Proceedings of the Institute of Mathematics and Mechanics,
National Academy of Sciences of Azerbaijan
Volume 48, Number 2, 2022, Pages 179–189
https://doi.org/10.30546/2409-4994.48.2.2022.179

REVERSE INEQUALITIES FOR THE BEREZIN NUMBER OF OPERATORS

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Abstract. For a bounded linear operator A on a reproducing kernel Hilbert space $\mathscr{H}(\Omega)$, with normalized reproducing kernel $\widehat{k}_\lambda = \frac{k_\lambda}{\|k_\lambda\|}$, the Berezin symbol, Berezin number and Berezin norm are defined respectively by $\widetilde{A}(\lambda) = \langle A\widehat{k}_\lambda, \widehat{k}_\lambda\rangle$, $ber(A) = \sup_{\lambda\in\Omega}\left|\widetilde{A}(\lambda)\right|$ and $\|A\|_{ber} = \sup_{\lambda\in\Omega}\left\|A\widehat{k}_\lambda\right\|$. A straightforward comparison between these characteristics yields the inequalities $ber(A) \leq \|A\|_{ber} \leq \|A\|$. In this paper, we prove further inequalities relating them, and give special care to the corresponding reverse inequalities. In particular, we refine the first one of the above inequalities, namely we prove that $ber(A) \leq \left(\|A\|_{ber}^2 - \inf_{\lambda\in\Omega}\left\|(A-\widetilde{A}(\lambda))\widehat{k}_\lambda\right\|^2\right)^{\frac{1}{2}}$.

1. Introduction

A reproducing kernel Hilbert space (RKHS) is a Hilbert space $\mathscr{H} = \mathscr{H}(\Omega)$ of complex valued functions on a (non-empty) set Ω with the property that the evaluation functional $f \to f(\lambda)$ is continuous on $\mathscr{H}$ for every $\lambda \in \Omega$. Then the Riesz representation theorem ensures the existence of a unique element $k_\lambda \in \mathscr{H}$, for each $\lambda \in \Omega$, such that

$$f(\lambda) = \langle f, k_\lambda\rangle \quad \text{for all} \quad f \in \mathscr{H}. \tag{1.1}$$

The function k_λ, $\lambda \in \Omega$, is called the reproducing kernel of $\mathscr{H}$. If $\{e_n\}$ is an orthonormal basis for a RKHS $\mathscr{H}$, then its reproducing kernel is given by $k_\lambda(z) = \sum_n \overline{e_n(\lambda)}e_n(z)$; see Aronzajn [1] and Saitoh and Sowano [15]. For $\lambda \in \Omega$, let $\widehat{k}_\lambda = \frac{k_\lambda}{\|k_\lambda\|}$ be the normalized reproducing kernel of $\mathscr{H}$. For a bounded linear operator $A \in \mathcal{B}(\mathscr{H})$, (with $\mathcal{B}(\mathscr{H})$ being the Banach algebra of all bounded linear operators on $\mathscr{H}$), the function $\widetilde{A} : \Omega \to \mathbb{C}$ defined by

$$\widetilde{A}(\lambda) := \langle A\widehat{k}_\lambda, \widehat{k}_\lambda\rangle \tag{1.2}$$

is the Berezin symbol of A, which was first introduced by Berezin [2, 3]. The Berezin set and the Berezin number of the operator A are defined, respectively,

2010 *Mathematics Subject Classification.* Primary: 47A30; Secondary: 47B20.
Key words and phrases. Berezin symbol, Berezin number, numerical radius, positive operator, hyponormal operator, invertible operator.

by (see [12, 13]):

$$Ber(A) := \text{Range}(\widetilde{A}) = \left\{\widetilde{A}(\lambda) : \lambda \in \Omega\right\}, \tag{1.3}$$

and

$$ber(A) = \sup_{\lambda \in \Omega} \left|\widetilde{A}(\lambda)\right|. \tag{1.4}$$

It is clear that the Berezin symbol $\widetilde{A}$ is a bounded function on Ω and that $Ber(A) \subset W(A)$ and $ber(A) \leq w(A)$ for all $A \in \mathcal{B}(\mathscr{H})$, where

$$W(A) := \{\langle Ax, x\rangle : x \in \mathscr{H} \quad \text{and} \quad \|x\| = 1\} \tag{1.5}$$

is the numerical range of the operator A and

$$w(A) := \sup\{|\langle Ax, x\rangle| : x \in \mathscr{H} \quad \text{and} \quad \|x\| = 1\} \tag{1.6}$$

is the numerical radius of A. It is well known that there are concrete examples of operators for which $Ber(A)$ is a proper subset of $W(A)$ and $ber(A) < w(A)$; and others satisfying $\overline{Ber(A)} = \sigma(A)$, $Ber(A) = W(A)$ and $ber(A) = w(A) = \|A\|$, (see the first author's paper [12]). The Berezin number of an operator A satisfies the following properties:

(i) $ber(A) \leq \|A\|$.
(ii) $ber(\alpha A) = |\alpha| ber(A)$ for all $\alpha \in \mathbb{C}$.
(iii) $ber(A + B) \leq ber(A) + ber(B)$.

Notice that, in general, the Berezin number does not define a norm. However, if $\mathscr{H}$ is a $RKHS$ of analytic functions, (for instance on the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$), then $ber(A)$ defines a norm on $\mathcal{B}(\mathscr{H}(\mathbb{D}))$; which follows from the following lemma (see, for instance, Zhu [16])

Lemma 1.1. *Let $\mathscr{H} = \mathscr{H}(\mathbb{D})$ be a RKHS of analytic functions on $\mathbb{D}$, and let $A \in \mathcal{B}(\mathscr{H})$ be an operator. Then the Berezin symbol $\widetilde{A}$ uniquely defines the operator A, i.e., $A = 0$ if and only if $\widetilde{A} = 0$.*

Now, for any operator A on the RKHS $\mathscr{H} = \mathscr{H}(\Omega)$, let us define the Berezin norm of the operator A by

$$\|A\|_{ber} := \sup_{\lambda \in \Omega} \left\|A\widehat{k}_\lambda\right\|_{\mathscr{H}}. \tag{1.7}$$

Clearly $\|A\|_{ber}$ shares the properties (i)-(iii) with $ber(A)$. Also, since the family $\{k_\lambda : \lambda \in \Omega\}$ is complete in $\mathscr{H}$, it is elementary to verify that $\|A\|_{ber} = 0$ if and only if $A = 0$. So, these properties together mean that $\|A\|_{ber}$ is a norm in $\mathcal{B}(\mathscr{H})$. Clearly $ber(A) \leq \|A\|_{ber}$ for any $A \in \mathcal{B}(\mathscr{H})$. However, it is known that in the case of the unit disk $\mathbb{D}$ these two new operator norms are not equivalent norms with respect to the usual operator norm $\|A\| := \sup\{\|Ax\| : x \in \mathscr{H} \quad \text{and} \quad \|x\| = 1\}$. Namely, Engliš [7] proved that

$$\|T_f\| \leq C \sup_{z \in \mathbb{D}} |\widetilde{T_f}(z)| = C\ ber(T_f),\ \forall f \in L^\infty(\mathbb{D}, dm_2), \tag{1.8}$$

can not hold for any constant $C > 0$, where T_f is the Toeplitz operator on the Bergman Hilbert space $L^2_a = L^2_a(\mathbb{D})$ and dm_2 is the usual normalized area measure on $\mathbb{D}$. Later, Nazarov showed the inequality (see Miao and Zheng [14] Section 6):

$$\|T_f\| \leq C\|T_f\|_{ber}, \quad \forall f \in L^\infty(\mathbb{D}, dm_2), \tag{1.9}$$

can not hold for any constant $C > 0$. These results show that in general there is no universal constants $C_1, C_2 > 0$ such that $\|A\| \leq C_1 ber(A)$ and $\|A\| \leq C_2 \|A\|_{ber}$. Dragomir [5, 6] obtained some elegant reverse inequalities related to the classical numerical radius power inequality

$$w^2(A) \leq w(A^2) + \inf_{\lambda \in \mathbb{C}} \|A - \lambda I\|^2, \quad \text{for } A \in \mathcal{B}(\mathscr{H}(\Omega)). \tag{1.10}$$

In this paper, by using some ideas of [5, 6], we prove several reverse inequalities involving $ber(A)$ and $\|A\|_{ber}$. In particular, we prove some analogue of the inequality (1.10) for the Berezin number of operators. We also discuss some problems related to invertible operators and hyponormal operators on a RKHS.

2. Reverse Berezin number and Berezin norm inequalities for invertible operators

In this section, we prove some new reverse inequalities for the Berezin number and the Berezin norm of two operators A, B on $\mathscr{H}(\Omega)$ with invertible B. Note that similar results for $\|A\|$ and $w(A)$, are proved by Dragomir [5].

Proposition 2.1. *Let $\mathscr{H} = \mathscr{H}(\Omega)$ be a RKHS, and let $A, B \in \mathcal{B}(\mathscr{H})$ be two operators, where B is invertible, satisfying the following inequality for a given $r > 0$:*

$$\|A - B\|_{ber} \leq r. \tag{2.1}$$

Then, we have

$$\|A\|_{ber} \leq \|B^{-1}\| \left[ber(B^*A) + \frac{1}{2} r^2 \right]. \tag{2.2}$$

Proof. Clearly (2.1) is equivalent to the inequality

$$\langle (A - B)\widehat{k}_\lambda, (A - B)\widehat{k}_\lambda \rangle \leq r^2, \tag{2.3}$$

which can be rephrased as

$$\|A\widehat{k}_\lambda\|^2 + \|B\widehat{k}_\lambda\|^2 \leq 2Re\langle B^*A\widehat{k}_\lambda, \widehat{k}_\lambda \rangle + r^2, \tag{2.4}$$

for all $\lambda \in \Omega$. Since B is invertible, we obtain

$$\|Bx\|^2 \geq \frac{1}{\|B^{-1}\|^2} \|x\|^2 \tag{2.5}$$

for all $x \in \mathscr{H}(\Omega)$. In particular, for $x = \widehat{k}_\lambda$, we have that $\|B\widehat{k}_\lambda\|^2 \geq \frac{1}{\|B^{-1}\|^2}$ for all $\lambda \in \Omega$. Now, by considering that $Re\langle B^*A\widehat{k}_\lambda, \widehat{k}_\lambda \rangle \leq |\langle B^*A\widehat{k}_\lambda, \widehat{k}_\lambda \rangle| = |\widetilde{B^*A}(\lambda)|$, we get from (2.4) that

$$\|A\widehat{k}_\lambda\|^2 + \frac{1}{\|B^{-1}\|^2} \leq 2|\widetilde{B^*A}(\lambda)| + r^2 \tag{2.6}$$

for all $\lambda \in \Omega$. Then taking the supremum over $\lambda \in \Omega$ in (2.6), we obtain

$$\|A\|_{ber}^2 + \frac{1}{\|B^{-1}\|^2} \leq 2ber(B^*A) + r^2. \tag{2.7}$$

By the elementary geometric-arithmetic mean inequality, from (2.7) we see that

$$\frac{2\|A\|_{ber}}{\|B^{-1}\|} \leq 2ber(B^*A) + r^2, \tag{2.8}$$

which implies the desired result. ☐

In what follows, we will use the short notation $A-\mu$ instead of $A-\mu I$, where I is the identity operator on $\mathscr{H}(\Omega)$. First, observe that we have the following consequence of Proposition 2.1:

Corollary 2.1. *For an operator $A \in \mathcal{B}(\mathscr{H})$, we have*

(i) $0 \leq \|A\|_{ber} - ber(A) \leq \frac{1}{2|\mu|} r^2$ *provided that* $\|A-\mu\|_{ber} \leq r$.

(ii) $\|A\|_{ber} \leq \|A^{-1}\| \left[ber(A^2) + \frac{1}{2|\mu|} r^2 \right]$ *provided that* $\|A-\mu A^*\|_{ber} \leq r$, $\mu \neq 0$.

It can be easily seen from the proof of Proposition 2.1 that the invertibility condition of the operator B can be replaced by the condition that

$$\widetilde{|B|^2}(\lambda) \geq C \tag{2.9}$$

for all $\lambda \in \Omega$ and for some $C > 0$, where $|B| := (B^*B)^{\frac{1}{2}} = \sqrt{B^*B}$ denotes the modulus (positive part) of the operator B. Namely, we can state without proof the following proposition.

Proposition 2.2. *Let $A, B \in \mathcal{B}(\mathscr{H})$ be such that (2.1) holds, with B satisfying inequality (2.9). Then*

$$\|A\|_{ber} = \sqrt{ber(|A|^2)} \leq \frac{1}{\sqrt{C}} \left[ber(B^*A) + \frac{1}{2} r^2 \right]. \tag{2.10}$$

Proposition 2.3. *Let $A, B \in \mathcal{B}(\mathscr{H}(\Omega))$ be two operators satisfying (2.1) and suppose that B is invertible. Then, we have*

$$\|A\|_{ber} \|B\| \leq ber(B^*A) + \frac{1}{2} \left[r^2 + \|B\|^2 - \|B^{-1}\|^{-2} \right]. \tag{2.11}$$

Proof. As in the proof of Proposition 2.1, the condition (2.1) is equivalent to (2.4), which is in turn equivalent to

$$\|A\widehat{k}_\lambda\|^2 + \|B\|^2 \leq 2Re\langle B^*A\widehat{k}_\lambda, \widehat{k}_\lambda\rangle + r^2 + \|B\|^2 - \|B\widehat{k}_\lambda\|^2. \tag{2.12}$$

Since $Re\langle B^*A\widehat{k}_\lambda, \widehat{k}_\lambda\rangle \leq |\langle B^*A\widehat{k}_\lambda, \widehat{k}_\lambda\rangle|$, $\|B\widehat{k}_\lambda\|^2 \geq \frac{1}{\|B^{-1}\|^2}$ and $\|A\widehat{k}_\lambda\|^2 + \|B\|^2 \geq 2\|B\| \, \|A\widehat{k}_\lambda\|$, for all $\lambda \in \Omega$, using (2.12) we get that

$$2\|B\| \, \|A\widehat{k}_\lambda\| \leq 2|\langle B^*A\widehat{k}_\lambda, \widehat{k}_\lambda\rangle| + r^2 + \|B\|^2 - \|B^{-1}\|^{-2} \tag{2.13}$$

for all $\lambda \in \Omega$. Taking the supremum over $\lambda \in \Omega$, we deduce the required result (2.11). ☐

Note that if, in Proposition 2.3, we choose $B = \mu A^*$, $\mu \neq 0$, and A is invertible, then we get

$$\|A\|_{ber}^2 - ber(A^2) \leq \frac{1}{2} \left[\frac{r^2}{|\mu|} + |\mu| \left(\|A\|^2 - \|A^{-1}\|^{-2} \right) \right], \tag{2.14}$$

provided that $\|A - \mu A^*\| \leq r$.
The following result can be proved using the same argument as in the proof of Proposition 2.3.

Proposition 2.4. *Let $A, B \in \mathcal{B}(\mathscr{H}(\Omega))$. If B is invertible and, for $r > 0$, we have*

$$\|A - B\|_{ber} \leq r < \|B\|, \tag{2.15}$$

then

$$\|A\|_{ber} \leq \frac{1}{\sqrt{\|B\|^2 - r^2}} \left[ber(B^*A) + \frac{1}{2}\left(\|B\|^2 - \|B^{-1}\|^{-2}\right)\right]. \tag{2.16}$$

Remark 2.1.

(a) The result of Proposition 2.4 is of particular interest. Indeed, if we choose $B = \mu I$ with $|\mu| > r$, then (2.15) is obviously fulfilled and by (2.16) we get

$$\|A\|_{ber} \leq \frac{ber(A)}{\sqrt{1 - \left(\frac{r}{|\mu|}\right)^2}}, \tag{2.17}$$

provided that $\|A - \mu I\| \leq r$.

(b) On the other hand, if we choose $B = \mu A^*$ with $\|A\| \geq \frac{r}{|\mu|}$ ($\mu \neq 0$), then by (2.16) we get

$$\|A\|_{ber} \leq \frac{1}{\sqrt{\|A\|^2 - \left(\frac{r}{|\mu|}\right)^2}} \left[ber(A^2) + \frac{|\mu|}{2}\left(\|A\|^2 - \|A^{-1}\|^{-2}\right)\right], \tag{2.18}$$

provided that $\|A - \mu A^*\| \leq r$.

Theorem 2.1. *Let $A, B \in \mathcal{B}(\mathscr{H})$. If B is invertible such that $\|A - B\| \leq r$, for $r > 0$, and*

$$\frac{1}{\sqrt{r^2 + 1}} \leq \|B^{-1}\| < \frac{1}{r}. \tag{2.19}$$

Then

$$\|A\|_{ber}^2 \leq ber^2(B^*A) + 2ber(B^*A)\frac{\|B^{-1}\| - \sqrt{1 - r^2\|B^{-1}\|^2}}{\|B^{-1}\|}. \tag{2.20}$$

Proof. Let $\lambda \in \Omega$. Then by (2.6) we have

$$\|A\widehat{k}_\lambda\|^2 + \frac{1}{\|B^{-1}\|^2} \leq 2|\langle B^*A\widehat{k}_\lambda, \widehat{k}_\lambda\rangle| + r^2, \tag{2.21}$$

and since $\frac{1}{\|B^{-1}\|^2} - r^2 > 0$, we can conclude that $|\langle B^*A\widehat{k}_\lambda, \widehat{k}_\lambda\rangle| > 0$, and thus obtain

$$\frac{\|A\widehat{k}_\lambda\|^2}{|\langle B^*A\widehat{k}_\lambda, \widehat{k}_\lambda\rangle|} \leq 2 + \frac{r^2}{|\langle B^*A\widehat{k}_\lambda, \widehat{k}_\lambda\rangle|} - \frac{1}{\|B^{-1}\|^2|\langle B^*A\widehat{k}_\lambda, \widehat{k}_\lambda\rangle|}. \tag{2.22}$$

Subtracting $|\langle B^*A\widehat{k}_\lambda, \widehat{k}_\lambda\rangle|$ from both sides of (2.22), we obtain

$$\begin{aligned}
\frac{\|A\widehat{k}_\lambda\|^2}{|\langle B^*A\widehat{k}_\lambda, \widehat{k}_\lambda\rangle|} &- |\langle B^*A\widehat{k}_\lambda, \widehat{k}_\lambda\rangle| \leq 2 - |\langle B^*A\widehat{k}_\lambda, \widehat{k}_\lambda\rangle| - \frac{1 - r^2\|B^{-1}\|^2}{\|B^{-1}\|^2|\langle B^*A\widehat{k}_\lambda, \widehat{k}_\lambda\rangle|} \\
&= 2 - \frac{\sqrt{1 - r^2\|B^{-1}\|^2}}{\|B^{-1}\|} \\
&- \left(\sqrt{|\langle B^*A\widehat{k}_\lambda, \widehat{k}_\lambda\rangle|} - \frac{\sqrt{1 - r^2\|B^{-1}\|^2}}{\|B^{-1}\|\sqrt{|\langle B^*A\widehat{k}_\lambda, \widehat{k}_\lambda\rangle|}}\right)^2 \\
&\leq 2\left(\frac{\|B^{-1}\| - \sqrt{1 - r^2\|B^{-1}\|^2}}{\|B^{-1}\|}\right),
\end{aligned}$$

which gives

$$\|A\widehat{k}_\lambda\|^2 \leq |\langle B^*A\widehat{k}_\lambda, \widehat{k}_\lambda\rangle|^2 + 2|\langle B^*A\widehat{k}_\lambda, \widehat{k}_\lambda\rangle|\frac{\|B^{-1}\| - \sqrt{1 - r^2\|B^{-1}\|^2}}{\|B^{-1}\|} \tag{2.23}$$

Notice that (2.19) guaranties the positivity of the nominator of the fraction in the right hand side of (2.23), while taking the supremum in (2.23) over $\lambda \in \Omega$, we deduce the desired inequality (2.20). □

Note that for $\mu \in \mathbb{C}$ with $0 < r \leq |\mu| \leq \sqrt{r^2 + 1}$ and $\|A - \mu I\| \leq r$, we can state that

$$\|A\|_{ber}^2 \leq |\mu|^2 ber(A^2) + 2|\mu|\left(1 - \sqrt{|\mu|^2 - r^2}\right) ber(A). \tag{2.24}$$

Also, if $\|A - A^*\| \leq r$, and A is invertible with $\frac{1}{\sqrt{r^2+1}} \leq \|A^{-1}\| \leq \frac{1}{r}$, then by (2.20) we have

$$\|A\|_{ber}^2 \leq ber^2(A^2) + 2ber(A^2)\frac{\|A^{-1}\| - \sqrt{1 - r^2\|A^{-1}\|^2}}{\|A^{-1}\|}. \tag{2.25}$$

Theorem 2.2. *Let $A, B \in \mathcal{B}(\mathscr{H})$. If B is invertible such that $\|A - B\| \leq r$, for $r > 0$, and $\|B^{-1}\| < \frac{1}{r}$, then*

$$0 \leq \|A\|_{ber}^2\|B\|^2 - ber^2(B^*A) \leq 2ber(B^*A)\frac{\|B\|}{\|B^{-1}\|}\left(\|B\|\ \|B^{-1}\| - \sqrt{1 - r^2\|B^{-1}\|^2}\right). \tag{2.26}$$

Proof. Subtracting the quantity $\frac{|\langle B^*A\widehat{k}_\lambda, \widehat{k}_\lambda\rangle|}{\|B\|^2}$ from both sides of (2.22), we obtain

$$\begin{aligned}
0 &\leq \frac{\|A\widehat{k}_\lambda\|^2}{|\langle B^*A\widehat{k}_\lambda, \widehat{k}_\lambda\rangle|} - \frac{|\langle B^*A\widehat{k}_\lambda, \widehat{k}_\lambda\rangle|}{\|B\|^2} \leq 2 - \frac{|\langle B^*A\widehat{k}_\lambda, \widehat{k}_\lambda\rangle|}{\|B\|^2} - \frac{1 - r^2\|B^{-1}\|^2}{\|B^{-1}\|^2|\langle B^*A\widehat{k}_\lambda, \widehat{k}_\lambda\rangle|} \\
&= 2 - 2\frac{\sqrt{1 - r^2\|B^{-1}\|^2}}{\|B\|\ \|B^{-1}\|} - \left(\frac{\sqrt{|\langle B^*A\widehat{k}_\lambda, \widehat{k}_\lambda\rangle|}}{\|B\|} - \frac{\sqrt{1 - r^2\|B^{-1}\|^2}}{\|B^{-1}\|\sqrt{|\langle B^*A\widehat{k}_\lambda, \widehat{k}_\lambda\rangle|}}\right)^2 \\
&\leq 2\left(\frac{\|B\|\ \|B^{-1}\| - \sqrt{1 - r^2\|B^{-1}\|^2}}{\|B\|\ \|B^{-1}\|}\right),
\end{aligned}$$

which is equivalent to

$$0 \leq \|A\|_{ber}^2\|B\|^2 - |\widetilde{B^*A}(\lambda)|^2 \leq 2\frac{\|B\|}{\|B^{-1}\|}|\widetilde{B^*A}(\lambda)|\left(\|B\|\,\|B^{-1}\| - \sqrt{1-r^2\|B^{-1}\|^2}\right), \tag{2.27}$$

for all $\lambda \in \Omega$. The inequality (2.27) also shows that $\|B\|\,\|B^{-1}\| \geq \sqrt{1-r^2\|B^{-1}\|^2}$, and then, by (2.27), we get

$$\|A\|_{ber}^2\|B\|^2 \leq |\widetilde{B^*A}(\lambda)|^2 + 2\frac{\|B\|}{\|B^{-1}\|}|\widetilde{B^*A}(\lambda)|\left(\|B\|\,\|B^{-1}\| - \sqrt{1-r^2\|B^{-1}\|^2}\right), \tag{2.28}$$

for all $\lambda \in \Omega$. Thus, taking the supremum in (2.28) we deduce the desired inequality (2.26). □

As mentioned above, Theorem 2.2 is of particular interest since putting $B = \mu I$ with $|\mu| \geq r > 0$ and assuming that $\|A - \mu I\| \leq r$, then by inequality (2.28) we get

$$0 \leq \|A\|_{ber}^2 - ber(A^2) \leq 2|\mu| ber(A)\left(1 - \sqrt{1 - \left(\frac{r}{|\mu|}\right)^2}\right). \tag{2.29}$$

Also, if A is invertible and $\|A - \mu A^*\| \leq r$ and $\|A^{-1}\| \leq \frac{|\mu|}{r}$, then by (2.26) we obtain

$$0 \leq \|A\|_{ber}^4 - ber^2(A^2) \leq 2|\mu| ber(A^2)\frac{\|A\|}{\|A^{-1}\|}\left(\|A\|\,\|A^{-1}\| - \sqrt{1 - \frac{r^2}{|\mu|^2}\|A^{-1}\|^2}\right). \tag{2.30}$$

3. Berezin number and Berezin norm inequalities for some operators

A bounded operator T acting on a complex infinite dimensional Hilbert space H is said to be normal if $T^*T = TT^*$; it is said to be positive if $\langle Tx, x\rangle \geq 0$ for all $x \in H$ (in this case we will write $T \geq 0$); and it is said to be hyponormal if its self-commutator $[T^*, T]$ is positive, that is $T^*T - TT^* \geq 0$. It is immediate from these definitions that every normal operator is hyponormal and that an operator T is hyponormal if and only if $\|T^*x\| \leq \|Tx\|$ for all $x \in H$. It is also obvious that every nonunitary isometry $V : H \to H$ (i.e. $V^*V = I$ and $VV^* \neq I$) is hyponormal, but not normal.
In this section, we will consider some operators, including hyponormal operators, and prove some inequalities for their Berezin numbers and Berezin norms.

Proposition 3.1. *Let $N \in \mathcal{B}(\mathscr{H}(\Omega))$ be a normal operator on the RKHS $\mathscr{H}(\Omega)$, and let $n \geq 1$ be any integer. Then*

$$ber\,(N^n) \leq \begin{cases} \|N^{\frac{n}{2}}\|_{ber} & if \quad n \quad is\ even, \\ \|N\|\,\|N^{\frac{n-1}{2}}\|_{ber} & if \quad n \quad is\ odd. \end{cases} \tag{3.1}$$

Proof. For any fixed integer $k \geq 0$, we have

$$\begin{aligned} |\langle N^{2k}\widehat{k}_\lambda, \widehat{k}_\mu\rangle| &= |\langle N^k\widehat{k}_\lambda, N^{k*}\widehat{k}_\mu\rangle| \leq \|N^k\widehat{k}_\lambda\| \, \|N^{k*}\widehat{k}_\mu\| \\ &\leq \sup_{\lambda\in\Omega}\|N^k\widehat{k}_\lambda\| \sup_{\mu\in\Omega}\|N^k\widehat{k}_\mu\| = \|N^k\|_{ber}^2 \end{aligned}$$

for all $\lambda, \mu \in \Omega$. So, we get

$$|\langle N^{2k}\widehat{k}_\lambda, \widehat{k}_\mu\rangle| \leq \|N^k\|_{ber}^2 \quad \forall k \geq 0. \tag{3.2}$$

In particular, for $\mu = \lambda$, we obtain

$$ber(N^{2k}) \leq \|N^k\|_{ber}^2 \quad \forall k \geq 0. \tag{3.3}$$

For $k = 1$, we have from (3.3) that

$$ber(N^2) \leq \|N\|_{ber}^2 \tag{3.4}$$

for any normal operator N in $\mathcal{B}(\mathscr{H}(\Omega)$.
On the other hand, we have

$$\begin{aligned} |\langle N^{2k+1}\widehat{k}_\lambda, \widehat{k}_\mu\rangle| &= |\langle N^{2k}N\widehat{k}_\lambda, \widehat{k}_\mu\rangle| = |\langle N^kN\widehat{k}_\lambda, N^{k*}\widehat{k}_\mu\rangle| \leq \|N^kN\widehat{k}_\lambda\| \, \|N^{k*}\widehat{k}_\mu\| \\ &\leq \sup_{\lambda\in\Omega}\|N^kN\widehat{k}_\lambda\| \sup_{\mu\in\Omega}\|N^k\widehat{k}_\mu\| \leq \|N\| \, \|N^k\|_{ber}^2 \end{aligned}$$

for all $\lambda, \mu \in \Omega$. In particular, by putting $\mu = \lambda$ we have from the last inequality that

$$ber(N^{2k+1}) \leq \|N\| \, \|N^k\|_{ber}^2 \quad \forall k \geq 0. \tag{3.5}$$

Now, the desired result follows from (3.3) and (3.5), which proves the proposition. □

The following result gives the exact relationship between the Berezin number and the Berezin norm of a concrete orthogonal projection (normal operator).

Example 3.1. Let S be the shift operator, $Sf = zf$, on the Hardy space $H^2(\mathbb{D})$ over the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, which consists of analytic functions on $\mathbb{D}$ having the sequence of Taylor coefficients belonging to the space l^2. Consider the operator A defined on $H^2(\mathbb{D})$ by

$$A = S(I - SS^*)S^*. \tag{3.6}$$

Then, $ber(A) = \|A\|_{ber}^2 = \frac{1}{4}$.

Proof. First, note that $\widehat{k}_\lambda(z) = \frac{(1-|\lambda|^2)^{\frac{1}{2}}}{1-\overline{\lambda}z}$ $(\lambda \in \mathbb{D})$ is the normalized reproducing kernel of $H^2(\mathbb{D})$. Then, for all $\lambda \in \mathbb{D}$, we have

$$\begin{aligned} \|A\widehat{k}_\lambda\| &= \|S(I - SS^*)S^*\widehat{k}_\lambda\| = \|(I - SS^*)\overline{\lambda}\widehat{k}_\lambda\| \\ &= |\lambda| \left\| \frac{(1-|\lambda|^2)^{\frac{1}{2}}}{1-\overline{\lambda}z} - \frac{\overline{\lambda}z(1-|\lambda|^2)^{\frac{1}{2}}}{1-\overline{\lambda}z} \right\| = |\lambda|(1-|\lambda|^2)^{\frac{1}{2}} \end{aligned} \tag{3.7}$$

On the other hand, for all $\lambda \in \mathbb{D}$, we have

$$\begin{aligned} \widetilde{A}(\lambda) &= \langle S(I - SS^*)S^*\widehat{k}_\lambda, \widehat{k}_\lambda\rangle = \langle (I - SS^*)S^*\widehat{k}_\lambda, S^*\widehat{k}_\lambda\rangle \\ &= |\lambda|^2\langle (I - SS^*)\widehat{k}_\lambda, \widehat{k}_\lambda\rangle = |\lambda|^2(1-|\lambda|^2) \end{aligned} \tag{3.8}$$

Thus, we get that $\widetilde{A}(\lambda) = \|A\widehat{k}_\lambda\|^2$ for all $\lambda \in \mathbb{D}$. This implies that one has $ber(A) = \|A\|_{ber}^2$. Also, by using that for the function $f(x) = x(1-x)$, $(0 \leq$

$x < 1$), $f_{\max} = f(\frac{1}{2})$, we deduce from the formula $\widetilde{A}(\lambda) = |\lambda|^2(1-|\lambda|^2)$ that $\sup_{\lambda\in\mathbb{D}}(\widetilde{A}(\lambda)) = \frac{1}{4}$, that is $ber(A) = \frac{1}{4}$. Hence $ber(A) = \|A\|_{ber}^2 = \frac{1}{4}$, as desired. □

For any hyponormal operator $T \in \mathcal{B}(\mathscr{H}(\Omega))$ it is easy to show that:

(i) $\|T^*\|_{ber} \leq \|T\|_{ber}$;

(ii)

$$ber\left([T^*, T]\right) \leq \|T\|_{ber}^2. \tag{3.9}$$

In fact, for any $\lambda \in \Omega$, we have that

$$0 \leq \widetilde{[T^*, T]}(\lambda) = \langle (T^*T - TT^*)\widehat{k}_\lambda, \widehat{k}_\lambda\rangle = \|T\widehat{k}_\lambda\|^2 - \|T^*\widehat{k}_\lambda\|^2 \tag{3.10}$$

From this, it is immediate that $\|T^*\|_{ber} \leq \|T\|_{ber}$. Since T is hyponormal, we have $\|T\widehat{k}_\lambda\| \geq \|T^*\widehat{k}_\lambda\|$, and hence we deduce from (3.10) $\widetilde{[T^*, T]}(\lambda) \leq \|T\widehat{k}_\lambda\|^2$ for all $\lambda \in \Omega$. Therefore, we get $ber\left([T^*, T]\right) \leq \|T\|_{ber}^2$, which proves inequality (3.9). Our next result is about more general operators.

Theorem 3.1. *Let $A \in \mathcal{B}(\mathscr{H}(\Omega))$. Then*

$$ber(A) \leq \left(\|A\|_{ber}^2 - \inf_{\mu\in\Omega}\left\|(A - \widetilde{A}(\mu))\widehat{k}_\mu\right\|^2\right)^{\frac{1}{2}}. \tag{3.11}$$

Proof. It is easy to see that

$$\left\|(A - \widetilde{A}(\lambda))\widehat{k}_\lambda\right\|^2 = \left\|A\widehat{k}_\lambda\right\|^2 - \left|\widetilde{A}(\lambda)\right|^2, \quad \forall\lambda \in \Omega. \tag{3.12}$$

Then

$$\left|\widetilde{A}(\lambda)\right|^2 = \left\|A\widehat{k}_\lambda\right\|^2 - \left\|(A - \widetilde{A}(\lambda))\widehat{k}_\lambda\right\|^2, \quad \forall\lambda \in \Omega. \tag{3.13}$$

Whence

$$\begin{aligned}\left|\widetilde{A}(\lambda)\right|^2 &\leq \left(\left\|A\widehat{k}_\lambda\right\|^2 - \inf_{\mu\in\Omega}\left\|(A - \widetilde{A}(\mu))\widehat{k}_\mu\right\|^2\right)^{\frac{1}{2}} \\ &\leq \left(\sup_{\lambda\in\Omega}\left\|A\widehat{k}_\lambda\right\|^2 - \inf_{\mu\in\Omega}\left\|(A - \widetilde{A}(\mu))\widehat{k}_\mu\right\|^2\right)^{\frac{1}{2}} \end{aligned} \tag{3.14}$$

for all $\lambda \in \Omega$, which implies the desired inequality (3.11). □

The next corollary gives an example of operators for which $ber(A) < \|A\|_{ber}$.

Corollary 3.1. *If* $\inf_{\mu\in\Omega}\left\|(A - \widetilde{A}(\mu))\widehat{k}_\mu\right\| > 0$, *then* $ber(A) < \|A\|_{ber}$.

In conclusion, note that conditions of the type

$$\left\|(A - \widetilde{A}(\lambda))\widehat{k}_\lambda\right\| \to 0 \quad \text{and} \quad \left\|(A - \widetilde{A}(\mu))^*\widehat{k}_\mu\right\| \to 0 \tag{3.15}$$

as $\lambda \to \partial\Omega$ define the so-called Engliš C*-operator algebras which are studied, for instance, in [8] and [11], and are closely related to an unsolved question of Engliš [8, Question 1] for Bergman space Toeplitz operators, where, in particular, he asked the following question:
Is it true that

$$T_\varphi \in \mathscr{A}_B :=$$

$$\left\{T \in \mathcal{B}(L_a^2(\mathbb{D})) : \left\|T\widehat{k}_{a,\lambda}\right\|^2 - \left|\widetilde{T}(\lambda)\right|^2 \to 0 \quad \text{radially, and similarly for } T^*\right\}, \tag{3.16}$$

for all $\varphi \in L^\infty(\mathbb{D})$?
Here, $\widehat{k}_{a,\lambda} = \frac{(1-|\lambda|^2)^2}{(1-\bar{\lambda}z)^2}$ is the normalized reproducing kernel of $L_a^2(\mathbb{D})$. In [9, 10], the authors discuss this question via the so-called maximal Berezin set and the Berezin number.

Acknowledgements. This project was funded by the National Plan for Science, Technology and Innovation (MAARIFAH), King Abdulaziz City for Science and Technology, Kingdom of Saudi Arabia, award number 13-MAT1276-02.

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Received: April 21, 2022; Accepted: May 31, 2022