consider the symmetry $X_6 = X_2 + \nu X_4$. This symmetry provides the invariant solution $u = \phi(\zeta)/\sqrt{t}$, $\zeta = e^{\nu x}/\sqrt{t}$, where ϕ solves the NODE

$$\phi' + \frac{1}{2}\zeta\phi'' + a\nu\phi^2\phi' + a\nu\zeta\phi^2\phi'' = 0.$$

Case 3.7. Subalgebra: $X_3 + X_4 = 3t\partial/\partial x + x\partial/\partial x - u\partial/\partial u$

The subalgebra $X_7 = X_3 + X_4$ gives the solution $u = t^{-1/3}\phi(t^{-1/3}x)$, where ϕ satisfies the NODE

$$\frac{2}{3}\phi' + \frac{1}{3}\zeta\phi'' + a\phi^2\phi'' = 0.$$

2.4. Solutions of the gSP equation (1.1) when $a, b = 0$. This equation is very well studied in the literature so we shall not discuss it here. However, its Lie symmetries are

$$X_1 = A(t)\frac{\partial}{\partial t},\ X_2 = B(x)\frac{\partial}{\partial x},\ X_3 = C_1 u\frac{\partial}{\partial u},\ X_4 = F(x)\frac{\partial}{\partial u},\ X_5 = G(t)\frac{\partial}{\partial u}$$

with $A(t)$, $G(t)$ functions of t and $B(x)$, $F(x)$ functions of x, while C_1 is an arbitrary constant.

3. Conservation laws of the gSP equation (1.1)

We now derive conservation laws of the gSP equation (1.1) by invoking the general multiplier method. Four different cases for the coefficients a and b are to be considered.

3.1. Conservation laws of (1.1) when $a, b \neq 0$. Let us seek first-order multiplier $Q = Q(t, x, u, u_t, u_x)$ for

$$u_{tx} - au^2 u_{xx} - buu_x^2 = 0$$

with $a, b \neq 0$. The determining equation for the multiplier is [6]

$$\frac{\delta}{\delta u}\left\{Q\left(u_{tx} - au^2 u_{xx} - buu_x^2\right)\right\} = 0 \tag{3.1}$$

with $\delta/\delta u$ being the Euler-Lagrange operator [28]

$$\frac{\delta}{\delta u} = \frac{\partial}{\partial u} - D_t\frac{\partial}{\partial u_t} - D_x\frac{\partial}{\partial u_x} + D_t D_x\frac{\partial}{\partial u_{tx}} + D_x^2\frac{\partial}{\partial u_{xx}} + \cdots .$$

Expanding (3.1) we get

$$\frac{\partial}{\partial u}\left(Qu_{tx} - aQu^2 u_{xx} - bQuu_x^2\right) - D_x\left(Q_{u_x} u_{tx} - au^2 u_{xx} Q_{u_x} - buu_x^2 Q_{u_x} - 2bQuu_x\right)$$
$$+ D_t D_x (Q) + D_x D_x\left(-au^2 Q\right) = 0. \tag{3.2}$$

Applying the total derivatives to the equation (3.2) gives

$$
\begin{aligned}
&Q_u u_{tx} - au^2 u_{xx} Q_u - 2auu_{xx} Q - buu_x^2 Q_u - bu_x^2 Q - u_{txx} Q_{u_x} - u_{tx} Q_{xu_x}\\
&- u_x u_{tx} Q_{uu_x} - u_{tx} u_{xx} Q_{u_x u_x} - u_{tx}^2 Q_{u_x u_t} + 2auu_x u_{xx} Q_{u_x} + au^2 Q_{u_x} u_{xxx}\\
&+ au^2 u_{xx} Q_{xu_x} + au^2 u_x u_{xx} Q_{uu_x} + au^2 u_{xx} u_{tx} Q_{u_x u_t} + au^2 u_{xx}^2 Q_{u_x u_x} + b(u_x)^2 Q_{u_x}\\
&+ 2buu_x u_{xx} Q_{u_x} + buu_x^2 Q_{xu_x} + buu_x^3 Q_{uu_x} + buu_x^2 u_{tx} Q_{u_x u_t} + buu_x^2 u_{xx} Q_{u_x u_x}\\
&+ 2bu_x^2 Q + 2buu_{xx} Q + 2buu_x Q_x + 2buu_x^2 Q_u + 2buu_x u_{tx} Q_{u_t} + 2buu_x u_{xx} Q_{u_x}\\
&+ Q_{tx} + u_t Q_{xu} + u_{tx} Q_{xu_x} + u_{tt} Q_{xu_t} + u_{tx} Q_u + u_x Q_{tu} + u_x u_t Q_{uu} + u_x u_{tx} Q_{uu_x}\\
&+ u_x u_{tt} Q_{uu_t} + u_{ttx} Q_{u_t} + u_{tx} Q_{tu_t} + u_t u_{tx} Q_{uu_t} + u_{tx}^2 Q_{u_t u_x} + u_{tx} u_{tt} Q_{u_t u_t}\\
&+ u_{xx} Q_{tu_x} + u_t u_{xx} Q_{uu_x} + u_{xx} u_{tx} Q_{u_x u_x} + u_{xx} u_{tt} Q_{u_x u_t} - 2au_x^2 Q - 2auu_{xx} Q\\
&- 2auu_x Q_x - 2auu_x^2 Q_u - 2auu_x u_{tx} Q_{u_t} - au^2 Q_{xx} - 2auu_x u_{xx} Q_{u_x} - 2auu_x Q_x\\
&- au^2 u_x Q_{xu} - au^2 u_{tx} Q_{xu_t} - au^2 u_{xx} Q_{xu_x} - 2auu_x^2 Q_u - au^2 u_{xx} Q_u - au^2 u_x Q_{xu}\\
&- au^2 u_x^2 Q_{uu} - au^2 u_x u_{tx} Q_{uu_t} - au^2 u_x u_{xx} Q_{uu_x} - 2auu_x u_{xt} Q_{u_t} - au^2 u_{txx} Q_{u_t}\\
&- au^2 u_{tx} Q_{xu_t} - au^2 u_x u_{tx} Q_{uu_t} - au^2 u_{tx}^2 Q_{u_t u_t} - au^2 u_{tx} u_{xx} Q_{u_t u_x} - 2auu_x u_{xx} Q_{u_x}\\
&- au^2 u_{xxx} Q_{u_x} - au^2 u_{xx} Q_{xu_x} - au^2 u_x u_{xx} Q_{uu_x} - au^2 u_{xx} u_{tx} Q_{u_x u_t}\\
&- au^2 u_{xx}^2 Q_{u_x u_x} = 0.
\end{aligned}
$$

Splitting on appropriate derivatives of u gives

$$Q_t = 0,\ Q_x = 0,\ Q_u = 0,\ Q_{u_t} = 0,\ bu_x Q_{u_x} + (b - 2a) Q = 0.$$

Solving these PDEs, we obtain one multiplier that is given by

$$Q = u_x^{\frac{2a-b}{b}}. \tag{3.3}$$

We now proceed to find the conserved vector corresponding to this multiplier. A multiplier for gSP equation (1.1) has the property that

$$Q(u_{tx} - au^2 u_{xx} - buu_x^2) = D_t T^t + D_x T^x, \tag{3.4}$$

where $T^t = T^t(t, x, u, u_x)$ represents the conserved density and $T^x = T^x(t, x, u, u_x)$ the spatial flux. Thus, after some calculations, we obtain the corresponding conserved vector (T^t, T^x) whose components are

$$T^t = \frac{b}{2a} u_x^{\frac{2a}{b}}, \quad T^x = -\frac{1}{2} bu^2 u_x^{\frac{2a}{b}}.$$

It should be noted that when $b = 2a$, we see from (3.3) that the multiplier becomes $Q = 1$ and $T^t = u_x$, $T^x = -au^2 u_x$, which shows that the gSP equation with $b = 2a$ is itself in the conserved form.

3.2. Conservation laws of (1.1) when $b = 2a$. In this case, the gSP equation (1.1) becomes

$$u_{tx} - au^2 u_{xx} - 2auu_x^2 = 0. \tag{3.5}$$

To look for the first-order multiplier Q, we invoke the determining equation

$$\frac{\delta}{\delta u} \left\{ Q \left(u_{tx} - au^2 u_{xx} - 2auu_x^2 \right) \right\} = 0, \tag{3.6}$$

where

$$\frac{\delta}{\delta u}=\frac{\partial}{\partial u}-D_x\frac{\partial}{\partial u_x}+D_tD_x\frac{\partial}{\partial u_{tx}}+D_xD_x\frac{\partial}{\partial u_{xx}}+\cdots$$

is Euler-Lagrange operator. Writing out equation (3.6) and splitting on appropriate derivatives of u, we get

$$Q_u=0,\ Q_{xx}=0,\ Q_{xu_x}=0,\ Q_{tu_x}=0, u_xQ_{u_xu_x}+3Q_{u_x}=0,\ Q_{tx}+2au_x^3Q_{u_x}=0,$$

whose solution is

$$Q=C_1xt+C_2x+F(t)+\frac{C_1}{4au_x^2}. \tag{3.7}$$

Thus, we obtain three conservation law multipliers

$$Q_1=F(t),\ Q_2=tx+\frac{1}{4au_x^2},\ Q_3=x,$$

The conserved vectors of equation (3.5) are constructed using the divergence identity

$$Q\left(u_{tx}-au^2u_{xx}-2auu_x^2\right)=D_tT^t+D_xT^x,$$

where T^t represents the conserved density and T^x is spatial flux. Thus, after some calculations, we obtain conserved vectors corresponding to the three multipliers Q_1, Q_2 and Q_3, respectively as

$$T^t=u_xF(t),\quad T^x=-au^2u_xF(t)-uF'(t);$$

$$T^t=xtu_x-\frac{1}{4au_x},\quad T^x=\frac{1}{3}atu^3+\left(\frac{1}{4u_x}-atxu_x\right)u^2-xu;$$

$$T^t=xu_x,\quad T^x=\frac{1}{3}au^3-axu^2u_x.$$

3.3. Conservation laws of (1.1) when $a=0, b\neq 0$. In this case, equation (1.1) becomes

$$u_{tx}-buu_x^2=0.$$

We apply the same algorithm to seek first-order multiplier Q. The multipliers are determined by solving

$$\frac{\delta}{\delta u}\left\{Q\left(u_{tx}-buu_x^2\right)\right\}=0, \tag{3.8}$$

where

$$\frac{\delta}{\delta u}=\frac{\partial}{\partial u}-D_x\frac{\partial}{\partial u_x}+D_tD_x\frac{\partial}{\partial u_{tx}}+\cdots$$

is the Euler-Lagrange operator. Expanding (3.8) and splitting on appropriate derivatives of u gives

$$Q_t=0,\ Q_x=0,\ Q_{u_t}=0,\ u_xQ_{u_x}+Q=0,$$

which yields one multiplier $Q = C_1/u_x$, where C_1 is a constant. Conserved vector corresponding to the multiplier Q is obtained by solving

$$\frac{1}{u_x}\left(u_{tx} - buu_x^2\right) = D_tT^t + D_xT^x,$$

for T^t and T^x. Expanding the above equation, we find that the components of the conserved vector (T^t, T^x) corresponding to the multiplier $Q = 1/u_x$ are given by

$$T^t = \ln u_x, \quad T^x = -\frac{1}{2}bu^2.$$

3.4. Conservation laws of (1.1) when $a \neq 0, b = 0$. In this case, we have multiplier $Q = 0$, which implies that equation (1.1) when $a \neq 0, b = 0$ has no conserved vectors.

3.5. Conservation laws of (1.1) when $a, b = 0$. In this case, equation (1.1) becomes

$$u_{tx} = 0. \tag{3.9}$$

Following the above procedure, we obtain four multipliers

$$Q_1 = I(u_t),\ Q_2 = J(u_x),\ Q_3 = C(x, u_x),\ Q_4 = E(t, u_t)$$

and the corresponding conserved vectors, respectively, are

$$T^t = 0, \quad T^x = \int I(u_t)du_t;$$

$$T^t = \int J(u_x)du_x, \quad T^x = 0;$$

$$T^t = \int C(x, u_x)du_x, \quad T^x = 0;$$

$$T^t = 0, \quad T^x = \int E(t, u_t)du_t.$$

4. Concluding remarks

The generalized short pulse equation (1.1) was studied in this paper from the group standpoint. Firstly, Lie symmetries of (1.1) were calculated and corresponding one parameter Lie group of transformations were presented. Using the Lie symmetries we constructed OSOSs and thereafter, group invariant solutions were derived under each subalgebra. Three different cases for the values of a and b were discussed in detail, namely, a and b not zero simultaneously; $a = 0$ but $b \neq 0$; $a \neq 0$ but $b = 0$. Graphical presentation of the dynamical behaviour of some of the obtained solutions was depicted in a bid to have a good apprehension of the physical phenomena of equation (1.1). Finally, we computed the conserved vectors of the generalized short pulse equation for different case of the constants a, b, by utilizing the general multiplier approach.

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