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# UNCERTAINTY INEQUALITIES FOR A FAMILY OF WEIGHTED DIRICHLET SPACES

FETHI SOLTANI

**Abstract**. In this paper, we introduce a family of weighted Dirichlet spaces $\{\mathcal{D}_{\alpha,n}\}_{n\in\mathbb{N}}$. This family satisfies the continuous inclusions $\mathcal{D}_{\alpha,n} \subset \ldots \subset \mathcal{D}_{\alpha,2} \subset \mathcal{D}_{\alpha,1} \subset \mathcal{D}_{\alpha,0} = \mathcal{D}_\alpha$, where $\mathcal{D}_\alpha$ is the classical weighted Dirichlet space. Next, we define and study operator $Xf(z) := f'(z) - f'(0)$ and its adjoint operator $Y_\alpha f(z) := z^2 f'(z) + \alpha z f(z) - \alpha \int_{[0,z]} f(s)\mathrm{d}s$ on the weighted Dirichlet space $\mathcal{D}_\alpha$, and we establish an uncertainty inequality of Heisenberg type for this space. A more general uncertainty inequality for the space $\mathcal{D}_{\alpha,n}$ is also given when we considered the operators $X_n = X^n$ and $Y_{\alpha,n} = Y_\alpha^n$.

## 1. Introduction

Heisenberg's uncertainty principle in quantum physics states that the position and momentum of a particle cannot be measured exactly at the same time [13]. More specifically, the product of the “uncertainty” for the position and the ”uncertainty” for the momentum of a particle is always greater than or equal to a tiny positive constant, namely, $h/4\pi$, where $h$ is Planck's constant. There exist many similar uncertainty principles, in physics [2, 6, 14, 19], and mathematics [5, 15], that are based on position, momentum, energy, time, and so on. In this paper we are going to prove a version of the uncertainty principle in the context of weighted Dirichlet space. The weighted Dirichlet space is one of the complex analysis tools used in harmonic analysis [3, 7].

Let $\mathbb{C}$ be the complex plane and $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ the open unit disk, and let $\alpha \geq 0$. The weighted Dirichlet space $\mathcal{D}_\alpha$ (see [1, 8, 18]) is the set of all analytic functions $f$ in the unit disk $\mathbb{D}$ with the finite Dirichlet integral

$$\int_{\mathbb{D}} |f'(z)|^2 \mathrm{d}v_\alpha(z),$$

where

$$\mathrm{d}v_\alpha(z) = (\alpha+1)(1-|z|^2)^\alpha \frac{\mathrm{d}x\mathrm{d}y}{\pi}, \quad z = x + iy,$$

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be the weighted Lebesgue measure on $\mathbb{D}$. It is a Hilbert space when equipped with the inner product

$$\langle f, g\rangle_{\mathcal{D}_\alpha} := f(0)\overline{g(0)} + \int_{\mathbb{D}} f'(z)\overline{g'(z)}\mathrm{d}v_\alpha(z).$$

If $f, g \in \mathcal{D}_\alpha$ with $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$, then

$$\langle f, g\rangle_{\mathcal{D}_\alpha} = a_0\overline{b_0} + (\alpha+1)\sum_{n=1}^{\infty} \frac{nn!}{(\alpha+1)_n} a_n\overline{b_n},$$

where $(\alpha+1)_n = \frac{\Gamma(n+\alpha+1)}{\Gamma(\alpha+1)}$, and the set $\left\{1, \sqrt{\frac{(\alpha+1)_n}{(\alpha+1)nn!}} z^n\right\}_{n=1}^{\infty}$ forms an orthonormal basis for the space $\mathcal{D}_\alpha$. The function $K_{z,\alpha}$ given for $z \in \mathbb{D}$, by

$$K_{z,\alpha}(w) = 1 + \frac{1}{\alpha+1}\sum_{n=1}^{\infty} \frac{(\alpha+1)_n}{nn!}(\overline{z}w)^n, \quad w \in \mathbb{D}, \tag{1.1}$$

is the reproducing kernel for the weighted Dirichlet space $\mathcal{D}_\alpha$, meaning that $K_{z,\alpha} \in \mathcal{D}_\alpha$, and for all $f \in \mathcal{D}_\alpha$, we have $\langle f, K_{z,\alpha}\rangle_{\mathcal{D}_\alpha} = f(z)$.

In the case $\alpha = 0$, the Dirichlet space $\mathcal{D}_0$ (see [1]) is the set of all analytic functions $f$ in the unit disk $\mathbb{D}$ equipped with the inner product

$$\langle f, g\rangle_{\mathcal{D}_0} = a_0\overline{b_0} + \sum_{n=1}^{\infty} n a_n\overline{b_n},$$

for all $f, g \in \mathcal{D}_0$ with $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$. And, the reproducing kernel of $\mathcal{D}_0$ is given by

$$K_{z,0}(w) = 1 + \log\left(\frac{1}{1-\overline{z}w}\right), \quad z, w \in \mathbb{D}.$$

Over the years, the applications of weighted Dirichlet space $\mathcal{D}_\alpha$ play an important role in various fields of mathematics [3, 10, 11, 16]. And this space is the background of some applications to our contribution, especially, we introduce on $\mathcal{D}_\alpha$ the operator $Xf(z) := f'(z) - f'(0)$ and its adjoint $Y_\alpha f(z) := z^2 f'(z) + \alpha z f(z) - \alpha \int_{[0,z]} f(s)\mathrm{d}s$; and therefore, we establish an uncertainty inequality of Heisenberg type on the space $\mathcal{D}_\alpha$. This version is given by

$$\|(X + Y_\alpha - a)f\|_{\mathcal{D}_\alpha}\|(X - Y_\alpha + ib)f\|_{\mathcal{D}_\alpha} \geq \|Y_\alpha f\|^2_{\mathcal{D}_\alpha} - \|Xf\|^2_{\mathcal{D}_\alpha}, \quad a, b \in \mathbb{R}.$$

Next, we deduce an uncertainty inequality of Heisenberg type on $\mathcal{D}_\alpha$ for the operators $X_n = X^n$ and $Y_{\alpha,n} = Y_\alpha^n$.

The analogous uncertainty inequalities are also proved, for the Fock space [4] in 2015, for the Bessel-Struve operator [20] in 2017, for the Fock space associated to Dunkl operators [21, 22] in 2017, and for the Fock space associated to higher-order Bessel operator [17, 23] in 2018.

The contents of the paper are as follows. In Section 2 we define and study the operators $X$ and $Y_\alpha$ on the space $\mathcal{D}_\alpha$. And in Section 3, we establish uncertainty inequalities on $\mathcal{D}_\alpha$.

## 2. Operators on $\mathcal{D}_\alpha$

For $z \in \mathcal{D}_\alpha$, the function $u(z) = K_{\overline{z},\alpha}(w)$ given by (1.1) is the unique analytic solution on $\mathbb{D}$ of the initial problem

$$u'(z) - u'(0) = w\left[zu'(z) + \alpha u(z) - \frac{\alpha}{z}\int_{[0,z]} u(s)\mathrm{d}s\right], \quad w \in \mathbb{D}, \quad u(0) = 1,$$

where $[0, z] = \{tz, t \in [0, 1]\}$ is the line segment joining 0 and $z$.

According to this equation we define on $\mathcal{D}_\alpha$ the two operators

$$Xf(z) := f'(z) - f'(0), \tag{2.1}$$

and

$$Y_\alpha f(z) := z^2 f'(z) + \alpha z f(z) - \alpha\int_{[0,z]} f(s)\mathrm{d}s. \tag{2.2}$$

We define the domain of $X$ denoted by $\mathrm{Dom}(X)$ as

$$\mathrm{Dom}(X) := \{f \in \mathcal{D}_\alpha : Xf \in \mathcal{D}_\alpha\}.$$

And as in the same we define $\mathrm{Dom}(Y_\alpha)$. The operators $X$ and $Y_\alpha$ defined by (2.1) and (2.2) satisfy the commutation relation

$$[X, Y_\alpha] := XY_\alpha - Y_\alpha X = 2Z_\alpha, \tag{2.3}$$

where

$$Z_\alpha f(z) := zf'(z) + \frac{\alpha}{2}(f(z) - f(0)).$$

Let $n \in \mathbb{N}$. We define the Hilbert space $\mathcal{D}_{\alpha,n}$ as the space of all analytic functions $f$ in the unit disk $\mathbb{D}$ such that

$$\|f\|^2_{\mathcal{U}_{\alpha,n}} := |f(0)|^2 + \int_{\mathbb{D}} |Z_0^{n+1} f(z)|^2 |z|^{-2}\mathrm{d}v_\alpha(z) < \infty,$$

where $Z_0 f(z) = zf'(z)$.

If $f \in \mathcal{D}_{\alpha,n}$ with $f(z) = \sum_{k=0}^\infty a_k z^k$ then

$$\|f\|^2_{\mathcal{D}_{\alpha,n}} = |a_0|^2 + (\alpha+1)\sum_{k=1}^\infty \frac{k^{2n+1}k!}{(\alpha+1)_k}|a_k|^2.$$

Thus, the spaces $\mathcal{D}_{\alpha,n}$, $n \in \mathbb{N}$ satisfy the following continuous inclusions

$$\mathcal{D}_{\alpha,n} \subset \ldots \subset \mathcal{D}_{\alpha,2} \subset \mathcal{D}_{\alpha,1} \subset \mathcal{D}_{\alpha,0} = \mathcal{D}_\alpha.$$

In particular, if $f \in \mathcal{D}_{\alpha,1}$ with $f(z) = \sum_{n=0}^\infty a_n z^n$ then

$$\begin{aligned}\|f\|^2_{\mathcal{D}_{\alpha,1}} &= |f(0)|^2 + \int_{\mathbb{D}} |f'(z) + zf''(z)|^2 \mathrm{d}v_\alpha(z)\\ &= |a_0|^2 + (\alpha+1)\sum_{n=1}^\infty \frac{n^3 n!}{(\alpha+1)_n}|a_n|^2.\end{aligned}$$

If $f \in \mathcal{D}_{\alpha,2}$ with $f(z) = \sum_{n=0}^\infty a_n z^n$ then

$$\begin{aligned}\|f\|^2_{\mathcal{D}_{\alpha,2}} &= |f(0)|^2 + \int_{\mathbb{D}} |f'(z) + 3zf''(z) + z^2 f'''(z)|^2 \mathrm{d}v_\alpha(z)\\ &= |a_0|^2 + (\alpha+1)\sum_{n=1}^\infty \frac{n^5 n!}{(\alpha+1)_n}|a_n|^2.\end{aligned}$$

**Theorem 2.1.** *(i) $Dom(X) = Dom(Y_\alpha) = Dom(Z_\alpha) = \mathcal{D}_{\alpha,1}$.*

*(ii) For $f, g \in \mathcal{D}_{\alpha,1}$ we have*

$$\langle Xf, g\rangle_{\mathcal{D}_\alpha} = \langle f, Y_\alpha g\rangle_{\mathcal{D}_\alpha}.$$

*(iii) For $f \in \mathcal{D}_{\alpha,1}$ we have*

$$\|Y_\alpha f\|^2_{\mathcal{D}_\alpha} = \|Xf\|^2_{\mathcal{D}_\alpha} + 2\langle Z_\alpha f, f\rangle_{\mathcal{D}_\alpha}.$$

**Proof.** (i) Let $f \in \mathcal{D}_\alpha$ with $f(z) = \sum_{n=0}^\infty a_n z^n$. Then

$$Xf(z) = \sum_{n=1}^\infty (n+1)a_{n+1}z^n, \quad Y_\alpha f(z) = \sum_{n=2}^\infty \frac{(n-1)(n+\alpha)}{n} a_{n-1}z^n,$$

and

$$Z_\alpha f(z) = \sum_{n=1}^\infty (n + \frac{\alpha}{2})a_n z^n. \tag{2.4}$$

Thus

$$\|Xf\|^2_{\mathcal{D}_\alpha} = (\alpha+1)\sum_{n=2}^\infty \frac{(n-1)nn!(n+\alpha)}{(\alpha+1)_n}|a_n|^2, \tag{2.5}$$

$$\|Y_\alpha f\|^2_{\mathcal{D}_\alpha} = (\alpha+1)\sum_{n=1}^\infty \frac{n^2 n!(n+\alpha+1)}{(\alpha+1)_n}|a_n|^2, \tag{2.6}$$

and

$$\|Z_\alpha f\|^2_{\mathcal{D}_\alpha} = (\alpha+1)\sum_{n=1}^\infty \frac{nn!(n+\frac{\alpha}{2})^2}{(\alpha+1)_n}|a_n|^2.$$

Therefore,

$$\|f\|^2_{\mathcal{D}_{\alpha,1}} - |f(0)|^2 - |f'(0)|^2 \le 2\|Xf\|^2_{\mathcal{D}_\alpha} \le 2(\alpha+1)\|f\|^2_{\mathcal{D}_{\alpha,1}},$$

$$\|f\|^2_{\mathcal{D}_{\alpha,1}} - |f(0)|^2 \le \|Y_\alpha f\|^2_{\mathcal{D}_\alpha} \le (\alpha+2)\|f\|^2_{\mathcal{D}_{\alpha,1}},$$

and

$$\|f\|^2_{\mathcal{D}_{\alpha,1}} - |f(0)|^2 \le \|Z_\alpha f\|^2_{\mathcal{D}_\alpha} \le (1+\frac{\alpha}{2})^2\|f\|^2_{\mathcal{D}_{\alpha,1}}.$$

Consequently, $\mathrm{Dom}(X) = \mathrm{Dom}(Y_\alpha) = \mathrm{Dom}(Z_\alpha) = \mathcal{D}_{\alpha,1}$.

(ii) For $f, g \in \mathcal{D}_{\alpha,1}$ with $f(z) = \sum_{n=0}^\infty a_n z^n$ and $g(z) = \sum_{n=0}^\infty b_n z^n$, one has

$$\begin{aligned}\langle Xf, g\rangle_{\mathcal{D}_\alpha} &= (\alpha+1)\sum_{n=1}^\infty \frac{n(n+1)!}{(\alpha+1)_n} a_{n+1}\overline{b_n} \\ &= (\alpha+1)\sum_{n=2}^\infty \frac{(n-1)n!(n+\alpha)}{(\alpha+1)_n} a_n \overline{b_{n-1}} \\ &= \langle f, Y_\alpha g\rangle_{\mathcal{D}_\alpha}.\end{aligned}$$

(iii) follows from (2.4), (2.5) and (2.6). □

In the following we consider the operators

$$X_n f(z) := X^n f(z) = \frac{\mathrm{d}^n}{\mathrm{d}z^n} f(z) - \frac{\mathrm{d}^n}{\mathrm{d}z^n} f(0), \quad Y_{\alpha,n} f(z) := Y_\alpha^n f(z).$$

**Theorem 2.2.** *$Dom(X_n) = Dom(Y_{\alpha,n}) = \mathcal{D}_{\alpha,n}$.*

**Proof.** Let $f \in \mathcal{D}_\alpha$ with $f(z) = \sum_{k=0}^{\infty} a_k z^k$. Then

$$X_n f(z) = \sum_{k=1}^{\infty} \frac{(k+n)!}{k!} a_{k+n} z^k, \quad Y_{\alpha,n} f(z) = \sum_{k=1}^{\infty} \frac{k(k+\alpha+1)_n}{k+n} a_k z^{k+n}.$$

Thus

$$\begin{aligned}
\|X_n f\|_{\mathcal{D}_\alpha}^2 &= (\alpha+1) \sum_{k=1}^{\infty} \frac{k((k+n)!)^2}{k!(\alpha+1)_k} |a_{k+n}|^2 \\
&= (\alpha+1) \sum_{k=1}^{\infty} \frac{k(k+n)! \prod_{j=1}^{n}(k+\alpha+j)(k+j)}{(\alpha+1)_{k+n}} |a_{k+n}|^2 \\
&\leq (\alpha+1)^{n+1} \sum_{k=1}^{\infty} \frac{(k+n)^{2n+1}(k+n)!}{(\alpha+1)_{k+n}} |a_{k+n}|^2 \\
&\leq (\alpha+1)^{n+1} \sum_{k=1}^{\infty} \frac{k^{2n+1} k!}{(\alpha+1)_k} |a_k|^2,
\end{aligned}$$

and

$$\begin{aligned}
\|X_{\alpha,n} f\|_{\mathcal{D}_\alpha}^2 &= (\alpha+1) \sum_{k=1}^{\infty} \frac{k^2 (k+n)!(k+\alpha+1)_n}{(k+n)(\alpha+1)_k} |a_k|^2 \\
&= (\alpha+1) \sum_{k=1}^{\infty} \frac{k^2 k! \prod_{j=1}^{n}(k+\alpha+j)(k+j)}{(k+n)(\alpha+1)_k} |a_k|^2 \\
&\leq (\alpha+1) \sum_{k=1}^{\infty} \frac{k^2 k!(k+\alpha+n)^n (k+n)^{n-1}}{(\alpha+1)_k} |a_k|^2 \\
&\leq (\alpha+1)(n+1)^{n-1}(n+\alpha+1)^n \sum_{k=1}^{\infty} \frac{k^{2n+1} k!}{(\alpha+1)_k} |a_k|^2.
\end{aligned}$$

Therefore,

$$\|f\|_{\mathcal{D}_{\alpha,n}}^2 - a_{n,\alpha}(f) \leq (n+1)^{2n+1} \|X_n f\|_{\mathcal{D}_\alpha}^2 \leq (n+1)^{2n+1}(\alpha+1)^n \|f\|_{\mathcal{D}_{\alpha,n}}^2,$$

where

$$a_{n,\alpha}(f) = |a_0|^2 + (\alpha+1) \sum_{k=1}^{n} \frac{k^{2n+1} k!}{(\alpha+1)_k} |a_k|^2.$$

and

$$\|f\|_{\mathcal{D}_{\alpha,n}}^2 - |f(0)|^2 \leq \|Y_{\alpha,n} f\|_{\mathcal{D}_\alpha}^2 \leq (n+1)^{n-1}(n+\alpha+1)^n \|f\|_{\mathcal{D}_{\alpha,n}}^2.$$

Consequently, $\mathrm{Dom}(X_n) = \mathrm{Dom}(Y_{\alpha,n}) = \mathcal{D}_{\alpha,n}$. □

## 3. Uncertainty inequalities on $\mathcal{D}_\alpha$

In this section, we establish some uncertainty inequalities of Heisenberg type for the weighted Dirichlet space $\mathcal{D}_\alpha$.

**Lemma 3.1.** *$Dom(XY_\alpha) = Dom(Y_\alpha X) = \mathcal{D}_{\alpha,2}$.*

**Proof.** Let $f \in \mathcal{D}_\alpha$ with $f(z) = \sum_{n=0}^{\infty} a_n z^n$. Then

$$XY_\alpha f(z) = \sum_{n=1}^{\infty} n(n+\alpha+1)a_n z^n, \quad Y_\alpha X f(z) = \sum_{n=1}^{\infty}(n-1)(n+\alpha)a_n z^n.$$

Thus

$$\|\Lambda X_\alpha f\|_{\mathcal{D}_\alpha}^2 = (\alpha+1)\sum_{n=1}^{\infty} \frac{n^3 n!(n+\alpha+1)^2}{(\alpha+1)_n}|a_n|^2,$$

and

$$\|X_\alpha \Lambda f\|_{\mathcal{D}_\alpha}^2 = (\alpha+1)\sum_{n=2}^{\infty} \frac{n n!(n-1)^2(n+\alpha)^2}{(\alpha+1)_n}|a_n|^2.$$

Therefore,

$$\|f\|_{\mathcal{D}_{\alpha,2}}^2 - |f(0)|^2 \leq \|XY_\alpha f\|_{\mathcal{D}_\alpha}^2 \leq (\alpha+2)^2\|f\|_{\mathcal{D}_{\alpha,2}}^2,$$

and

$$\|f\|_{\mathcal{D}_{\alpha,2}}^2 - |f(0)|^2 - |f'(0)|^2 \leq 4\|Y_\alpha X f\|_{\mathcal{D}_\alpha}^2 \leq 4(\alpha+1)^2\|f\|_{\mathcal{D}_{\alpha,2}}^2.$$

Consequently, $\mathrm{Dom}(XY_\alpha) = \mathrm{Dom}(Y_\alpha X) = \mathcal{D}_{\alpha,2}$. □

We will use the following result of functional analysis.

**Lemma 3.2.** *(See [9, 12]). Let $A$ and $B$ be self-adjoint operators on a Hilbert space $H$ ($A^* = A$, $B^* = B$). Then*

$$\|(A-a)f\|_H\|(B-b)f\|_H \geq \frac{1}{2}|\langle [A,B]f, f\rangle_H|,$$

*for all $f \in Dom(AB) \cap Dom(BA)$, and all $a, b \in \mathbb{R}$.*

**Theorem 3.1.** *Let $f \in \mathcal{D}_{\alpha,2}$. For all $a, b \in \mathbb{R}$, we have*

$$\|(X+Y_\alpha - a)f\|_{\mathcal{D}_\alpha}\|(X - Y_\alpha + ib)f\|_{\mathcal{D}_\alpha} \geq 2\langle Z_\alpha f, f\rangle_{\mathcal{D}_\alpha}. \tag{3.1}$$

**Proof.** Let us consider the lowering and raising type operators

$$A := X + Y_\alpha, \quad B := i(X - Y_\alpha).$$

By Theorem 2.1 (ii) and Lemma 3.1, the operators $A$ and $B$ possess the following properties.

(i) $A^* = A$ and $B^* = B$.

(ii) $\mathrm{Dom}(AB) = \mathrm{Dom}(BA) = \mathcal{D}_{\alpha,2}$.

(iii) $[A, B] = -2i[X, Y_\alpha]$.

Thus the inequality (3.1) follows from (2.3) and Lemma 3.2. □

*Remark* 3.1. Let $f \in \mathcal{D}_{\alpha,2}$, and let $a, b \in \mathbb{R}$. By Theorem 2.1 (iii) we obtain

$$\|(X+Y_\alpha - a)f\|_{\mathcal{D}_\alpha}\|(X - Y_\alpha + ib)f\|_{\mathcal{D}_\alpha} \geq \|Y_\alpha f\|_{\mathcal{D}_\alpha}^2 - \|Xf\|_{\mathcal{D}_\alpha}^2.$$

And by Theorem 2.1 (i), this uncertainty inequality can be extended to the space $\mathcal{D}_{\alpha,1}$.

**Lemma 3.3.** $Dom(X_n Y_{\alpha,n}) = Dom(Y_{\alpha,n} X_n) = \mathcal{D}_{\alpha,2n}$.

**Proof.** Let $f \in \mathcal{D}_\alpha$ with $f(z) = \sum_{k=0}^{\infty} a_k z^k$. Then

$$X_n Y_{\alpha,n} f(z) = \sum_{k=n}^{\infty} \frac{k(k+n)!(k+\alpha+1)_n}{(k+n)k!} a_k z^k, \tag{3.2}$$

and

$$Y_{\alpha,n} X_n f(z) = \sum_{k=n+1}^{\infty} \frac{(k-1)!(k-n+\alpha+1)_n}{(k-n-1)!} a_k z^k. \tag{3.3}$$

Thus

$$\begin{aligned}
\|X_n Y_{\alpha,n} f\|_{\mathcal{D}_\alpha}^2 &= (\alpha+1) \sum_{k=n}^{\infty} \frac{k^3((k+n)!)^2((k+\alpha+1)_n)^2}{k!(k+n)^2(\alpha+1)_k} |a_k|^2 \\
&= (\alpha+1) \sum_{k=n}^{\infty} \frac{k^3 k! \prod_{j=1}^{n}(k+\alpha+j)^2(k+j)^2}{(k+n)^2(\alpha+1)_k} |a_k|^2 \\
&\leq (\alpha+1) \sum_{k=n}^{\infty} \frac{k^3 k!(k+\alpha+n)^{2n}(k+n)^{2n-2}}{(\alpha+1)_k} |a_k|^2 \\
&\leq (\alpha+1)(n+1)^{2n-2}(n+\alpha+1)^{2n} \sum_{k=n}^{\infty} \frac{k^{4n+1} k!}{(\alpha+1)_k} |a_k|^2,
\end{aligned}$$

and

$$\begin{aligned}
\|Y_{\alpha,n} X_n f\|_{\mathcal{D}_\alpha}^2 &= (\alpha+1) \sum_{k=n+1}^{\infty} \frac{kk!((k-1)!)^2((k-n+\alpha+1)_n)^2}{(\alpha+1)_k((k-n-1)!)^2} |a_k|^2 \\
&= (\alpha+1) \sum_{k=n+1}^{\infty} \frac{kk! \prod_{j=1}^{n}(k-n+\alpha+j)^2(k-j)^2}{(\alpha+1)_k} |a_k|^2 \\
&\leq (\alpha+1) \sum_{k=n+1}^{\infty} \frac{kk!(k+\alpha)^{2n}(k-1)^{2n}}{(\alpha+1)_k} |a_k|^2 \\
&\leq (\alpha+1)^{2n+1} \sum_{k=n+1}^{\infty} \frac{k^{4n+1} k!}{(\alpha+1)_k} |a_k|^2.
\end{aligned}$$

Therefore,

$$\|f\|_{\mathcal{D}_{\alpha,2n}}^2 - b_{n,\alpha}(f) \leq \|X_n Y_{\alpha,n} f\|_{\mathcal{D}_\alpha}^2 \leq (n+1)^{2n-2}(n+\alpha+1)^{2n} \|f\|_{\mathcal{D}_{\alpha,2n}}^2,$$

where

$$b_{n,\alpha}(f) = |a_0|^2 + (\alpha+1) \sum_{k=1}^{n-1} \frac{k^{4n+1} k!}{(\alpha+1)_k} |a_k|^2,$$

and

$$\|f\|_{\mathcal{D}_{\alpha,2n}}^2 - c_{n,\alpha}(f) \leq \|Y_{\alpha,n} X_n f\|_{\mathcal{D}_\alpha}^2 \leq (n+1)^{4n}(\alpha+1)^{2n} \|f\|_{\mathcal{D}_{\alpha,2n}}^2,$$

where

$$c_{n,\alpha}(f) = |a_0|^2 + (\alpha+1) \sum_{k=1}^{n} \frac{k^{4n+1} k!}{(\alpha+1)_k} |a_k|^2.$$

Consequently, $\mathrm{Dom}(X_n Y_{\alpha,n}) = \mathrm{Dom}(Y_{\alpha,n} X_n) = \mathcal{D}_{\alpha,2n}$. □

We define the operator $\Delta_{\alpha,n}$ by

$$\Delta_{\alpha,n} := [X_n, Y_{\alpha,n}] = X_n Y_{\alpha,n} - Y_{\alpha,n} X_n.$$

This operator satisfies the following property.

**Lemma 3.4.** *If $f \in \mathcal{D}_{\alpha,2n}$, with $f(z) = \sum_{k=0}^{\infty} a_k z^k$, we have*

$$\Delta_{\alpha,n} f(z) = \sum_{k=n}^{\infty} r_k(\alpha, n) a_k z^k, \quad \textit{where} \quad r_k(\alpha, n) \geq 0, \quad \textit{for} \quad k \geq n.$$

**Proof.** If $f \in \mathcal{D}_{2n}$, with $f(z) = \sum_{k=0}^{\infty} a_k z^k$, by (3.2) and (3.3) we have

$$\Delta_{\alpha,n} f(z) = \sum_{k=n}^{\infty} r_k(\alpha, n) a_k z^k,$$

where

$$r_k(\alpha, n) = \frac{k(k+n)!(k+\alpha+1)_n}{(k+n)k!} - \frac{(k-n)k!(k-n+\alpha+1)_n}{k(k-n)!}.$$

Then for $k \geq n$ we have

$$r_k(\alpha, n) = \frac{k^2(k+n)!(k-n)!(k+\alpha+1)_n - (k^2-n^2)(k!)^2(k-n+\alpha+1)_n}{k(k+n)k!(k-n)!}.$$

Since $(k+n)!(k-n)! \geq (k!)^2$ and $k!(k-n)! \leq (k!)^2$ we obtain

$$r_k(\alpha, n) \geq \frac{n^2(k+\alpha)^n}{k(k+n)} \geq 0.$$

□

By using Lemma 3.3, Lemma 3.4 and as in the same way of Theorem 3.1 we deduce the following result.

**Theorem 3.2.** *Let $f \in \mathcal{D}_{\alpha,2n}$. For all $a, b \in \mathbb{R}$, we have*

$$\|(X_n + Y_{\alpha,n} - a)f\|_{\mathcal{D}_\alpha} \|(X_n - Y_{\alpha,n} + ib)f\|_{\mathcal{D}_\alpha} \geq \langle \Delta_{\alpha,n} f, f \rangle_{\mathcal{D}_\alpha}.$$

*Remark* 3.2. For $f \in \mathcal{D}_{\alpha,2n}$ we have

$$\|Y_{\alpha,n} f\|^2_{\mathcal{D}_\alpha} = \|X_n f\|^2_{\mathcal{D}_\alpha} + \langle \Delta_{\alpha,n} f, f \rangle_{\mathcal{D}_\alpha}.$$

Then, for all $a, b \in \mathbb{R}$, we obtain

$$\|(X_n + Y_{\alpha,n} - a)f\|_{\mathcal{D}_\alpha} \|(X_n - Y_{\alpha,n} + ib)f\|_{\mathcal{D}_\alpha} \geq \|Y_{\alpha,n} f\|^2_{\mathcal{D}_\alpha} - \|X_n f\|^2_{\mathcal{D}_\alpha}.$$

And by Theorem 2.2, this uncertainty inequality can be extended to the space $\mathcal{D}_{\alpha,n}$.

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Fethi Soltani

*1. Université de Tunis El Manar, Faculté des Sciences de Tunis, Laboratoire d'Analyse Mathématique et Applications LR11ES11, Tunis 2092, Tunisia*

*2. Université de Carthage, Ecole Nationale d'Ingénieurs de Carthage, Tunis 2035, Tunisia*

E-mail address: `fethi.soltani@fst.utm.tn`

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