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PHRAGMEN-LINDELÖF THEOREM FOR A CLASS OF NON-UNIFORMLY ELLIPTIC EQUATION

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Abstract. The non-divergent structure elliptic equations of second order with non-uniform degeneration at infinity is considered in this paper. For a class of non-uniformly elliptic equation we obtain sufficient conditions on a domain, subordinate term coefficients, degeneration behavior of eigenvalues of the leading term coefficients of the equation near infinity. This allows to obtain a version of the Phragmen-Lindelöf theorem.

1. Introduction

Let E_n be the n-dimensional Euclidean space of points $x=(x_1,...,x_n), n\geq 2$ and D be an unbounded domain in E_n with boundary $\partial D\in C^2$ lied in the cone

$$G=\left\{x:\left(\sum_{i=1}^{n-1}x_i^2\right)^{1/2}<kx_n,\ 0<x_n<\infty,\ 0<k\leq\frac{1}{16}\right\}.$$

Consider in D the problem

$$Lu=\sum_{i,j=1}^{n}a_{ij}(x)u_{ij}+\sum_{i=1}^{n}b_i(x)u_i+c(x)u(x)=0, \tag{1.1}$$

$$u|_{\partial D}=0. \tag{1.2}$$

Assume that for $x\in D$ and $\forall\xi\in E_n$

$$\gamma\sum_{i=1}^{n}\lambda_i(x)\xi_i^2\leq\sum_{i,j=1}^{n}a_{ij}(x)\xi_i\xi_j\leq\gamma^{-1}\sum_{i=1}^{n}\lambda_i(x)\xi_i^2, \tag{1.3}$$

where $\gamma\in(0,1]$ is a constant, $\|a_{ij}(x)\|$ is a real symmetric matrix, $u_i=\frac{\partial u(x)}{\partial x_i}$, $u_{ij}=\frac{\partial^2u(x)}{\partial x_i\partial x_j}$, and

$$\lambda_i(x)=\left(\frac{\omega_i^{-1}(1+\rho(x))}{1+\rho(x)}\right)^2,\quad \rho(x)=\sum_{i=1}^{n}\omega_i(|x_i|);\quad i,j=1,...,n. \tag{1.4}$$

Further, $\omega_i(t)$, $i=1,..,n$ are strongly monotone positive functions on $[0,\infty)$ such that $\omega_i(t)\to\infty$ as $t\to+\infty$. By $\omega_i^{-1}(t)$ denote the inverse function of $\omega_i(t)$ on

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$[0,\infty)$. Assume that $\omega_1^{-1}(t)\geq ...\geq \omega_n^{-1}(t)$. There exist constants $\alpha,p,\eta\in(1,\infty)$ and $q>n,\ A>0$ such that for sufficiently large $R>0$ and $i=1,..,n$ the following inequalities hold

$$\alpha\omega_i(R)\leq\omega_i(\eta R)\leq p\omega_i(R), \tag{1.5}$$

and

$$\left(\frac{\omega_i^{-1}(R)}{R}\right)^{q-1}\int\limits_0^{\omega_i^{-1}(R)}\left(\frac{\omega_i(\tau)}{\tau}\right)^q d\tau\leq AR. \tag{1.6}$$

Also assume that

$$-C_0\leq c(x)\leq 0, \tag{1.7}$$

where C_0 is a positive constant. Denote

$$b_R(x)=\left(\frac{b_1(x)}{(\omega_1^{-1}(R))^2},...,\frac{b_n(x)}{(\omega_n^{-1}(R))^2}\right).$$

Consider also the "shortened" operator $L_c=L-c(x)$. Let $x^0\in E_n, R>0, k>0$ and let $E_R^{x^0}(k)$ be the ellipsoid

$$\left\{x:\sum_{i=1}^n\frac{(x_i-x_i^0)^2}{(\omega_i^{-1}(R))^2}<k^2\right\}.$$

Denote by $E_R^{x^0}(k_1,k_2)$ the ellipsoid layer $E_R^{x^0}(k_2)\backslash E_R^{x^0}(k_1)$ with $k_2>k_1$. Denote $D_R=D\cap E_R^0(1,17)$ and for $z\in\partial E_R^0(9)\cap\overline{G}$ set $D_R^z=D\cap E_R^z(8)$, $x^0(z)\in\partial E_R^z(1)\cap\partial E_R^0(9)$, $A_R=\bigcup\limits_z x^0(z)$, and set

$$G_S^{(R)}(x)=\left(\sum_{i=1}^n\frac{(x_i-x_i^0)^2}{(\omega_i^{-1}(R))^2}\right)^{-s/2},\ s>0.$$

For the small term coefficients and the domain D we assume the condition

$$(b_R(x),x-x^0)\leq 0,\quad x\in D_R,\ x^0\in A_R. \tag{1.8}$$

at infinity.

We derive a growth estimate from below for vanishing on boundary positive solutions of the non-uniformly elliptic equations on the unbounded domain. Such estimates are historically called Phragmen-Lindelöf type theorem. The Dirichlet problem for elliptic equations is not unique solvable on unbounded domains. The mentioned estimates depending on geometry of domain near infinity is a main tool to prove the uniqueness theorems for elliptic equations.

For elliptic equations the Pragmen-Lindelöf theorem was proved in [6] and [8] for the first time. For further discussions of Phragmen-Lindelöf's principle we refer to the books [15], [1], [5], and [7]. Also we cite[10], [4], [11], [14], [13], [17] on this subject; for the degenerate elliptic equations see e.g. [16], [12]. In [9, 2, 3], the non-uniformly elliptic equation with power type degeneration $\lambda_i(x)=|x|_\alpha^{\alpha_i}$ was considered without small term coefficients. In this paper, we consider a general non-uniformly elliptic equation with small terms and a general form degeneration.

2. Notation and auxiliary results

Definition 2.1. Let $\lambda = (\lambda_1(x), ..., \lambda_n(x))$. Denote by $W^2_{2,\lambda}(D)$ a Banach space of functions $u(x)$ in D having the finite norm

$$\|u\|_{W^2_{2,\lambda}(D)} = \left(\int\limits_D \left(u^2 + \sum_{i=1}^{n} \lambda_i(x) u_i^2 + \sum_{i,j=1}^{n} \lambda_i(x)\lambda_j(x) u_{i,j}^2 \right) dx \right)^{\frac{1}{2}}.$$

Set $\overset{\circ}{W}{}^2_{2,\lambda}(D)$ closure of the functions class $u(x) \in C_0^\infty(\overline{D})$ under the norm $W^2_{2,\lambda}(D)$.

Definition 2.2. A function $u(x) \in \overset{\circ}{W}{}^2_{2,\lambda}(D)$ is called a strong solution of problem (1.1)-(1.2) if it satisfies (1.1) a.e. in D.

Lemma 2.1. *Let* $z \in \partial E^0_R(9) \cap \overline{G}$, $x^0(z) \in \partial E^z_R(1) \cap \partial E^0_R(9)$ *where* $R \geq 1$, $x_1^0(z) > 0$, $x_2^0(z) = ... = x_{n-1}^0(z) = 0$, $x_n^0(z) > 0$. *Then* $x^0(z) = \left(x_1^0(z), x_2^0(z), ..., x_n^0(z)\right) \overline{\in} G$.

Proof. To shorten the records we will write x_i^0 in place of $x_i^0(z)$, $i = 1, ..., n$. We have

$$\begin{cases} \frac{(x_1^0)^2}{(\omega_1^{-1}(R))^2} + \frac{(x_n^0)^2}{(\omega_n^{-1}(R))^2} = 81, \\ \frac{(x_1^0 - z_1)^2}{(\omega_1^{-1}(R))^2} + \frac{z_2^2}{(\omega_2^{-1}(R))^2} + ... + \frac{z_{n-1}^2}{(\omega_{n-1}^{-1}(R))^2} + \frac{(x_n^0 - z_n)^2}{(\omega_n^{-1}(R))^2} = 1, \\ \frac{z_1^2}{(\omega_1^{-1}(R))^2} + ... + \frac{z_n^2}{(\omega_{n-1}^{-1}(R))^2} = 81. \end{cases} \tag{2.1}$$

Inserting third equation into the second and subtracting it from the first equation, we obtain

$$\frac{2x_1^0 z_1}{(\omega_1^{-1}(R))^2} + \frac{2x_n^0 z_1}{(\omega_n^{-1}(R))^2} = 161.$$

Therefore,

$$\frac{2x_n^0 z_n}{(\omega_n^{-1}(R))^2} \leq 161 + \frac{2x_1^0 |z_n|}{(\omega_n^{-1}(R))^2}.$$

On the other hand

$$\frac{|z_1|}{\omega_1^{-1}(R)} \leq \frac{|z_1|}{\omega_n^{-1}(R)} < \frac{k z_n}{\omega_n^{-1}(R)},$$

$$\frac{x_1^0}{\omega_1^{-1}(R)} = \sqrt{81 - \left(\frac{x_n^0}{\omega_n^{-1}(R)}\right)^2}.$$

Denoting $\frac{x_n^0}{\omega_n^{-1}(R)} = v$, $\frac{z_n}{\omega_n^{-1}(R)} = \omega$ we have

$$2v\omega \leq 161 + 2k\omega\sqrt{81 - v^2}. \tag{2.2}$$

From the third equation it follows that

$$81 \leq \left(\frac{z_1}{\omega_1^{-1}(R)}\right)^2 + ... + \left(\frac{z_n}{\omega_n^{-1}(R)}\right)^2 \leq k^2\omega^2 + \omega^2 = (k^2+1)\omega^2,$$

or

$$\omega \geq \frac{9}{\sqrt{k^2+1}}.$$

Taking into the account this from (2.2) it follows that

$$v \leq \frac{8,9 + k\sqrt{0,94}}{\sqrt{1+k^2}}.$$

Also from the first equation of (2.1) it follows that

$$\left(\frac{x_1^0}{\omega_1^{-1}(R)} \right)^2 \geq 81 - \frac{\left(8,9 + k\sqrt{0,94}\right)^2}{1+k^2},$$

or

$$x_1^0 \geq \omega_1^{-1}(R) \sqrt{81 - \frac{\left(8,9 + k\sqrt{0,94}\right)^2}{1+k^2}} \geq$$

$$\geq \sqrt{81 - \frac{(8,9 + k\sqrt{0,94})^2}{1+k^2}} \cdot \frac{\sqrt{1+k^2}}{\left(8,9 + k\sqrt{0,94}\right)} x_n^0. \tag{2.3}$$

In order to have $x^0(z)\overline{\in}G$ it suffices that

$$81(1+k^2) - \left(8,9 + k\sqrt{0,94}\right)^2 > k^2 \left(8,9 + k\sqrt{0,94}\right)^2.$$

Solving this inequality we obtain the condition $k < \frac{1}{\sqrt{94}}$. Since $K \in \left(0, \frac{1}{16}\right]$, the last condition is verified.

This completes the proof of Lemma 2.1.

□

Lemma 2.2. *Let $x^0 \in A_R$. Then for any $x \in D_R^z$ there exists a constant $\beta = \beta(k) > 0$ such that*

$$\left(\sum_{i=1}^n \frac{(x_i - x_i^0)^2}{(\omega_i^{-1}(R))^2} \right)^{1/2} \geq \beta.$$

Proof. There are two possibilities:

$$1) \qquad \left(\sum_{i=1}^n \frac{(x_i - z_i)^2}{(\omega_i^{-1}(R))^2} \right)^{1/2} \geq 1 + \mu,$$

$$2) \qquad \left(\sum_{i=1}^n \frac{(x_i - z_i)^2}{(\omega_i^{-1}(R))^2} \right)^{1/2} < 1 + \mu,$$

where the constant $\mu \in (0, 3)$ will be determined later.

Let the case 1) take place. Then using Minkowsky's inequality, we get

$$\left(\sum_{i=1}^n \frac{(x_i - x_i^0)^2}{(\omega_i^{-1}(R))^2} \right)^{1/2} \geq \left(\sum_{i=1}^n \frac{(x_i - z_i)^2}{(\omega_i^{-1}(R))^2} \right)^{1/2} - \left(\sum_{i=1}^n \frac{(x_i^0 - z_i)^2}{(\omega_i^{-1}(R))^2} \right)^{1/2} \geq 1 + \mu - 1 = \mu.$$

Let now the case 2) take place. For $x \in D \cap E_R^z(1+\mu)$ we have

$$\begin{cases} \sum\limits_{i=1}^n \frac{(x_i - z_i)^2}{(\omega_i^{-1}(R))^2} < (1+\mu)^2, \\ \sum\limits_{i=1}^{n-1} x_i^2 < K^2 x_n^2. \end{cases}$$

From the first inequality we get

$$\left(\frac{x_n - z_n}{\omega_n^{-1}(R)}\right)^2 < (1+\mu)^2 \quad \text{and} \quad \left(\frac{x_n}{\omega_n^{-1}(R)}\right)^2 < (1+\mu)^2 + \frac{2x_n z_n - z_n^2}{(\omega_n^{-1}(R))^2}.$$

Set $\overline{x}_n = \sup\limits_{x \in D \cap E_R^z(1+\mu)} x_n$. Evidently, $\overline{x}_n \geq z_n$. Therefore, the function $\varphi(z_n) = 2\overline{x}_n z_n - z_n^2$ does not decrease. Also we have

$$\frac{\overline{x}_n^2}{(\omega_n^{-1}(R))^2} \leq (1+\mu)^2 + \frac{2\overline{x}_n 9\omega_n^{-1}(R) - 81(\omega_n^{-1}(R))^2}{(\omega_n^{-1}(R))^2} =$$

$$= (1+\mu)^2 + 18\frac{\overline{x}_n}{\omega_n^{-1}(R)} - 81.$$

Denoting $t = \frac{\overline{x}_n}{\omega_n^{-1}(R)}$, we obtain

$$t^2 - 18t + 81 - (1+\mu)^2 \leq 0.$$

Solving this inequality we obtain that $t \leq 10 + \mu$ or $\overline{x}_n \leq (10+\mu)\omega_n^{-1}(R)$. From the second inequality (2.1) we get

$$|x_1| < k(10+\mu)\omega_n^{-1}(R) \leq k(10+\mu)\omega_1^{-1}(R).$$

From (2.3) it follows that

$$\left(\sum_{i=1}^{n} \frac{(x_i - x_i^0)^2}{\omega_i^{-1}(R))^2}\right)^{1/2} \geq \frac{x_1^0 - |x_1|}{\omega_1^{-1}(R)} \geq \sqrt{81 - \frac{(8,9 + k\sqrt{0,94})^2}{1+k^2}} - k(10+\mu).$$

It easily seen that for $k \in \left(0, \frac{1}{16}\right]$

$$J(k) = \sqrt{81 - \frac{(8,9 + k\sqrt{0,94})^2}{1+k^2}} > 10k.$$

Therefore, there exists $\varepsilon_0 = \varepsilon_0(k) > 0$ such that $J(k) = (10 + \varepsilon_0)k$. Choose $\mu = \varepsilon_0/2$ and fix it. Then in case 2)

$$\left(\sum_{i=1}^{n} \frac{(x_i - x_i^0)^2}{(\omega_i^{-1}(R))^2}\right)^{1/2} \geq k\varepsilon_0 - \frac{k\varepsilon_0}{2} = \frac{k\varepsilon_0}{2}$$

Set $\beta = \frac{k\varepsilon_0}{2}$ and the proof of Lemma 2.2 is ready. $\square$

Lemma 2.3. *Let the conditions* (1.3)-(1.8) *be satisfied . Then for any fixed* $x^0 \in A_R$ *there exists* $s = s(\gamma, n) > 0$ *such that for any* $x \in D_R$ *we have*

$$L_c G_s^{(R)}(x) \geq 0.$$

Proof. Denote $r(x) = \left(\sum\limits_{i=1}^{n} \frac{(x_i - x_i^0)^2}{(\omega_i^{-1}(R))^2}\right)^{1/2}$. For $G_s^{(R)}(x) = r^{-s}$ it is easily seen that $\frac{\partial G_s^{(R)}(x)}{\partial x_i} = -sr^{-s-2} \cdot \frac{x_i - x_i^0}{(\omega_i^{-1}(R))^2}$, $\frac{\partial^2 G_s^{(R)}(x)}{\partial x_i^2} = s(s+2)r^{-s-4} \cdot \frac{(x_i - x_i^0)^2}{(\omega_i^{-1}(R))^4} - sr^{-s-2} \cdot \frac{1}{(\omega_i^{-1}(R))^2}$, $\frac{\partial^2 G_s^{(R)}(x)}{\partial x_i \partial x_j} = s(s+2)r^{-S-4} \cdot \frac{(x_i - x_i^0)(x_j - x_j^0)}{(\omega_i^{-1}(R))^2(\omega_j^{-1}(R))^2}$.

Therefore,

$$L_c G_s^{(R)}(x) = s r^{-s-2} \left[\frac{s+2}{r^2} \sum_{i,j=1}^{n} a_{ij}(x) \frac{(x_i - x_i^0)(x_j - x_j^0)}{(\omega_i^{-1}(R))^2 (\omega_j^{-1}(R))^2} - \right.$$

$$\left. - \sum_{i=1}^{n} a_{ii}(x) \frac{1}{(\omega_i^{-1}(R))^2} - \sum_{i=1}^{n} b_i(x) \frac{x_i - x_i^0}{(\omega_i^{-1}(R))^2} \right].$$

Make use the conditions (1.3) and (1.8). Then we have

$$L_c G_s^{(R)}(x) \geq s r^{-s-2} \left(\frac{\gamma(s+2)}{r^2} \sum_{i=1}^{n} \frac{\lambda_i(x)(x_i - x_i^0)^2}{(\omega_i^{-1}(R))^4} - \gamma^{-1} \sum_{i=1}^{n} \frac{\lambda_i(x)}{(\omega_i^{-1}(R))^2} \right).$$

It is possible to show that (see [3, Lemma 1]) there exist positive constants $C_1(n)$ and $C_2(n)$ such that

$$C_1(n) \left(\frac{\omega_i^{-1}(R)}{R} \right)^2 \leq \lambda_i(x) \leq C_2(n) \left(\frac{\omega_i^{-1}(R)}{R} \right)^2, i = 1, ..., n.$$

From this we deduce that

$$L_c G_s^{(R)}(x) \geq \frac{s}{R^2 r^{s+2}} \left(\gamma(s+2) C_1(n) - \frac{C_2(n) n}{\gamma} \right).$$

To finish the proof it suffices to choose $s \geq \frac{C_2(n)n}{C_1(n)\gamma^2} - 2$.

This completes the proof of Lemma 2.3. □

Corollary 2.1. *Let* $z \in D \cap \partial E_R^0(9)$, $x^0 = x^0(z)$, $x \in D \cap E_R^z(8)$, $g_s^{(R)}(x) = \beta^s G_s^{(R)}(x)$. *Then* $g_s^{(R)}(x) \leq 1$.

Lemma 2.4. *Let* $z \in D \cap E_R^0(9)$, $x^0 = x^0(z)$ *and in* $H_R = D \cap E_R^z(8)$ *a positive solution* $u(x)$ *of equation* (1.1) *is defined, which is continuous in* $\overline{H_R}$ *and vanishes on the part* Γ *of the boundary* ∂H_R *lied strongly interior in* $E_R^z(8)$. *Then if the conditions* (1.3)-(1.8) *are fulfilled there exists a constant* $\eta = \eta(\gamma, n)$ *such that*

$$\sup_{H_R} u(x) \geq (1+\eta) \sup_{H_R \cap E_R^z(1)} u(x).$$

Proof. First, show that if $Lu(x) = 0$ then $L_c u^2(x) \geq 0$. Indeed,

$$L_c u^2(x) = 2u(x) \sum_{i,j=1}^{n} a_{ij}(x) u_{ij} + 2 \sum_{i,j=1}^{n} a_{ij}(x) u_i u_j +$$

$$+2u(x) \sum_{i,j=1}^{n} b_i(x) u_i \geq 2u(x) \left(\sum_{i,j=1}^{n} a_{ij}(x) u_{ij} + \sum_{i=1}^{n} b_i(x) u_i \right) =$$

$$= 2u(x)(-c(x)u(x)) = -2c(x)u^2(x) \geq c_0 u^2(x) \geq 0.$$

Set $\sup_{H_R} u^2(x) = M$. Consider the auxiliary function

$$U(x) = M \left[1 - g_s^{(R)}(x) + \sup_{x \in \overline{H}_R \cap \partial E_R^z(8)} g_s^{(R)}(x) \right].$$

It is easily seen that

$$L_c(U(x) - u^2(x)) \le 0 \text{ in } H_R, \ \ (U(x) - u^2(x))\big|_{\Gamma} \ge 0, \ \ (U(x) - u^2(x))\big|_{\partial H_R \setminus \Gamma} \ge 0.$$

Using the maximum principle we have $U(x) \ge u^2(x)$ in H_R and, in particular,

$$\sup_{H_R \cap E_R^z(1)} u^2(x) \le M \left(1 - \left(\inf_{H_R \cap E_R^z(1)} g_s^{(R)}(x) - \sup_{x \in \overline{H}_R \cap \partial E_R^z(8)} g_s^{(R)}(x)\right)\right).$$

Let $x \in \overline{H}_R \cap \partial E_R^z(8)$. Then

$$\left(\sum_{i=1}^n \frac{(x_i - x_i^0)^2}{(\omega_i^{-1}(R))^2}\right)^{1/2} \ge \left(\sum_{i=1}^n \frac{(x_i - z_i)^2}{(\omega_i^{-1}(R))^2}\right)^{1/2} - \left(\sum_{i=1}^n \frac{(x_i^0 - z_i)^2}{(\omega_i^{-1}(R))^2}\right)^{1/2} \ge 8 - 1 = 7.$$

Therefore, $\sup\limits_{H_R \cap E_R^z(8)} g_s^{(R)}(x) \le 7^{-s}\beta^s$. If $x \in H_R \cap E_R^z(1)$ then

$$\left(\sum_{i=1}^n \frac{(x_i - x_i^0)^2}{(\omega_i^{-1}(R))^2}\right)^{1/2} \le \left(\sum_{i=1}^n \frac{(x_i - z_i)^2}{(\omega_i^{-1}(R))^2}\right)^{1/2} +$$

$$+ \left(\sum_{i=1}^n \frac{(z_i - x_i^0)^2}{(\omega_i^{-1}(R))^2}\right)^{1/2} \le 1 + 1 = 2.$$

Hence

$$\inf_{H_R \cap E_R^z(1)} g_s^{(R)}(x) \le 2^{-s}\beta^s,$$

We get

$$\sup_{H_R \cap E_R^z(1)} u^2(x) \le M(1 - \beta^s(2^{-s} - 7^{-s})).$$

This completes the proof of Lemma 2.4. □

Lemma 2.5. *Let $u(x)$ be a positive solution of the equation* (1.1) *in $H_R^1 = D \cap E_R^0(17)$ which is continuous in $\overline{H}_R^1$ and vanishes on the part of boundary ∂H_R^1 that lies strongly interior in $E_R^0(1.17)$. If the conditions* (1.3)-(1.8) *are fulfilled then*

$$\sup_{H_R^1} u(x) \ge (1 + \eta) \sup_{H_R^1 \cap \partial E_R^0(9)} u(x).$$

Proof. Let z be a point from the set $\overline{H}_R^1 \cap \partial E_R^0(9)$ for which $u(z) = \sup\limits_{H_R^1 \cap \partial E_R^0(9)} u(x)$.

By Lemma 2.4,

$$\sup_{H_R^1 \cap E_R^z(8)} u(x) \ge (1 + \eta) \sup_{H_R^1 \cap \partial E_R^z(1)} u(x).$$

On the other hand,

$$\sup_{H_R^1 \cap \partial E_R^z(1)} u(x) \ge u(z) \ \text{ and } \ E_R^z(8) \subset E_R^0(1 : 17).$$

This completes the proof of Lemma 2.5. □

Corollary 2.2. *Let in* $H_R^2 = D \cap E_R^0(17)$ *it is defined a positive solution* $u(x)$ *of the equation* (1.1)*, which is continuous in* $\overline{H}_R^2$ *and vanishes on the part of the boundary* ∂H_R^2 *that lies strongly interior in* $E_R^0(17)$*. If the conditions* (1.3)-(1.8) *are fulfilled then*

$$\sup_{H_R^2} u(x) \geq (1+\eta) \sup_{H_R^2 \cap E_R^0(9)} u(x).$$

Proof. According to Lemma 2.5

$$\sup_{H_R^2 \cap E_R^0(1:17)} u(x) \geq (1+\eta) \sup_{H_R^2 \cap \partial E_R^0(9)} u(x).$$

On the other hand,

$$\sup_{H_R^2 \cap \partial E_R^0(9)} u(x) = \sup_{H_R^2 \cap E_R^0(9)} u(x)$$

and

$$\sup_{H_R^2 \cap E_R^0(1:17)} u(x) \leq \sup_{H_R^2} u(x).$$

This completes the proof of Corollary 2.2. □

3. Main result

Theorem 3.1. *Let the coefficients of operator* L *are defined on a domain* $D \subset G$ *and satisfy the conditions* (1.3)-(1.8)*. Let* $u(x)$ *be a solution of the problem* (1.1)-(1.2)*. Then for* $M(r) = \sup_{D \cap \partial E_r^0(1)} |u(x)|$ *we have:*

$$1) \quad \textit{either} \quad u(x) \equiv 0 \quad \textit{in} \quad D \quad \textit{or} \quad 2) \quad \varliminf_{r\to\infty} \frac{M(r)}{r^\delta} > 0,$$

where $\delta > 0$ *depend on* γ, n, k*.*

Proof. Let there exists a point $y \in D$ on which $u(y) = \eta_1 \neq 0$, $\eta_1 = const.$ Without loss of generality we may assume that $\eta_1 > 0$. Let $D^+ = \{x : x \in D, u(x) > 0\}$ and D' be a connected component of D^+ that contains the point y. It follows from the maximum principle that this component is an unbounded set, on boundary of which $u(x)$ vanishes. Let $\gamma_0 < p$, then for any $R > 0$ the inclusion $E_{\gamma_0 R}^0(1) \subset E_R^0(9/17)$ takes place. Hence, for any $R \geq 1$ Corollary 2.2 asserts the inequality

$$\sup_{D \cap E_R^0(1)} u(x) \geq (1+\eta) \sup_{D \cap E_{\gamma_0 R}^0(1)} u(x).$$

Let m_0 be a minimal natural number such that $y \in E_{\gamma_0^{-m_0}}^0(1) \cap D'$. Let further, $r > 1$ be arbitrary real number and the natural number $m > m_0$ be such that

$$\gamma_0^{-m} \leq r < \gamma_0^{-m-1},$$

i.e.

$$m \ln \frac{1}{\gamma_0} \leq \ln r < (m+1) \ln \frac{1}{\gamma_0}$$

and hence

$$m > \frac{\ln r}{\ln \frac{1}{\gamma_0}} - 1.$$

We assume also r is so large that

$$\frac{\ln r}{\ln \frac{1}{\gamma_0}} - 1 \geq \frac{\ln r}{2 \ln \frac{1}{\gamma_0}}.$$

Denote $N(r) = \sup\limits_{D' \cap E_r^0(1)} u(x)$. Applying sequentially Corollary 2.2, we get

$$N(r) \geq (1+\eta)^{m-m_0} N(\gamma_0^{-m_0}) \geq (1+\eta)^{m-m_0} \eta_1 = (1+\eta)^m \frac{\eta_1}{(1+\eta)^{m_0}} =$$

$$= (1+\eta)^m \eta_0 \geq \eta_0 (1+\eta)^{\frac{\ln r}{2 \ln \frac{1}{\gamma_0}}} = \eta_0 \eta_2^{\ln r} = \eta_0 \exp(\ln \eta_2 \ln r) =$$
$$= \eta_0 \exp(\ln r^\delta) = \eta_0 r^\delta,$$

where

$$\eta_0 = \frac{\eta_1}{(1+\eta)^{m_0}}, \ \eta_1 = N(\gamma_0^{-m_0}), \ \eta_2 = (1+\eta)^{\frac{1}{2 \ln \frac{1}{\gamma_0}}}, \ \delta = \ln \eta_2.$$

Therefore, for sufficiently large r it holds the inequality

$$\frac{N(r)}{r^\delta} \geq \eta_0.$$

Using the maximum principle, this completes the proof of Theorem 3.1.

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□

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