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NON-COERCIVE SOLVABILITY OF SOME BOUNDARY VALUE PROBLEMS FOR SECOND ORDER ELLIPTIC DIFFERENTIAL-OPERATOR EQUATIONS WITH QUADRATIC COMPLEX PARAMETER

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Abstract. In a separable Hilbert space H, solvability of boundary value problems for a second order elliptic differential-operator equation with quadratic complex parameter is investigated. The complex parameter enters linearly into a boundary condition and the boundary conditions are non-separable. An application of the obtained abstract results to elliptic boundary value problems is given.

1. Introduction

In the monograph by S.Yakubov and Ya.Yakubov [14, chapter 5, section 5.4] (see also S.Yakubov [13], [15]), in a separable Hilbert space H, solvability of the following boundary value problem for second order elliptic differential-operator equations was investigated:

$$L(\lambda)u := \lambda u(x) - u''(x) + Au(x) = f(x), \quad x \in (0,1)\,, \tag{1.1}$$

$$L_k u := \alpha_k u^{(m_k)}(0) + \beta_k u^{(m_k)}(1) = f_k, \;\; k = 1,2, \tag{1.2}$$

where λ is a complex parameter; A is a linear closed operator with dense domain $D(A)$ in H and with resolvent decreasing as $|\lambda|^{-1}$ for large enough λ from some angles containing the negative semiaxis; $m_k \in \{0,1\}$; α_k, β_k are complex numbers which satisfy $(-1)^{m_1}\alpha_1\beta_2 - (-1)^{m_2}\alpha_2\,\beta_1 \neq 0$. It was proved that for large enough λ from the angles $|\arg\lambda| \leq \varphi < \pi$, an isomorphism theorem for the problem (1.1), (1.2) takes place between the solution of the problem, belonging to $W_p^2\left((0,1)\,;H(A),H\right)$, and the right-hand side of the problem, belonging to $L_p(0,1);H) \dot{+} (H(A),H)_{\theta_1,p} \dot{+} (H(A),H)_{\theta_2,p}$, where $\theta_k = (H(A),H)_{\frac{m_k}{2}+\frac{1}{2},p}$, $k=1,2$, $p \in (1,\infty)$. It was also established some estimate for the solution of the problem (1.1), (1.2) (with respect to u and λ) in the space $L_p\left((0,1)\,;H\right)$, $1 < p < \infty$. In this case, we say that the problem (1.1), (1.2) is coercive solvable in the space $L_p\left((0,1)\,;H\right)$ with respect to u. In fact, it is maximal L_p-regularity. The corresponding established estimate is called the coercive estimate.

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Further, solvability of boundary value problems for second order elliptic differential - operator equations with a complex parameter has been investigated in [1-9][11], [16] and others in various cases:
a) boundary conditions may contain linear unbounded operators;
b) the complex parameter, entering into the equation (1.1), may appear in the boundary conditions, as well;
c) boundary conditions may contain, in addition to the complex parameter, linear bounded (or unbounded) operators.

Let us mention some works related to our this paper by problem's formulation.

In [4], in a separable Hilbert space H, solbability of the following problem with separated boundary conditions was studied

$$L\left(\lambda\right)u := \lambda^2 u(x) - u''(x) + Au(x) = f(x),\ x \in \left(0,1\right), \tag{1.3}$$

$$\begin{aligned} L_1(\lambda)u &:= u'(1) + \lambda u(1) = f_1, \\ L_2 u &:= u(0) = f_2, \end{aligned} \tag{1.4}$$

where λ is a complex parameter; A is a linear unbounded selfadjoint, positive-definite operator in H. It was proved that for large enough λ from some angle $|\arg\lambda| \le \varphi < \frac{\pi}{2}$, the problem (1.3), (1.4) is coercive solvable (i.e., there is maximal L_p-regularity) in the space $L_p\left((0,1)\,;H\right),\ p \in (1,+\infty)$. Note that the coercive solvability of (1.3), (1.4) takes place also if the operator A in (1.3) is taken from a more wide class of operators in H, so-called, φ -positive operators in H (the definition of φ-positive operators is given below).

In [7], solvability of the following boundary value problem in H was studied

$$L\left(\lambda\right)u := \lambda^2 u(x) - u''(x) + Au(x) = f(x),\ x \in \left(0,1\right), \tag{1.5}$$

$$\begin{aligned} L_1\left(\lambda\right)u &:= u'\left(1\right) + \lambda Bu\left(0\right) = f_1, \\ L_2 u &:= u'\left(0\right) = f_2, \end{aligned} \tag{1.6}$$

where A is a φ-positive operator in H; B is a linear bounded or unbounded operator in H. It was proved that if in (1.6) the operator B is linear bounded in the spaces H and $H(A)$ (in particular, B can be the identical operator), then for large enough λ from some angle $|\arg\lambda| \le \varphi < \frac{\pi}{2}$ the problem (1.5), (1.6) is coercive solvable (i.e., there is maximal L_p-regularity) with respect to u in the space $L_p\left((0,1)\,;H\right)$, $p \in (1,+\infty)$.

Let us also mention [5], where solvability of boundary value problems for the equation (1.5) with the following non-separated boundary conditions was studied:

$$L_1\left(\lambda\right)u := u'\left(0\right) + \lambda u\left(1\right) = f_1,\ L_2\left(\lambda\right)u := u'\left(1\right) + \lambda u\left(0\right) = f_2. \tag{1.7}$$

It was proved that for large enough λ from some angle $|\arg\lambda| \le \varphi < \frac{\pi}{2}$, the coercive solvability in the space $L_p\left((0,1)\,;H\right)$, $p \in (1,\infty)$ holds for the problem (1.5), (1.7) (again, we have maximal L_p-regularity).

In this paper, in a separable Hilbert space H, solvability of a boundary value problem for the equation (1.5) with the following non-separated boundary conditions is treated:

$$L_1\left(\lambda\right)u := u'\left(1\right) + \lambda u\left(0\right) = f_1, \quad L_2 u := u\left(1\right) = f_2, \tag{1.8}$$

where λ is a complex parameter; A is a φ–positive operator in H. Very small modifications in (1.4) or (1.6) and so "dramatical" changes in results! It is proved that for large enough λ from some angle $|\arg\lambda| \le \varphi < \frac{\pi}{2}$, the problem (1.5), (1.8) is non-coercive solvable with respect to u in the space $L_p\left((0,1)\,;H\right)$ (i.e., there is no maximal L_p-regularity!. Non-coercivity of the problem (1.5), (1.8) is characterized by the following fact. When we look for a solution of the problem (1.5), (1.8), belonging to $W_p^2\left((0,1)\,;\,H(A),H\right)$, the elements f_1 and f_2 cannot be taken from the natural interpolation spaces $(H(A),\ H)_{\frac{1}{2}+\frac{1}{2p},p}$ and $(H(A),\ H)_{\frac{1}{2p},p}$, respectively, that follows from the trace theorem. The elements are taken from more narrow interpolation spaces $\left(H(A^2),\ H\right)_{\frac{1}{4}+\frac{1}{4p},p}$ and $\left(H(A^2),\ H\right)_{\frac{1}{4p},p}$, respectively. By this reason, one cannot take the function $f(x)$ from the space $L_p\left((0,1);\ H\right)$. It is needed to take the function from a more narrow space, namely, from $L_p\left((0,1); H\left(A\right)\right),\ p\in(1,\infty)$. As a result, there is no an isomoprhism between the solution, belonging to $W_p^2((0,1);\ H(A),H)$, and the right-hand side of the problem (1.5) , (1.8). So, for the solution of the problem (1.5), (1.8), some non-coercive estimate in the space $L_p\left((0,1);\ H\right),\ p\in(1,\infty)$ is established.

Non-coercive phenomena was previously discovered for some problems (see [11], [7]). In [11], the reason of that was in irregular boundary conditions. In [7], if the operator B in the problem (1.5), (1.6) is linear unbounded and subordinate to the operator $A^{1/2}$ in some sense, then the problem (1.5), (1.6) becomes non-coercive solvable in the space $L_p\left((0,1)\,;H\right)$, $p\in(1,\infty)$. In [11] it was also constructed an example showing that a boundary value problem for the equation (1.1) with irregular boundary conditions does not have a solution from $W_p^2((0,1);\ H(A),H)$ for some concrete $f\in L_p\left((0,1)\,;H\right)$.

The abstract results, obtained in the present paper, allow us to investigate non-coercive solvability for a new class of boundary value problems for second order elliptic partial differential equations in non-smooth domains. At the end of the paper, one such application is shown for elliptic equations in a square.

Let us introduce definitions and notions used in the paper.

Let E_1 and E_2 be Banach spaces. The set $E_1\dot{+}E_2$ of all vectors of the form (u,v), where $u\in E_1,\ v\in E_2$ with ordinary coordinate-linear operations and with the norm

$$\|(u,v)\|_{E_1\dot{+}E_2} := \|u\|_{E_1} + \|v\|_{E_2}$$

is a Banach space and is called the direct sum of Banach spaces E_1 and E_2.

By $B\left(E_1,E_2\right)$ we denote a Banach space of all linear bounded operators acting from E_1 into E_2 with standard operator norm. In particular $B(E_1) := B(E_1,E_1)$.

Definition 1.1. A linear closed operator A in a Hilbert space H is called φ-positive if its domain $D\left(A\right)$ is dense in H and for some $\varphi\in[0,\pi)$, for all points from the angle $|\arg\mu|\le\varphi$ (including $\mu=0$), there exist the operators $\left(A+\mu I\right)^{-1}$ for which the estimate takes place

$$\left\|\left(A+\mu I\right)^{-1}\right\|_{B(H)} \le C\left(1+|\mu|\right)^{-1},\quad |\arg\mu|\le\varphi,$$

where I is the identity operator in H, $C = const > 0$. If $\varphi = 0$ then the operator A is called positive.

Note that if A is φ-positive then A^α, $\alpha\in(0,1)$ is also φ-positive. A simple example of a φ-positive operator is a selfadjoint, positive-definite operator in a Hilbert space.

Let A be φ-positive operator in H. Since the inverse operator A^{-1} is bounded in H, then

$$H\left(A^n\right) := \left\{u: \ u \in D\left(A^n\right), \|u\|_{H(A^n)} = \|A^n u\|_H\right\}, \ n \in N$$

is a Hilbert space the norm of which is equivalent to the graph norm of A^n.

It is also known that $-A$ is a generator of an analytic for $t > 0$ semigroup e^{-tA}, which exponentially decreasing, i.e., there exist $C > 0$, $\delta_0 > 0$ such that $\left\|e^{-tA}\right\| \leq Ce^{-\delta_0 t}$, $0 \leq t < +\infty$.

As it was shown in the monograph by H. Triebel [15], there are different but equivalent definitions of interpolation spaces. One of definitions of interpolation spaces of two Banach spaces is given using the theory of analytic semigroups which is useful in the theory of differential-operators equations

Let E_0 and E_1 be two Banach spaces continuously embedded into a Banach space $E : E_0 \subset E, E_1 \subset E$. Two such spaces are called an interpolation pair and it is denoted by $\{E_0, E\}$.

Definition 1.2. (see [12, theorem 1.14.5]). Let A be φ-positive operator in H. Then interpolation spaces $\left(H\left(A^n\right), H\right)_{\theta,p}$ of Hilbert spaces $H\left(A^n\right)$ and H are defined by the equality

$$(H(A^n),\, H)_{\theta,p} := \Bigg\{u : u \in H,\ \|u\|_{(H(A^n),H)_{\theta,p}} :=$$
$$= \left(\int_0^{+\infty} t^{-1+n\theta p} \left\|A^n e^{-tA} u\right\|_H^p dt\right)^{\frac{1}{p}} < \infty, \Bigg\},\quad \theta \in (0,1)\,,\ p > 1,\ n \in N.$$

Define also $(H(A^n),\, H)_{0,p} := H(A^n)$ and $(H(A^n),\, H)_{1,p} := H$.

By $L_p\left((0,1)\,; H\right)$ $1 < p < \infty$, denote a Banach space (for $p = 2$, a Hilbert space) of vector-functions $x \to u\left(x\right) : [0,1] \to H$, strongly measurable and summable in p-th power with the norm

$$\|u\|_{L_p((0,1);H)} := \left(\int_0^1 \|u\left(x\right)\|_H^p \,dx\right)^{1/p} < \infty.$$

In accordance, by $W_p^{2n}\left((0,1)\,; H\left(A^n\right), H\right) := \left\{u : A^n u,\ u^{(2n)} \in L_p\left((0,1)\,; H\right)\right\}$ denote a Banach space (for $p = 2$, a Hilbert space) of vector-functions with norm

$$\|u\|_{W_p^{2n}((0,1);H(A^n),H)} := \|A^n u\|_{L_p((0,1);H)} + \left\|u^{(2n)}\right\|_{L_p((0,1);H)} < \infty.$$

By Ff and $F^{-1}f$ denote the Fourier transform and the inverse Fourier transform, respectively, of the function f from $L_p\left(R; H\right)$, $R = (-\infty, +\infty)$:

$$Ff := \left(Ff\right)\left(\sigma\right) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-i\sigma x} f(x) dx,$$

$$F^{-1}f := \left(F^{-1}f\right)\left(x\right) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{i\sigma x} f(\sigma) d\sigma\,.$$

Definition 1.3. The mapping $\sigma \to T(\sigma) : R \to B(H)$ is called a Fourier multiplier in the space $L_p(R;H)$ if $\exists C > 0$ such that

$$\left\|F^{-1}TFf\right\|_{L_p(R;H)} \le C\,\|f\|_{L_p(R;H)}\,,\ \forall f \in L_p(R;H)\,.$$

2. Non-coercive solvability in the case when a non-separated boundary condition contains the spectral parameter.

First, consider in a separable Hilbert space H the following boundary value problem:

$$L(\lambda)\,u := \lambda^2 u(x) - u''(x) + Au(x) = 0,\ \ x \in (0,\,1)\,, \tag{2.1}$$

$$\begin{array}{l} L_1(\lambda)\,u := u'(1) + \lambda u(0) = f_1\,, \\ L_2 u := u(1) = f_2, \end{array} \tag{2.2}$$

where λ is a complex parameter; A is a $\varphi-$positive operator in H.

Theorem 2.1. *Let A be φ-positive operator in H with $\varphi \in [0, \frac{\pi}{2})$.*

Then, for $f_k \in \left(H(A^2), H\right)_{\frac12-\frac{k}{4}+\frac{1}{4p},p}$, $p \in (1,+\infty)$, $k = 1,2$ and for large enough λ from the angle $|\arg\lambda| \le \varphi < \frac{\pi}{2}$, the problem (2.1), (2.2) *has a unique solution $u \in W_p^2((0,1);H(A),H)$ and for the solution the following non-coercive estimate is satisfied*

$$|\lambda|^2\,\|u\|_{L_p((0,1);H)} + \left\|u''\right\|_{L_p((0,1);H)} + \|Au\|_{L_p((0,1);H)} \le$$

$$\le C\sum_{k=1}^{2}\left(\|f_k\|_{(H(A^2),H)_{\frac12-\frac{k}{4}+\frac{1}{4p},p}} + |\lambda|^{2+k-\frac1p}\,\|f_k\|_H\right),$$

where $C > 0$ is a constant which does not depend on λ.

Proof. By virtue of [14, lemma 5.4.2/6], under $|\arg\lambda| \le \varphi < \frac{\pi}{2}$, there exists an analytic for $x > 0$ and strongly continuous for $x \ge 0$ semigroup $e^{-x(A+\lambda^2 I)^{1/2}}$. By virtue of [14, lemma 5.3.2/1], for $u(x)$ to be a solution of the equation (2.1), belonging to $W_p^2((0,1);H(A),H)$, $p \in (1,\infty)$, it is necessary and sufficient that, under $|\arg\lambda| \le \varphi < \frac{\pi}{2}$,

$$u(x) = e^{-x(A+\lambda^2 I)^{1/2}}g_1 + e^{-(1-x)(A+\lambda^2 I)^{1/2}}g_2, \tag{2.3}$$

where $g_1, g_2 \in (H(A),H)_{\frac{1}{2p},p}$.

Claim that a function of the form (2.3) satisfies the boundary conditions in (2.2). Then, for g_1 and g_2, one gets the following system

$$\left[-(A+\lambda^2 I)^{1/2}e^{-(A+\lambda^2 I)^{1/2}} + \lambda I\right]g_1 + \left[(A+\lambda^2 I)^{1/2} + \lambda e^{-(A+\lambda^2 I)^{1/2}}\right]g_2 = f_1,$$

$$e^{-(A+\lambda^2 I)^{1/2}}g_1 + g_2 = f_2. \tag{2.4}$$

Define $\vartheta_1 := (A+\lambda^2 I)^{1/2}g_1$, $\vartheta_2 := (A+\lambda^2 I)^{1/2}g_2$. Then, from (2.4), we have

$$\left[-e^{-(A+\lambda^2 I)^{1/2}} + \lambda(A+\lambda^2 I)^{-1/2}\right]\vartheta_1 + \left[I + \lambda(A+\lambda^2 I)^{-1/2}e^{-(A+\lambda^2 I)^{1/2}}\right]\vartheta_2 = f_1,$$

$$(A+\lambda^2 I)^{-1/2}e^{-(A+\lambda^2 I)^{1/2}}\vartheta_1 + (A+\lambda^2 I)^{-1/2}\vartheta_2 = f_2. \tag{2.5}$$

The coefficients of the system (2.5) are linear combinations of bounded operators which commute each of other I, $(A+\lambda^2 I)^{-1/2}$, $e^{-\left(A+\lambda^2 I\right)^{1/2}}$, $(A+\lambda^2 I)^{-1/2}e^{-(A+\lambda^2 I)^{1/2}}$. Therefore, the system (2.5) one can solve like in a scalar case. The "determinant" of the system (2.5) has the form

$$D(\lambda)=\lambda(A+\lambda^2 I)^{-1}\left[I-\lambda^{-1}(A+\lambda^2 I)R(\lambda)\right],$$

where

$$R(\lambda):=\lambda(A+\lambda^2 I)^{-1}e^{-2(A+\lambda^2 I)^{1/2}}+2(A+\lambda^2 I)^{-1/2}e^{-(A+\lambda^2 I)^{1/2}}.$$

By virtue of [14, lemma 5.4.2/6], $\left\|\lambda^{-1}(A+\lambda^2 I)R(\lambda)\right\|_{B(H)}\to 0$ for $|\arg\lambda|\le\varphi<\frac{\pi}{2}$, $|\lambda|\to\infty$. Hence, for large enough λ from the angle $|\arg\lambda|\le\varphi<\frac{\pi}{2}$, the system (2.5) has a unique solution with respect to $\vartheta_k,\ k=1,2$, and

$$\vartheta_1=\left[\lambda^{-1}(A+\lambda^2 I)^{1/2}+R_{11}(\lambda)\right]f_1+\left[-\lambda^{-1}(A+\lambda^2 I)+R_{21}(\lambda)\right]f_2,$$

$$\vartheta_2=R_{12}(\lambda)f_1+\left[(A+\lambda^2 I)^{1/2}+R_{22}(\lambda)\right]f_2,$$

where $\|R_{jk}(\lambda)\|_{B(H)}\to 0,\ j,k=1,2,$ under $|\arg\lambda|\le\varphi, |\lambda|\to\infty$.

Hence,

$$\begin{aligned} g_1&=\left[\lambda^{-1}I+(A+\lambda^2 I)^{-1/2}R_{11}(\lambda)\right]f_1+\\ &+\left[-\lambda^{-1}(A+\lambda^2 I)^{1/2}+(A+\lambda^2 I)^{-1/2}R_{21}(\lambda)\right]f_2, \\ g_2&=(A+\lambda^2 I)^{-1/2}R_{12}(\lambda)f_1+\left[I+(A+\lambda^2 I)^{-1/2}R_{22}(\lambda)\right]f_2. \end{aligned} \tag{2.6}$$

Substituting (2.6) into (2.3), we have

$$\begin{aligned} u(x)&=e^{-x(A+\lambda^2 I)^{1/2}}\left[\left(\lambda^{-1}I+(A+\lambda^2 I)^{-1/2}R_{11}(\lambda)\right)f_1+\right.\\ &\left.+\left(-\lambda^{-1}(A+\lambda^2 I)^{1/2}+(A+\lambda^2 I)^{-1/2}R_{21}(\lambda)\right)f_2\right]+\\ &+e^{-(1-x)(A+\lambda^2 I)^{1/2}}\left[(A+\lambda^2 I)^{-1/2}R_{12}(\lambda)f_1+\left(I+(A+\lambda^2 I)^{-1/2}R_{22}(\lambda)\right)f_2\right]. \end{aligned}$$

Then, using the Minkovskii inequality for large enough λ from the angle $|\arg\lambda|\le\varphi<\frac{\pi}{2}$, we have

$$|\lambda|^2\|u\|_{L_p((0,1);H)}+\|u''\|_{L_p((0,1);H)}+\|Au\|_{L_p((0,1);H)}\le$$

$$\le|\lambda|^2\left[\left(\int_0^1\left\|e^{-x(A+\lambda^2 I)^{1/2}}\lambda^{-1}f_1\right\|_H^p dx\right)^{1/p}+\right.$$

$$+\|R_{11}(\lambda)\|\left(\int_0^1\left\|e^{-x(A+\lambda^2 I)^{1/2}}(A+\lambda^2 I)^{-1/2}f_1\right\|_H^p dx\right)^{1/p}+$$

$$+\left(\int_0^1\left\|e^{-x(A+\lambda^2 I)^{1/2}}\lambda^{-1}(A+\lambda^2 I)^{1/2}f_2\right\|_H^p dx\right)^{1/p}+$$

$$+\|R_{21}(\lambda)\|\left(\int_0^1\left\|e^{-x(A+\lambda^2I)^{1/2}}(A+\lambda^2I)^{-1/2}f_2\right\|_H^p dx\right)^{1/p}+$$

$$+\|R_{12}(\lambda)\|\left(\int_0^1\left\|e^{-(1-x)(A+\lambda^2I)^{1/2}}(A+\lambda^2I)^{-1/2}f_1\right\|_H^p dx\right)^{1/p}+$$

$$+\left(\int_0^1\left\|e^{-(1-x)(A+\lambda^2I)^{1/2}}f_2\right\|_H^p dx\right)^{1/p}+$$

$$+\|R_{22}(\lambda)\|\left(\int_0^1\left\|e^{-(1-x)(A+\lambda^2I)^{1/2}}(A+\lambda^2I)^{-1/2}f_2\right\|_H^p dx\right)^{1/p}\Bigg]+$$

$$+\left(1+\left\|A(A+\lambda^2I)^{-1}\right\|\right)\Bigg[\left(\int_0^1\left\|(A+\lambda^2I)e^{-x(A+\lambda^2I)^{1/2}}\lambda^{-1}f_1\right\|_H^p dx\right)^{1/p}+$$

$$+\|R_{11}(\lambda)\|\left(\int_0^1\left\|(A+\lambda^2I)^{1/2}e^{-x(A+\lambda^2I)^{1/2}}f_1\right\|_H^p dx\right)^{1/2}+$$

$$+\left(\int_0^1\left\|(A+\lambda^2I)e^{-x(A+\lambda^2I)^{1/2}}\lambda^{-1}(A+\lambda^2I)^{1/2}f_2\right\|_H^p dx\right)^{1/p}+$$

$$+\|R_{21}(\lambda)\|\left(\int_0^1\left\|(A+\lambda^2I)^{1/2}e^{-x(A+\lambda^2I)^{1/2}}f_2\right\|_H^p dx\right)^{1/p}+$$

$$+\|R_{12}(\lambda)\|\left(\int_0^1\left\|(A+\lambda^2I)^{1/2}e^{-(1-x)(A+\lambda^2I)^{1/2}}f_1\right\|_H^p dx\right)^{1/p}+$$

$$+\left(\int_0^1\left\|(A+\lambda^2I)e^{-(1-x)(A+\lambda^2I)^{1/2}}f_2\right\|_H^p dx\right)^{1/p}+$$

$$+\|R_{22}(\lambda)\|\left(\int_0^1\left\|(A+\lambda^2I)^{1/2}e^{-(1-x)(A+\lambda^2I)^{1/2}}f_2\right\|_H^p dx\right)^{1/p}\Bigg]. \tag{2.7}$$

In order to get the estimate (2.7)(and further continuation), we have used the following calculation which is true in view of [14, lemma 5.4.2./6] for large enough λ from the angle $|\arg\lambda|\le\varphi<\frac{\pi}{2}$:

$$\|Au\|_{L_p((0,1);H)}=\left\|A(A+\lambda^2I)^{-1}(A+\lambda^2I)u\right\|_{L_p((0,1);H)}\le$$

$$\leq \left\|A(A+\lambda^2 I)^{-1}\right\|_{B(H)} \left\|(A+\lambda^2 I)u\right\|_{L_p((0,1);H)} \leq C \left\|(A+\lambda^2 I)u\right\|_{L_p((0,1);H)}.$$

Estimate the integrals in the right-hand side of the inequality (2.7). It is enough to illustrate the estimation of the integrals

$$J_1 := \left(\int_0^1 \left\|(A+\lambda^2 I)e^{-x(A+\lambda^2 I)^{1/2}}\lambda^{-1}(A+\lambda^2 I)^{1/2} f_2\right\|_H^p dx\right)^{1/p}$$

and

$$J_2 := \left(\int_0^1 \left\|(A+\lambda^2 I)e^{-x(A+\lambda^2 I)^{1/2}}\lambda^{-1} f_1\right\|_H^p dx\right)^{1/p}.$$

From the representation

$$\frac{1}{\lambda^2}\left(A+\lambda^2 I\right)^{1/2} A^{-1/2} = A^{1/2}\left(A+\lambda^2 I\right)^{-1/2}\frac{1}{\lambda^2}\left(A+\lambda^2 I\right)A^{-1},$$

by virtue of [14, lemma 5.4.2/6], for large enough λ from the angle $|\arg\lambda| \leq \varphi < \frac{\pi}{2}$, it follows that the operator $\frac{1}{\lambda^2}\left(A+\lambda^2 I\right)^{1/2} A^{-1/2}$ is bounded from H into H and the estimate holds

$$\left\|\frac{1}{\lambda^2}\left(A+\lambda^2 I\right)^{1/2} A^{-1/2}\right\|_{B(H)} \leq \left\|A^{1/2}\left(A+\lambda^2 I\right)^{-1/2}\right\|_{B(H)} \left\|\frac{1}{\lambda^2} I + A^{-1}\right\|_{B(H)} \leq$$

$$\leq C\left(\frac{1}{|\lambda|^2} + \|A^{-1}\|\right) \leq C, \quad \exists C > 0. \tag{2.8}$$

By (2.8), for large enough λ from the angle $|\arg\lambda| \leq \varphi < \frac{\pi}{2}$, the operator $\frac{1}{\lambda^2}\left(A+\lambda^2 I\right)^{1/2} A^{-1/2}$ is bounded from $H(A^2)$ into $H(A^2)$ and the estimate takes place

$$\left\|\frac{1}{\lambda^2}\left(A+\lambda^2 I\right)^{1/2} A^{-1/2}\right\|_{B(H(A^2))} = \left\|\frac{1}{\lambda^2}\left(A+\lambda^2 I\right)^{-1/2} A^{-1/2}\right\|_{B(H)} \leq C, \ \exists C > 0. \tag{2.9}$$

From (2.8) and (2.9), by virtue of [12, theorem 1.3.3/(a)] (see also [14, section 1.7.9]), it follows that for large enough λ from the angle $|\arg\lambda| \leq \varphi < \frac{\pi}{2}$, the operator $\frac{1}{\lambda^2}\left(A+\lambda^2 I\right)^{1/2} A^{-1/2}$ is bounded from $\left(H\left(A^2\right), H\right)_{\theta,p}$ into $\left(H\left(A^2\right), H\right)_{\theta,p}$ for any $\theta \in (0,1)$ and the estimate holds

$$\left\|\frac{1}{\lambda^2}\left(A+\lambda^2 I\right)^{1/2} A^{-1/2}\right\|_{B\left((H(A^2),H)_{\theta,p}\right)} \leq C, \quad \exists C > 0. \tag{2.10}$$

Then, by [14, lemma 5.4.2/6], [14, theorem 5.4.2/1] and estimates (2.8), (2.10), for large enough λ from the angle $|\arg\lambda| \leq \varphi < \frac{\pi}{2}$, we have

$$J_1 \leq |\lambda| \left\|(A+\lambda^2 I)^{-1/2}\right\|_{B(H)} \left\|A^{1/2}(A+\lambda^2 I)^{-1/2}\right\|_{B(H)} \times$$

$$\times \left(\int_0^1 \left\|(A+\lambda^2 I)^2 e^{-x(A+\lambda^2 I)^{1/2}}\frac{1}{\lambda^2}(A+\lambda^2 I)^{1/2}A^{-1/2} f_2\right\|_H^p dx\right)^{1/p} \leq$$

$$\leq C\left(\left\|\frac{1}{\lambda^2}(A+\lambda^2 I)^{1/2}A^{-1/2}f_2\right\|_{(H(A^2),H)_{\frac{1}{4p},p}}+|\lambda|^{4-\frac{1}{p}}\left\|\frac{1}{\lambda^2}(A+\lambda^2 I)^{1/2}A^{-1/2}f_2\right\|_H\right)\leq$$

$$\leq C\left(\|f_2\|_{(H(A^2),H)_{\frac{1}{4p},p}}+|\lambda|^{4-\frac{1}{p}}\|f_2\|_H\right);$$

$$J_2\leq|\lambda|\left\|(A+\lambda^2 I)^{-1/2}\right\|_{B(H)}\left\|A^{1/2}(A+\lambda^2 I)^{-1/2}\right\|_{B(H)}\times$$

$$\times\left(\int_0^1\left\|(A+\lambda^2 I)^{3/2}e^{-x(A+\lambda^2 I)^{1/2}}\frac{1}{\lambda^2}(A+\lambda^2 I)^{1/2}A^{-1/2}f_1\right\|_H^p dx\right)^{1/p}\leq$$

$$\leq C\left(\|f_1\|_{(H(A^2),H)_{\frac{1}{4}+\frac{1}{4p},p}}+|\lambda|^{3-\frac{1}{p}}\|f_1\|_H\right).$$

Similarly, one can estimate the rest part of the right-hand side of the inequality (2.7). Theorem 2.1 is proved. □

Consider now a boundary value problem for a non-homogeneous equation with a quadratic complex parameter in H, i.e., the problem

$$L(\lambda)u:=\lambda^2u(x)-u''(x)+Au(x)=f(x),\quad x\in(0,1),\tag{2.11}$$

$$L_1(\lambda)u:=u'(1)+\lambda u(0)=f_1,\ L_2u:=u(1)=f_2.\tag{2.12}$$

Theorem 2.2. *Let A be φ-positive operator in H, where $\varphi\in[0,\frac{\pi}{2})$.*

Then, for $f\in L_p((0,1);H(A))$, $f_k\in\left(H\left(A^2\right),H\right)_{\frac{1}{2}-\frac{k}{4}+\frac{1}{4p},p}$, $k=1,2$, $p\in(1,+\infty)$, and for large enough λ from the angle $|\arg\lambda|\leq\varphi<\frac{\pi}{2}$, the problem (2.11), (2.12) has a unique solution $u\in W_p^2((0,1);H(A),H)$ and for the solution the following non-coercive estimate holds

$$|\lambda|^2\|u\|_{L_p((0,1);H)}+\left\|u''\right\|_{L_p((0,1);H)}+\|Au\|_{L_p((0,1);H)}\leq$$

$$\leq C\left[|\lambda|^2\|f\|_{L_p((0,1);H(A))}+\sum_{k=1}^{2}\left(\|f_k\|_{(H(A^2),H)_{\frac{1}{2}-\frac{k}{4}+\frac{1}{4p},p}}+|\lambda|^{2+k-\frac{1}{p}}\|f_k\|_H\right)\right].\tag{2.13}$$

Proof. Uniqueness follows from theorem 2.1. Represent a solution of the problem (2.11), (2.12), belonging to $W_p^2((0,1);H(A),H)$, as a sum $u(x)=u_1(x)+u_2(x)$, where $u_1(x)$ is the restriction on $[0,1]$ of the solution of the equation

$$L(\lambda)\tilde{u}_1(x):=\lambda^2\tilde{u}_1(x)-\tilde{u}_1''(x)+A\tilde{u}_1(x)=\tilde{f}(x),\ x\in R=(-\infty,+\infty),\tag{2.14}$$

where $\tilde{f}(x):=f(x)$ if $x\in[0,1]$ and $\tilde{f}(x)=0$ if $x\notin[0,1]$, and $u_2(x)$ is the solution of the problem

$$\begin{gathered}L(\lambda)u_2=0,\quad x\in(0,1),\\ L_1(\lambda)u_2=f_1-L_1(\lambda)u_1,\ \ L_2u_2=f_2-L_2u_1.\end{gathered}\tag{2.15}$$

As it was shown in the proof of theorem 5.4.4 in [14], a solution of the equation (2.14) is given by the formula

$$\tilde{u}_1(x)=\frac{1}{\sqrt{2\pi}}\int_R e^{i\mu x}L(\lambda,i\mu)^{-1}F\tilde{f}(\mu)\,d\mu,\tag{2.16}$$

where $F\tilde{f}$ is the Fourier transform of the function $\tilde{f}(x)$, and $L(\lambda,\sigma)$ is a characteristic operator pencil of the equation (2.14), i.e.,

$$L(\lambda,\sigma)=-\sigma^2 I+A+\lambda^2 I,\ \ |\arg\lambda|\le\varphi<\frac{\pi}{2}.$$

From (2.16) it follows that

$$\|\tilde{u}_1\|_{W_p^2(R;H(A^2),H(A))}=\|\tilde{u}_1\|_{L_p(R;H(A^2))}+\left\|\tilde{u}_1''\right\|_{L_p(R;H(A))}=$$
$$=\left\|\left(F^{-1}L(\lambda,i\mu)^{-1}F\tilde{f}(\mu)\right)(\cdot)\right\|_{L_p(R;H(A^2))}+$$
$$+\left\|\left(F^{-1}(i\mu)^2L(\lambda,i\mu)^{-1}F\tilde{f}(\mu)\right)(\cdot)\right\|_{L_p(R;H(A))}\le$$
$$\le\left\|\left(F^{-1}AL(\lambda,i\mu)^{-1}F\tilde{f}(\mu)\right)(\cdot)\right\|_{L_p(R;H(A))}+$$
$$+\left\|\left(F^{-1}(i\mu)^2L(\lambda,i\mu)^{-1}F\tilde{f}(\mu)\right)(\cdot)\right\|_{L_p(R;H(A))}. \tag{2.17}$$

Show that the operator-functions (with respect to μ)

$$T_{k+1}(\lambda,\mu):=(i\mu)^{2k}A^{1-k}L(\lambda,i\mu)^{-1},\ k=0,1, \tag{2.18}$$

are the Fourier multipliers in the space $L_p(R;H(A))$. For that, it is enough to check conditions of a theorem in [10, ch. XI, §§11.28,11.29] (see also [14, theorem 1.3.7/1]) for the operator-functions $\mu\to T_k(\lambda,\mu):R\to B(H(A)),k=1,2$. Obviously, for $|\arg\lambda|\le\varphi<\frac{\pi}{2}$ and $\mu\in R$, one has $|\arg(\lambda^2+\mu^2)|\le 2\varphi<\pi$. Since A is φ-positive in H then, for $|\arg\lambda|\le\varphi<\frac{\pi}{2}$ and $\mu\in R$, the estimates take place

$$\|L(\lambda,i\mu)^{-1}\|_{B(H)}=\left\|\left(A+\left(\lambda^2+\mu^2\right)I\right)^{-1}\right\|_{B(H)}\le\frac{C}{1+|\lambda^2+\mu^2|}\le\frac{C}{\mu^2}; \tag{2.19}$$

$$\|AL(\lambda,i\mu)^{-1}\|_{B(H)}=\left\|A\left(A+\left(\lambda^2+\mu^2\right)I\right)^{-1}\right\|_{B(H)}=$$
$$=\left\|I-\left(\lambda^2+\mu^2\right)\left(A+\left(\lambda^2+\mu^2\right)I\right)^{-1}\right\|_{B(H)}\le 1+\left|\lambda^2+\mu^2\right|\frac{C}{1+|\lambda^2+\mu^2|}\le C, \tag{2.20}$$

uniformly on λ in the angle $|\arg\lambda|\le\varphi<\frac{\pi}{2}$. From the estimates (2.19), (2.20), for $|\arg\lambda|\le\varphi<\frac{\pi}{2}$ and $\mu\in R$, we have

$$\|T_1(\lambda,\mu)\|_{B(H(A))}=\left\|AL(\lambda,i\mu)^{-1}\right\|_{B(H(A))}=\left\|AL(\lambda,i\mu)^{-1}\right\|_{B(H)}\le C; \tag{2.21}$$

$$\|T_2(\lambda,\mu)\|_{B(H(A))}=\left\|(i\mu)^2L(\lambda,i\mu)^{-1}\right\|_{B(H(A))}=\left\|(i\mu)^2L(\lambda,i\mu)^{-1}\right\|_{B(H)}=$$
$$=|\mu|^2\left\|L(\lambda,i\mu)^{-1}\right\|_{B(H)}\le C, \tag{2.22}$$

uniformly on λ in the angle $|\arg\lambda|\le\varphi<\frac{\pi}{2}$.

Since

$$\frac{\partial}{\partial\mu}T_1(\lambda,\mu)=A\frac{\partial}{\partial\mu}L(\lambda,i\mu)^{-1}=-AL(\lambda,i\mu)^{-1}\cdot 2\mu\,L(\lambda,i\mu)^{-1};$$

$$\frac{\partial}{\partial\mu}T_2(\lambda,\mu)=-2\mu\,L(\lambda,i\mu)^{-1}+\mu^2\,L(\lambda,i\mu)^{-1}\,2\mu\,L(\lambda,i\mu)^{-1}\ ,$$

then, by (2.19) , (2.20), we have

$$\left\| \frac{\partial}{\partial \mu} T_1 (\lambda, \mu) \right\|_{B(H(A))} = \left\| AL(\lambda, i\mu)^{-1} 2\mu \, L(\lambda, i\mu)^{-1} \right\|_{B(H(A))} =$$

$$= \left\| AL(\lambda, i\mu)^{-1} 2\mu \, L(\lambda, i\mu)^{-1} \right\|_{B(H)} \leq \frac{C}{|\mu|}; \tag{2.23}$$

$$\left\| \frac{\partial}{\partial \mu} T_2 (\lambda, \mu) \right\|_{B(H(A))} \leq 2|\mu| \left\| L(\lambda, i\mu)^{-1} \right\|_{B(H(A))} +$$

$$+ |\mu|^2 \left\| L(\lambda, i\mu)^{-1} \right\|_{B(H(A))} 2|\mu| \left\| L(\lambda, i\mu)^{-1} \right\|_{B(H(A))} =$$

$$= 2|\mu| \left\| L(\lambda, i\mu)^{-1} \right\|_{B(H)} + |\mu|^2 \left\| L(\lambda, i\mu)^{-1} \right\|_{B(H)} 2|\mu| \left\| L(\lambda, i\mu)^{-1} \right\|_{B(H)} \leq \frac{C}{|\mu|}. \tag{2.24}$$

By [10, ch. XI, §§11.28, 11.29] (see also [14, theorem 1.3.7/1]), from estimates (2.21)-(2.24), it follows that the operator-functions $\mu \to T_{k+1}(\lambda, \mu)$, $k = 0, 1$, which are defined by (2.18), are the Fourier multipliers in $L_p((0,1); H(A))$. Then, from (2.17), it follows that

$$\|\tilde{u}_1\|_{W_p^2(R; H(A^2), H(A))} \leq C \left\| \tilde{f} \right\|_{L_p(R; H(A))}, \tag{2.25}$$

uniformly on λ.

From (2.25) it follows that $u_1 \in W_p^2\left((0,1); H\left(A^2\right), H(A)\right)$ and the estimate takes place

$$\|u_1\|_{W_p^2((0,1); H(A^2), H(A))} \leq C \|f\|_{L_p((0,1); H(A))}. \tag{2.26}$$

From (2.26), by the continuous embedding

$$W_p^2\left((0,1); H\left(A^2\right), H(A)\right) \subset W_p^2\left((0,1); H(A), H\right),$$

one has

$$\|u_1\|_{W_p^2((0,1); H(A), H)} \leq C \|f\|_{L_p((0,1); H(A))}. \tag{2.27}$$

Then, from (2.14) and (2.27), for λ from the angle $|\arg \lambda| \leq \varphi < \frac{\pi}{2}$, we have

$$|\lambda|^2 \|u_1\|_{L_p((0,1); H)} + \left\| u_1'' \right\|_{L_p((0,1); H)} + \|Au_1\|_{L_p((0,1); H)} \leq$$

$$\leq C \|f\|_{L_p((0,1); H(A))}. \tag{2.28}$$

Indeed, from (2.14) for $u_1(x)$ we have

$$\lambda^2 u_1(x) = f(x) + u_1''(x) - Au_1(x), \ \ x \in (0,1).$$

Hence, by (2.27),

$$|\lambda|^2 \|u_1\|_{L_p((0,1); H)} \leq \|f\|_{L_p((0,1); H)} + \left\| u_1'' \right\|_{L_p((0,1); H)} + \|Au_1\|_{L_p((0,1); H)} \leq$$

$$\leq \|f\|_{L_p((0,1); H)} + C \|f\|_{L_p((0,1); H(A))} \leq C \|f\|_{L_p((0,1); H(A))}. \tag{2.29}$$

So, (2.27) and (2.29) imply (2.28).

By the trace theorem [12, theorem 1.8.2] (see also [14, theorem 1.7.7/1]), for any fixed $x_0 \in [0,1]$ and $s = 0, 1$,

$$u_1^{(s)}(x_0) \in \left(H\left(A^2\right), H(A)\right)_{\frac{s}{2} + \frac{1}{2p}, p}.$$

By virtue of [12, theorem 1.3.3] and [12, formula 1.15.4/2] one gets

$$\left(H\left(A^2\right),H(A)\right)_{\frac{1+sp}{2p},p}=\left(H\left(A^2\right),H\right)_{\frac{1+sp}{4p},p}. \tag{2.30}$$

Hence, for any fixed $x_0\in[0,1]$,

$$u_1'\left(x_0\right)\in\left(H\left(A^2\right),H\right)_{\frac{1}{4}+\frac{1}{4p},p},\ u_1\left(x_0\right)\in\left(H\left(A^2\right),H\right)_{\frac{1}{4p},p}.$$

Since the embedding $\left(H\left(A^2\right),H\right)_{\frac{1}{4p},p}\subset\left(H\left(A^2\right),H\right)_{\frac{1}{4}+\frac{1}{4p},p}$ is continuous, then

$$L_1\left(\lambda\right)u_1\in\left(H\left(A^2\right),H\right)_{\frac{1}{4}+\frac{1}{4p},p},\ L_2u_1\in\left(H\left(A^2\right),H\right)_{\frac{1}{4p},p}.$$

Therefore, by virtue of theorem 2.1, for large enough λ from the angle $|\arg\lambda|\le\varphi<\frac{\pi}{2}$, for the solution of the problem (2.15) one has

$$\begin{gathered}|\lambda|^2\left\|u_2\right\|_{L_p((0,1);H)}+\left\|u_2''\right\|_{L_p((0,1);H)}+\left\|Au_2\right\|_{L_p((0,1);H)}\le\\ \le C\left(\left\|f_1\right\|_{(H(A^2),H)_{\frac{1}{4}+\frac{1}{4p},p}}+\left\|f_2\right\|_{(H(A^2),H)_{\frac{1}{4p},p}}+|\lambda|^{3-\frac{1}{p}}\left\|f_1\right\|_H+\right.\\ +|\lambda|^{4-\frac{1}{p}}\left\|f_2\right\|_H+\left\|u_1'(1)\right\|_{(H(A^2),H)_{\frac{1}{4}+\frac{1}{4p},p}}+|\lambda|\left\|u_1(0)\right\|_{(H(A^2),H)_{\frac{1}{4}+\frac{1}{4p},p}}+\\ \left.+\left\|u_1(1)\right\|_{(H(A^2),H)_{\frac{1}{4p},p}}+|\lambda|^{3-\frac{1}{p}}\left\|u_1'(1)\right\|_H+|\lambda|^{4-\frac{1}{p}}\left(\left\|u_1(0)\right\|_H+\left\|u_1(1)\right\|_H\right)\right).\end{gathered} \tag{2.31}$$

Estimate the norm $|\lambda|\left\|u_1(0)\right\|_{(H(A^2),H)_{\frac{1}{4}+\frac{1}{4p},p}}$. Taking into account the continuity of the embedding $\left(H\left(A^2\right),H\right)_{\frac{1}{4p},p}\subset\left(H\left(A^2\right),H\right)_{\frac{1}{4}+\frac{1}{4p},p}$, by virtue of [12, theorem 1.8.2] (see also [14, theorem 1.7.7/1]), (2.26), and (2.30), for λ from the angle $|\arg\lambda|\le\varphi<\frac{\pi}{2}$, we have

$$\begin{gathered}|\lambda|\left\|u_1(0)\right\|_{(H(A^2),H)_{\frac{1}{4}+\frac{1}{4p},p}}\le C|\lambda|\ \left\|u_1(0)\right\|_{(H(A^2),H)_{\frac{1}{4p},p}}\le\\ \le C|\lambda|\ \left\|u_1\right\|_{W_p^2((0,1);H(A^2),H(A))}\le C|\lambda|\left\|f\right\|_{L_p((0,1);H(A))}.\end{gathered} \tag{2.32}$$

Moreover, by the same considerations, the following estimates hold

$$\left\|u_1'(1)\right\|_{(H(A^2),H)_{\frac{1}{4}+\frac{1}{4p},p}}\le C\left\|u_1\right\|_{W_p^2((0,1);H(A^2),H(A))}\le C\left\|f\right\|_{L_p((0,1);H(A))}; \tag{2.33}$$

$$\left\|u_1(1)\right\|_{(H(A^2),H)_{\frac{1}{4p},p}}\le C\left\|u_1\right\|_{W_p^2((0,1);H(A^2),H(A))}\le C\left\|f\right\|_{L_p((0,1);H(A))}. \tag{2.34}$$

By [14, theorem 1.7.7/2], for $\lambda\in\mathbb{C}$ and $u_1\in W_p^2\left((0,1);H\right)$, the following inequality holds

$$|\lambda|^{2-s}\left\|u_1^{(s)}\left(x_0\right)\right\|_H\le C\left(|\lambda|^{\frac{1}{p}}\left\|u_1\right\|_{W_p^2((0,1);H)}+|\lambda|^{2+\frac{1}{p}}\left\|u_1\right\|_{L_p((0,1);H)}\right), \tag{2.35}$$

where $x_0\in[0,1]$, $s=\{0,1\}$, $p\in(1,+\infty)$.

Dividing (2.35) by $|\lambda|^{\frac{1}{p}}$, for $\lambda\in\mathbb{C}$, $u_1\in W_p^2\left((0,1);H\right)$, one has

$$|\lambda|^{2-s-\frac{1}{p}}\left\|u_1^{(s)}\left(x_0\right)\right\|_H\le C\left(\left\|u_1\right\|_{W_p^2((0,1);H)}+|\lambda|^2\left\|u_1\right\|_{L_p((0,1);H)}\right). \tag{2.36}$$

Take in (2.36) $s=1,\ x_0=1$ and multiply the obtained inequality by $|\lambda|^2$. Then, in view of (2.28), for λ from the angle $|\arg\lambda|\le\varphi<\frac{\pi}{2}$, we have

$$|\lambda|^{3-\frac{1}{p}}\left\|u_1'(1)\right\|_H\le C|\lambda|^2\left(\|u_1\|_{W_p^2((0,1);H)}+|\lambda|^2\|u_1\|_{L_p((0,1);H)}\right)\le$$

$$\le C|\lambda|^2\left(\|u_1\|_{W_p^2((0,1);H(A),H)}+|\lambda|^2\|u_1\|_{L_p((0,1);H)}\right)\le C|\lambda|^2\|f\|_{L_p((0,1);H(A))}. \tag{2.37}$$

Take now in (2.36) $s=0$ and $x_0=0,\ x_0=1$, consequently. Multiply the obtained inequalities by $|\lambda|^2$ and sum them. Then, by (2.28), for λ from the angle $|\arg\lambda|\le\varphi<\frac{\pi}{2}$, one gets

$$|\lambda|^{4-\frac{1}{p}}\left(\|u_1(0)\|_H+\|u_1(1)\|_H\right)\le C|\lambda|^2\left(\|u_1\|_{W_p^2((0,1);H)}+|\lambda|^2\|u_1\|_{L_p((0,1);H)}\right)\le$$

$$\le C|\lambda|^2\|f\|_{L_p((0,1);H(A))}. \tag{2.38}$$

Taking into account the estimates (2.32), (2.33), (2.34), (2.37) and (2.38) in (2.31), for large enough λ from the angle $|\arg\lambda|\le\varphi<\frac{\pi}{2}$, we have

$$|\lambda|^2\|u_2\|_{L_p((0,1);H)}+\left\|u_2''\right\|_{L_p((0,1);H)}+\|Au_2\|_{L_p((0,1);H)}\le$$

$$\le C\left[|\lambda|^2\|f\|_{L_p((0,1);H(A))}+\sum_{k=1}^{2}\left(\|f_k\|_{(H(A^2),H)_{\frac12-\frac{k}{4}+\frac{1}{4p},p}}+|\lambda|^{2+k-\frac{1}{p}}\|f_k\|_H\right)\right]. \tag{2.39}$$

From (2.28) and (2.39) the estimate (2.13) follows. Theorem 2.2 is proved. □

3. Application of the obtained abstract results to partial differential equations.

In the square $\Omega=[0,1]\times[0,1]$, consider a boundary value problem for a second order elliptic differential equation with a quadratic complex parameter

$$L(\lambda)u:=\lambda^2u(x,y)-D_x^2u(x,y)-D_y\left(a(y)D_yu(x,y)\right)=f(x,y),\quad (x,y)\in\Omega, \tag{3.1}$$

$$\begin{aligned}&L_1(\lambda)u:=D_xu(1,y)+\lambda u(0,y)=f_1(y),\ \ y\in[0,1],\\&L_2u:=u(1,y)=f_2(y),\ \ y\in[0,1],\end{aligned} \tag{3.2}$$

$$L_3u:=u(x,0)=0,\quad L_4u:=u(x,1)=0,\quad x\in[0,1], \tag{3.3}$$

where λ is the parameter; $a(y)$ is a continuously differentiable function on $[0,1]$; $D_x:=\frac{\partial}{\partial x},\ D_y:=\frac{\partial}{\partial y}$.

Denote the interpolation space of Sobolev spaces by $B_{p,q}^s(0,1):=$ $=\left(W_p^{s_0}(0,1),W_p^{s_1}(0,1)\right)_{\theta,q}$, where $s_0,s_1\ge0$ are integers $0<\theta<1, 1<q<\infty, 1<p<\infty$ and $s=(1-\theta)s_0+\theta s_1$. In particular, $W_p^s(0,1):=$ $B_{p,p}^s(0,1):=\left(W_p^{s_0}(0,1),W_p^{s_1}(0,1)\right)_{\theta,p}$, if $s>0$ is not integer. Denote also $W_{p,q}^{\ell,s}(\Omega):=W_p^\ell\left((0,1);W_q^s(0,1),L_q(0,1)\right)$, where $0\le l,s$ are integres, $1<p<\infty,\ 1<q<\infty$. If $p=q$ and $l=s$ then $W_{p,q}^{\ell,s}(\Omega)=W_p^\ell(\Omega)$. Finally, denote

$L_{p,q}(\Omega) := W_{p,q}^{0,0}(\Omega) = L_p((0,1); L_q(0,1))$. We have that $L_{p,q}(\Omega)$ is a Banach space of measurable on $(0,1)\times(0,1)$ functions $u(x,y)$ such that

$$\|u\|_{L_{p,q}(\Omega)} = \left(\int_0^1 \left(\int_0^1 |u(x,y)|^q \, dy\right)^{p/q} dx\right)^{1/p} < \infty,$$

and $W_{p,q}^{\ell,s}(\Omega)$ is a Banach space of measurable on $(0,1)\times(0,1)$ functions $u(x,y)$ which have generalized derivatives $\frac{\partial^\ell u(x,y)}{\partial x^\ell}$, $\frac{\partial^s u(x,y)}{\partial y^s}$ on $(0,1)\times(0,1)$ and

$$\|u\|_{W_{p,q}^{\ell,s}(\Omega)} = \|u\|_{L_{p,q}(\Omega)} + \left\|D_x^\ell u\right\|_{L_{p,q}(\Omega)} + \left\|D_y^s u\right\|_{L_{p,q}(\Omega)} < \infty.$$

Theorem 3.1. *Let $a(y) \in C^3[0,1]$, $a(y) > 0$ for $y \in [0,1]$ and $a'(0) = a'(1) = 0$. Then, for $f(x,y) \in L_p((0,1); W_2^2((0,1); L_\nu f = 0, \nu = 3,4))$, $p \in (1,\infty)$, $f_k(y) \in B_{2,p,*}^{2+k-\frac{1}{p}}(0,1)$ (these spaces are defined in the proof) and for large enough λ from the angle $|\arg\lambda| \le \varphi < \frac{\pi}{2}$, the problem (3.1)-(3.3) has a unique solution u from*

$$W_p^2\left((0,1); W_2^2((0,1); u(0) = u(1) = 0), L_2(0,1)\right)$$

and for the solution the following estimate takes place

$$|\lambda|^2 \|u(x,y)\|_{L_p((0,1);L_2(0,1))} + \left\|D_x^2 u(x,y)\right\|_{L_p((0,1);L_2(0,1))} +$$

$$+ \left\|D_y\left(a(y) D_y u(x,y)\right)\right\|_{L_p((0,1);L_2(0,1))} \le C\left[|\lambda|^2 \|f(x,y)\|_{L_p\left((0,1);W_2^2(0,1)\right)} + \right.$$

$$\left. + \sum_{k=1}^2 \left(\|f_k\|_{B_{2,p}^{2+k-\frac{1}{p}}(0,1)} + |\lambda|^{2+k-\frac{1}{p}} \|f_k\|_{L_2(0,1)}\right)\right].$$

Proof. In the space $H := L_2(0,1)$, consider an operator A, defined by the equalities

$$D(A) := W_2^2((0,1); u(0) = u(1) = 0), \; Au := -\left(a(y) u'(y)\right)'. \tag{3.4}$$

Rewrite the problem (3.1)-(3.3) in the operator form

$$\lambda^2 u(x) - u''(x) + Au(x) = f(x), \; x \in (0,1), \tag{3.5}$$

$$u'(1) + \lambda u(0) = f_1, \quad u(1) = f_2, \tag{3.6}$$

where $u(x) := u(x,\cdot), f(x) := f(x,\cdot)$ are vector-functions with values from the Hilbert space $L_2(0,1)$ and $f_k := f_k(\cdot)$. Obviously, the proof of theorem 3.1 is reduced to the checking of the conditions of theorem 2.2 for the problem (3.5), (3.6). It follows from the conditions of theorem 3.1 that the operator A, defined by (3.4), is selfadjoint, positive-definite in $L_2(0,1)$. Therefore, the condition of theorem 2.2 is satisfied for any fixed $0 \le \varphi < \pi/2$. Obviously, $A^2 u = (a(y)(a(y)u'(y))'')'$, $D(A^2) = W_2^4((0,1); L_\nu u = 0, L_\nu Au = 0, \nu = 3,4)$. It is also clear that $L_\nu Au = 0$, $\nu = 3,4$, coincide with boundary conditions $u''(0) = u''(1) = 0$. Since the order of boundary conditions $L_\nu u = 0$, $\nu = 3,4$, is equal to 0 and the order of boundary conditions $L_\nu Au = 0$, $\nu = 3,4$, is equal to two then, by [12, theorem 4.3.3], we have

$$\left(H\left(A^2\right), H\right)_{\theta,p} = \left(W_2^4\left((0,1); \; u^{(j)}(0) = u^{(j)}(1) = 0, \; j = 0,2\right), L_2(0,1)\right)_{\theta,p} =$$

$$= B_{2,p}^{4(1-\theta)}\Bigg((0,1)\,;\,u(0)=u(1)=0 \text{ if } 0<4(1-\theta)-\frac{1}{2}<2;\; u\,(0)=u\,(1)=$$

$$=u''(0)=u''(1)=0 \text{ if } 2<4(1-\theta)-\frac{1}{2};\;\; u\,(0)=u\,(1)=0,$$

$$\int_0^1 (\min\{x,\,1-x\})^{-1}\left|u''\,(x)\right|^2 dx<\infty \text{ if } 4\,(1-\theta)-\frac{1}{2}=2\Bigg).$$

Hence, for $\theta=\frac{1}{4}+\frac{1}{4p}$ and $\theta=\frac{1}{4p}$, we get implicit forms of the interpolation spaces $\left(H\left(A^2\right),H\right)_{\frac{1}{4}+\frac{1}{4p},p}$ and $\left(H\left(A^2\right),H\right)_{\frac{1}{4p},p}$:

$$\left(H\left(A^2\right),H\right)_{\frac{1}{4}+\frac{1}{4p},p}=B_{2,p,*}^{3-\frac{1}{p}}(0,1)=$$

$$=\begin{cases} B_{2,p}^{3-\frac{1}{p}}\left((0,1)\,;u\,(0)=u\,(1)=0\right), & 1<p<2,\\ W_2^{\frac{5}{2}}\left((0,1)\,;\,u\,(0)=u\,(1)=0,\,\int\limits_0^1(\min\{x,\,1-x\})^{-1}\left|u''\,(x)\right|^2dx<\infty\right), & p=2,\\ B_{2,p}^{3-\frac{1}{p}}\left((0,1)\,;\,u^{(j)}\,(0)=u^{(j)}\,(1)=0,\,j=0,2\right), & p>2;\end{cases}$$

$$\left(H\left(A^2\right),H\right)_{\frac{1}{4p},p}=B_{2,p}^{4-\frac{1}{p}}\left((0,1)\,;\,u^{(j)}(0)=u^{(j)}(1)=0,\;j=0,2\right)=$$

$$=B_{2,p,*}^{4-\frac{1}{p}}(0,1),\;\; p>1.$$

□

Theorem 3.1. is proved.

References

[1] A. Aibeche, A. Favini, Ch. Mezoued, Deficient coerciveness estimate for an abstract differential equation with a parameter dependent boundary conditions. *Boll Unione Mat. Ital. Sez. B Artic. Ric. Mat.* **10** (2007), No 3, 535-547.

[2] B.A. Aliev, Ya. Yakubov, Elliptic differential-operator problems with a spectral parameter in both the equation and boundary-operator conditions. *Adv. Differential Equat.,* **11** (2006), no.10, 1081-1110. (Erratum in 12 (2007), No.9, 1079).

[3] B.A. Aliev, Ya. Yakubov, Second order elliptic differential-operator equations with unbounded operator boundary conditions in UMD Banach Spaces. *Inregral Equations and Operator Theory* , **69** (2011), 269-300.

[4] B.A. Aliev, N.K. Kurbanova, Ya. Yakubov, Solvability of the abstract Regge boundary value problem and asymptotic behavior of eigenvalues of one abstract spectral problem. *Rivista Di Mathematica Della Universita Di Parma,* **6** (2015), 241-265.

[5] B.A. Aliev, N.K. Kurbanova, Ya. Yakubov, One boundary value problem for elliptic differential-operator equations of the second order with quadratic spectral parameter. *Ukrainian Mathematical Journal,* **69** (2017), no.6, 857-875.

[6] B.A. Aliev, N.K. Kurbanova, Ya. Yakubov, Solvability of a boundary value problem for second-order elliptic differential-operator equations with spectral parameter in the equation and in the boundary conditions. *Differential Equations,* **54** (2018), no.1, 67-85.

[7] B.A. Aliev, V.Z. Kerimov, Ya. Yakubov, Solvability of a boundary value problem for elliptic differential-operator equations of the second order with quadratic complex parameter *Differential Equations,* **56**(2020), no 10, 1306-1317.

[8] B.A. Aliev, V.Z.Kerimov, Ya. Yakubov, Solvability of a boundary value problem for a second-order elliptic differential-operator equation with a complex parameter. *Proc.Inst.Math.Mech.Natl.Acad.Sci. Azerb.* **47** (2021), no. 2, 309-326.

[9] M. Denche, Abstract differential equation with a spectral parameter in the boundary conditions. *Result. Math.,* **35** (1999), 216-227.

[10] N. Dunford, T.J. *Schwartz, Linear Operators, Part II, Spectral Theory, Wiley-Interscinece,* New York, 1963.

[11] A. Favini, Ya. Yakubov, Irregular boundary value problems for second order elliptic differential-operator equations in UMD Banach spaces. *Mathemtische Annalen,* **348** (2010), 601-632.

[12] H. Tribel, *Interpolation Theory. Functions spaces. Differential Operators*, North-Holland, Amsterdam, 1978.

[13] S. Yakubov, A nonlocal boundary value problem for elliptic differential-operator equations and applications. *Integral Equations and Operator Theory*, **35** (1999), 485-506.

[14] S. Yakubov, Ya. Yakubov, *Differential-Operator Equations. Ordinary and Partial Differential Equations,* Chapman and Hall / CRC, Boca Raton, 2000.

[15] S. Yakubov, Problems for elliptic equations with operator-boundary conditions. *Integral Equations and Operator Theory* , **43** (2002), 215-236.

[16] Ya. Yakubov, Elliptic differential-operator problems with the spectral parameter in both the equation and boundary conditions and the corresponding abstract parabolic initial boundary value problems. *New Prospects in Direct, Inverse and Control Problems for Evolution Equations, Springer INDAM series,* (2014), no 10, 437-471.

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