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k-ALMOST NEWTON-EINSTEIN SOLITONS ON HYPERSURFACES IN GENERALIZED SASAKIAN SPACE FORMS

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Abstract. This research article is based on the study of k-Almost Newton-Einstein solitons (k-ANES) immersed into a generalized Sasakian space forms (GSS-forms). We obtain the minimal and totally geodesic condition for the hypersurface of generalized Sasakian space forms in terms of k-ANES. Besides, we show that a hypersurface $\mathcal{M}^n$ of generalized Sasakian space forms admits the steady k-Almost Newton-Einstein solitons. A few applications of generalized Sasakian space forms that allow k-Almost Newton-Einstein soliton are also explained. We explore the triviality of the Schur's type inequality and show that the gradient Newton-Einstein soliton on GSS-manifold is compact.

1. Introduction

In 2011, Barros et al. studied the immersed almost Ricci soliton on the Riemannian manifold [7]. In particular, if M^{n+p} has non-positive sectional curvature, an almost Ricci soliton is a Ricci soliton and the vector field V has integrable norm on M^n, then M^n can not be minimal. Wylie [34] explained that a complete Riemannian manifold with a shrinking soliton must be compact. If M^{n+p} is a space form of non-positive sectional curvature, then such immersions can not be minimal. Cunha et al.[11] introduced the notion of r-almost Newton-Ricci soliton in Riemannian manifolds by using Newton transformation P_k with second order differential operator $\mathcal{L}_k$ for $0 \leq k \leq n$, (briefly k-ANRS). Siddiqi et al. also discussed about this notion named Newton-Ricci-Bourguignon almost solitons on Lagraigian submanifolds in complex space form (for more details see [33, 23]).

In recent years much effort has been devoted to the classification of self-similar solutions of geometric flows. In 2016, Catino and Mazzieri introduced the notion of Einstein solitons [16], which generate self-similar solutions to Einstein flow

$$\frac{\partial g}{\partial t} = -2\left(Ric - \frac{\rho}{2}d\right), \tag{1.1}$$

where ρ is the scalar curvature of the Riemannian metric d. The interest in studying this equation from different points of view arises from concrete physical

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problems. On the other hand, gradient vector fields play a central role in Morse-Smale theory.

Motivated by the notion of Ricci soliton, Catino and Mazzieri [16] developed the notions of Einstein solitons (for more details see [15] [16]), which satisfies

$$\mathfrak{L}_V d + 2Ric + (2\Lambda - \rho)d = 0, \tag{1.2}$$

where $\mathfrak{L}_V$ is the Lie derivative along the vector field V on M and Λ is a real scalar. An Einstein soliton on (M, d) is said to be shrinking, steady or expanding according as Λ is negative, zero, and positive, respectively.

If the vector field V is the gradient of a potential function $-\psi$, where ψ is some smooth function $\psi : M \to \mathcal{R}$, then d is called a *gradient Einstein soliton* and equation (1.2) assumes the form

$$\nabla^2\psi + Ric = (\Lambda - \frac{1}{2}\rho)d, \tag{1.3}$$

where $\nabla^2\psi$ is the *Hessian* of ψ and ∇ is the covariant derivative operator. According to Pigola et al. [29], if we replace the constant λ in (1.2) with a smooth function $\lambda \in C^\infty(M)$, called soliton function, then we can say that (g, V, λ) on (M, g) is an almost Einstein soliton. Others geometers have extensively discussed the Einstein solitons. For instance, we refer ([15], [20]-[22], [18], [32]) and the references therein.

On other hand, a Riemannian manifold with constant sectional curvature c is known as a real space-form and its curvature tensor is given by

$$\mathcal{R}(U, V)W = c\{d(V, W)U - d(U, W)V\}.$$

A Sasakian manifold with constant ϕ-sectional curvature is a Sasakian space-form and it has a specific form of its curvature tensor. Similar notion also holds for Kenmotsu and cosymplectic space-forms. In order to generalized such space-forms in common frame Alegre et al.[2] developed and studied generalized Sasakian space-forms (briefly GSS). Many geometers have studied generalized Sasakian space forms in the papers (for more details see [3, 4, 5, 6]).

The present article is inspired with the above literature. In this frame work, we explore the study of k-almost Newton-Einstein solitons on hypersurface of generalized Sasakian space forms.

2. Generalized Sasakian space forms

A $(2n+1)$-dimensional differentiable manifold $\overline{\mathcal{M}}$ is said to have an almost contact structure (ϕ, ξ, η, d) if there exists on $\overline{\mathcal{M}}$ a tensor field ϕ of type (1, 1), a vector field ξ, a 1-form η and a Riemannian metric d such that [2]

$$\phi^2 = -I + \eta \otimes \xi, \quad \phi\xi = 0, \quad \eta(\xi) = 1, \quad \eta(\phi) = 0, \quad \eta(U) = d(U, \xi) \tag{2.1}$$

$$d(\phi U, \phi V) = d(U, V) - \eta(U)\eta(V), \quad d(\phi U, V) + d(U, \phi V) = 0 \tag{2.2}$$

Here U, V, W denote arbitrary vector fields on $\overline{\mathcal{M}}$. The fundamental 2-form φ on $\overline{\mathcal{M}}$ is defined by

$$\varphi(U, V) = d(\phi U, V)$$

An almost contact metric manifold $(\overline{\mathcal{M}}, \phi, \xi, \eta, d)$ is said to be a *generalized Sasakian space form* (GSS-forms) if there exist differentiable functions f_1, f_2, f_3 such that curvature tensor R of $\overline{\mathcal{M}}$ is given by

$$\begin{aligned}\mathcal{R}(U,V)W &= f_1\big[d(V,W)U - d(U,W)V\big] + f_2\big[d(U,\phi W)\phi V - d(V,\phi W)\phi U \\ &\quad +2d(U,\phi V)\phi W\big] + f_3\big[\eta(U)\eta(W)V - \eta(V)\eta(W)U + d(U,W)\eta(Y)\xi \\ &\quad -d(V,W)\eta(U)\xi\big] \qquad (2.3)\end{aligned}$$

for all vector fields $U, V, W \in T\overline{\mathcal{M}}$.

The GSS-form generalizes the concept of Sasakian space form, Kenmotsu space form and cosymplectic space form as follows:

(i) A Sasakian space form is the generalized Sasakian space form with $f_1 = \frac{c+3}{4}$ and $f_2 = f_3 = \frac{c-1}{4}$.

(ii) A Kenmotsu space form is the generalized Sasakian space form with $f_1 = \frac{c-3}{4}$ and $f_2 = f_3 = \frac{c+1}{4}$.

(iii) A cosymplectic space form is the generalized Sasakian space form with $f_1 = f_2 = f_3 = \frac{c}{4}$.

In the following we consider $\overline{\mathcal{M}}$ as a generalized Sasakian space form $\overline{\mathcal{M}}(f_1, f_2, f_3)$ of dimension $(2n+1)$ and let $\mathcal{M}$ be an $n$dimensional submanifold of $\overline{\mathcal{M}}(f_1, f_2, f_3)$. Let $T\mathcal{M}$ and $T^{\perp}\mathcal{M}$ denote the Lie algebra of vector fields and set of all normal vector fields on $\mathcal{M}$ respectively. The operator of covariant differentiation with respect to the Levi-Civita connection in $\mathcal{M}$ and $\overline{\mathcal{M}}$ is denoted by ∇ and $\overline{\nabla}$, respectively. Let $\overline{R}$ and R be the curvature tensor of $\overline{\mathcal{M}}(f_1, f_2, f_3)$ and $\mathcal{M}$, respectively.

3. k-almost Newton-Einstein soliton

We recall that an oriented and connected hypersurface $f : \mathcal{M}^n \longrightarrow \overline{\mathcal{M}}^{2n+1}$ is to be immersed into an $(2n+1)$-GSS-forms manifold $\overline{\mathcal{M}}^{n+1}$. Then $\mathcal{M}^n$ is called an k-ANES, for some $0 \leq k \leq m$, if there exists a function $\psi : \mathcal{M}^n \longrightarrow \mathbb{R}$ such that ([16], [11])

$$Ric + P_k \circ Hess\psi = \Big(\Lambda - \frac{\rho}{2}\Big)\, d, \qquad (3.1)$$

where ψ and Λ both are smooth functions on $\mathcal{M}^n$ and $P_k \circ Hess\psi$ stands for tensor given by

$$P_k \circ Hess\psi(U,W) = d(P_k \nabla_U \nabla_\psi, W), \qquad (3.2)$$

$U, W \in \mathcal{X}(\mathcal{M})$. For $k = 0$, equation (3.1) reduces to a gradient almost Einstein soliton. Here P_k denotes the k-th Newton transformation $P_k : \mathcal{X}(\mathcal{M}) \longrightarrow \mathcal{X}(\mathcal{M})$ such that $P_0 = I$ (identity operator).

Example 3.1. Let us consider the standard immersion of $\mathcal{M}^n$ in $\mathbb{S}^{2n+1}(1)$, which we know that its is totally geodesic. In particular, $P_r = 0$ for all $1 \leq r \leq n$, and choosing $\Lambda = \frac{(n-1)}{n-\frac{1}{2}}$, we obtain that the immersion satisfies equation (3.1).

Also we can see that if $\mathcal{M}$ is constant scalar curvature then the equation (3.1) become

$$\mathrm{Ric} + \mathrm{P_r} \circ \mathrm{Hess} f = \mu g,$$

where $\mu = \Lambda - \frac{1}{2}\rho$. So, we can recall to the Example 2 of [11] to another example of gradient r-almost-Newton-Einstein soliton.

The Gauss equation implies that

$$\mathcal{R}(U,W)Z = (\bar{\mathcal{R}}(U,W)Z)^T + d(BU,Z)BW - d(BW,Z)BU \tag{3.3}$$

for every tangent vector fields $U, W, Z \in \mathcal{X}(\mathcal{M}^n)$, where $()^T$ denotes the tangential components of a vector field in $\mathcal{X}(\mathcal{M}^n)$ along $\mathcal{M}^n$. Here the second fundamental form (or shape operator) B of $\mathcal{M}^n$ in $\overline{\mathcal{M}}^{2n+1}$ is related with the second fundamental form h by the relation

$$d(h(U,W),\alpha) = d(B_\alpha U, W) \tag{3.4}$$

for a normal vector field α on $\mathcal{M}^n$.

Let $\bar{\mathcal{R}}$ and $\mathcal{R}$ represent the Riemannian curvature tensors of $\overline{\mathcal{M}}^{2n+1}$ and $\mathcal{M}^n$, respectively.
The scalar curvature ρ of the of the hypersurface $\mathcal{M}^n$ satisfies

$$\rho = \sum_{i,j}^{m} d(\bar{R}(E_i,E_j)E_j,E_i) + n^2H^2 - \|B\|^2, \tag{3.5}$$

where $\{E_1,\dots,E_m\}$ is an orthonormal frame on $T(M)$ and $\|B\|$ indicates the Hilbert-Schmidt norm. If $\overline{\mathcal{M}}^{2n+1}$ is a GSS-forms with functions f_1, f_2, f_3, then the scalar curvature ρ is given by

$$\rho = 2n(2n+1)f_1 + 6nf_2 - 4nf_3 + n^2H^2 - \|B\|^2. \tag{3.6}$$

There exist n algebraic invariants corresponding to the second fundamental form B of the hypersurface $\mathcal{M}^n$, which are the elementary symmetric functions ρ_k of its principal curvatures $r_1,\dots,r_m$, and are given by

$$\rho_0 = 1, \quad \rho_k = \sum_{i_1<\dots<i_k} r_1 \dots r_n. \tag{3.7}$$

The k-th mean curvature H_k of the immersion is defined by $\binom{n}{k}H_k = \rho_k$. If $k=0$, we have $H_1 = \frac{1}{n}Tr(A) = H$, the mean curvature of $\mathcal{M}^n$. Here Tr stands for trace. For each $0 \le k \le m$, we define the Newton transformation $P_k : \mathcal{X}(\mathcal{M}^n) \longrightarrow \mathcal{X}(\mathcal{M}^n)$ of the hypersurface $\mathcal{M}^n$ by setting $P_0 = I$ and for $0 \le k \le m$, by the recurrence relation

$$P_k = \sum_{j=0}^{k} (-1)^{k-j}\tbinom{m}{j} H_j A^{k-j}, \tag{3.8}$$

where B^j represents the composition of B with itself j times ($B^0 = I$). The second order linear differential operator $\mathcal{L}_k : C^\infty(\mathcal{M}^n) \longrightarrow C^\infty(\mathcal{M}^n)$ is defined by

$$\mathcal{L}_k u = Tr(P_k \circ Hess u). \tag{3.9}$$

If we take $k = 0$, then we have the Laplacian operator $\mathcal{L}_0$. Also, we have

$$div_{\mathcal{M}}(P_k \nabla u) = \sum_{i=1}^{m} d((\nabla_{E_i} P_k)\nabla_u, E_i) + \sum_{i=1}^{m} d(P_k(\nabla_{E_i}\nabla_u), E_i) \tag{3.10}$$

$$= d(div_{\mathcal{M}} P_k, \nabla_u) + \mathcal{L}_k u,$$

where

$$div_{\mathcal{M}} P_k = Tr(\nabla P_k) = \sum_{i=1}^{m} (\nabla_{E_i} P_k) E_i. \tag{3.11}$$

If the ambient space possesses the constant sectional curvatures, then equation (3.10) takes the form

$$\mathcal{L}_k u = div_{\mathcal{M}}(P_k \nabla u) \tag{3.12}$$

because $div_{\mathcal{M}} P_k = 0$ (see [28] for more details).

The trace-less second fundamental form of the hypersurface is given by

$$\Phi = BHI, \qquad Tr(\Phi) = 0 \tag{3.13}$$

and

$$|\Phi|^2 = Tr(\Phi^2) = \|B\|^2 - nH^2 \geq 0. \tag{3.14}$$

The manifold $\mathcal{M}^n$ is totally umbilical if and only $|\Phi|^2 = 0$.

To establish our results let us adopt the following maximum principle (for more details see [10]). We follows that, for all $s \geq 1$, adopt the notation

$$\mathcal{L}^s(L) = \left\{ u : \mathcal{M}^n \longrightarrow \mathcal{R};\ \int_{\mathcal{M}} |u|^s \, dL < +\infty \right\}. \tag{3.15}$$

Also, we have the following lemma:

Lemma 3.1. *Let $\mathcal{M}^n$ be an n-dimensional, complete, non-compact, oriented Riemannian manifold and $div_{\mathcal{M}} U$ does not alter the sign on $\mathcal{M}^n$ for a smooth vector field U. If $|U| \in \mathcal{L}^1(\mathcal{M})$, then $div_{\mathcal{M}} U = 0$.*

The following results further generalize Theorem 1.2 of [8].

Theorem 3.1. *Let (d, ψ, Λ, k) denote a complete k-ANES on hypersurface $\mathcal{M}^n$ of GSS-forms $\overline{\mathcal{M}}^{2n+1}$ of functions f_1, f_2, f_3 with bounded B and potential function $\psi : \mathcal{M}^n \longrightarrow \mathcal{R}$ such that $|\nabla \psi| \in \mathcal{L}^1(\mathcal{M})$. If*

1. *$f_1, f_2, f_3 \leq 0$, and $\Lambda > 0$, then $\mathcal{M}^n$ can not be minimal,*
2. *$f_1, f_2, f_3 < 0$, and $\Lambda \geq 0$, then $\mathcal{M}^n$ can not be minimal.*
3. *$f_1, f_2, f_3 = 0$, $\Lambda \geq 0$ and $\mathcal{M}^n$ is minimal, then $\mathcal{M}^n$ is isometric to the $\mathbb{R}^n$.*

Proof. Since f_1, f_2 and f_3 are functions in terms of the the constant sectional curvature c on the ambient space, then from (3.12) we notice that the operator $\mathcal{L}_k$ is divergent type operator. Also, the second fundamental form is bounded on $\mathcal{M}^n$ and thus from (3.8) we notice that the Newton transformation P_k has bounded norm, that is,

$$|P_k \nabla \psi| \leq |P_k| \, |\nabla \psi| \in \mathcal{L}^1(\mathcal{M}). \tag{3.16}$$

To prove (1) and (2), let us consider by contradiction that $\mathcal{M}^n$ is minimal. Then, equation (3.6) together with the consideration $f_1, f_2, f_3 \leq 0$ and $f_1, f_2, f_3 < 0$ imply that the scalar curvature of $\mathcal{M}^n$ satisfies $\rho \leq 0\,(\rho < 0)$. Hence, by contracting (3.1) we have $\mathcal{L}_r\psi = n\Lambda - \frac{(n+2)\rho}{2} > 0$ in both cases, which contradicts Lemma 3.1. This completes the proof of the assertions (1) and (2).

For the (3) assertion, Since c_1 and c_2 are the the constant sectional curvatures of the ambient space and $\mathcal{M}^n$ is minimal, then equation (3.6) becomes

$$\rho = -\frac{2\parallel B\parallel^2}{(n+2)} \leq 0. \tag{3.17}$$

Since $\Lambda \geq 0$ therefore we have $\mathcal{L}_r(\psi) = n\Lambda - \frac{(n+2)\rho}{2} \geq 0$. Now, using the fact that $\mathcal{L}_r u = div_{\mathcal{M}}(P_k \nabla u)$ and $|P_k\nabla\psi| \in \mathcal{L}^1(\mathcal{M})$, we have again from Lemma (3.1) that $\mathcal{L}_r\psi = 0$ on $\mathcal{M}^n$. Thus, we observe that $0 \geq \frac{(n+2)\rho}{2} = n\Lambda \geq 0$, that is, $\rho = \lambda = 0$. This shows that $\|B\|^2 = 0$ and therefore the k-ANES $\mathcal{M}^n$ is geodesic and flat. □

To prove our next theorems we need the following lemma, which corresponds to Theorem 3 of [37].

Lemma 3.2. *Let a complete Riemannian manifold M^n admits a non-negative smooth subharmonic function u. If $u \in \mathcal{L}^s(\mathcal{M})$, then u is constant for some $s > 1$.*

Further, we state the following:

Theorem 3.2. *Let (d, ψ, Λ, k) denote a complete k-ANES on hypersurface $\mathcal{M}^n$ of GSS-forms $\overline{\mathcal{M}}^{2n+1}$ of functions f_1, f_2, f_3 with sectional curvature $K_{\mathcal{M}}$, P_k is bounded from above (in the sense of quadratic forms) and potential function $\psi : \mathcal{M}^n \longrightarrow \mathcal{R}$ is non-negative such that $\psi \in \mathcal{L}^s(\mathcal{M})$ for some $s > 1$. If*

1. *$K_{\mathcal{M}} \leq 0$ and $\Lambda > 0$, then $\mathcal{M}^n$ can not be minimal,*
2. *$K_{\mathcal{M}} < 0$ and $\Lambda \geq 0$, then $\mathcal{M}^n$ can not be minimal,*
3. *$K_{\mathcal{M}} \leq 0$, $\Lambda \geq 0$ and $\mathcal{M}^n$ is minimal, then $\mathcal{M}^n$ is flat and totally geodesic.*

Proof. For proving (1), we begin with a contradiction that $\mathcal{M}^n$ is minimal. By the hypothesis and equation (3.5) we have $\rho \leq 0$. The contraction of equation (3.1) gives

$$\mathcal{L}_k\psi = n\Lambda - \frac{(n+2)\rho}{2} > 0. \tag{3.18}$$

Since we considered that P_k is bounded from above, therefore there exists a positive constant ω such that

$$\omega\Delta\psi \geq \mathcal{L}_k\psi > 0. \tag{3.19}$$

Thus, from Lemma 3.2 we conclude that ψ is constant, which is inadmissible. By using the similar process of the proof of Theorem 3.1, we can easily obtain (2) and (3). □

In our next theorem, we generalize the Theorem 1.5 of [7] for $U = \nabla\psi$. We also give the conditions for an k-ANES on hypersurface of GSS-forms to be totally umbilical, provided $\mathcal{M}^n$ has bounded second fundamental form. Thus we state the following:

Theorem 3.3. *If the data (d, ψ, Λ, k) be a complete k-ANES on hypersurface $\mathcal{M}^n$ of GSS-forms $\overline{\mathcal{M}}^{2n+1}$ of functions f_1, f_2, f_3 with bounded second fundamental form and the potential function $\psi : \mathcal{M}^n \longrightarrow \mathcal{R}$ such that $|\nabla\psi| \in \mathcal{L}^1(\mathcal{M})$. Then for*

(1) $\Lambda \geq (2n+1)(n+2)f_1 - 3(n+2)f_2 + 2(n+2)f_3 + \frac{n(n+2)}{2}H^2$ *is totally geodesic with* $\Lambda = (2n+1)(n+2)f_1 - 3(n+2)f_2 + 2(n+2)f_3$, *and the scalar curvature* $\rho = 2n(2n+1)f_1 + 6n(n+2)f_2 - 4n(n+2)f_3$,

(2) $\mathcal{M}^n$ *is compact and* $\Lambda \geq (2n+1)(n+2)f_1 + 3(n+2)f_2 - 2(n+2)f_3 - \frac{n(n+2)}{2}H^2$, $\mathcal{M}^n$ *is isometric to a Euclidean sphere,*

(3) $\Lambda \geq (n+2)\left\{(2n+1)f_1 - 3f_2 + f_3 + \frac{n}{2}H^2\right\}$, $\mathcal{M}^n$ *is totally umbilical. Particularly, the scalar curvature* $\rho = n(n+2)\left\{(2n+1)f_1 - 3f_2 + f_3\right\}K_{\mathcal{M}}$ *is constant, where* $K_{\mathcal{M}} = \frac{2\Lambda}{(n+2)}$ *is the sectional curvature of* $\mathcal{M}^n$.

Proof. Using the equations (3.1) and (3.6), we obtain

$$\mathcal{L}_r\psi = n\left[\Lambda - (2n+1)(n+2)f_1 + 3(n+2)f_2 - 2(n+2)f_3 - \frac{n(n+2)}{2}H^2\right] + \frac{\|B\|^2}{2}, \tag{3.20}$$

For our consideration on λ, we can easily get that $\mathcal{L}_r\psi$ is non-negative function on $\mathcal{M}^n$. By Lemma 3.1 we find that $\mathcal{L}_k\psi$ vanishes identically. Thus, from (3.20) we arrive that $\mathcal{M}^n$ is totally geodesic and we turn up

$$\Lambda = (2n+1)(n+2)f_1 + 3(n+2)f_2 - 2(n+2)f_3. \tag{3.21}$$

Moreover, it is clear form (3.6) that $\rho = 2n(2n+1)f_1 + 6nf_2 - 4nf_3$, which complete the proof of (1).

If $\mathcal{M}^n$ is compact, since it is totally geodesic, then the ambient space must be a sphere $\mathcal{S}^{2n+1}$ and $\overline{\mathcal{M}}^{2n+1}$ is isometric to the Euclidean sphere $\mathcal{S}^{2n+1}$, proving (2). From equation (3.20), we have

$$\mathcal{L}_k\psi = n[\Lambda - (n+2)\left\{(2n+1)f_1 - 3f_2 + f_3 + \frac{n}{2}H^2\right\}] + |\Phi|^2. \tag{3.22}$$

Therefore, our assumption on Λ gives $\mathcal{L}_k\psi \geq 0$. From Lemma (3.1) we have $\mathcal{L}_k\psi = 0$. This shows that $\mathcal{M}^n$ is a totally umbilical. In particular, the principal curvature κ of $\mathcal{M}^n$ is constant and hence $\mathcal{M}^n$ possesses a constant sectional curvature
$K_{\mathcal{M}} = (n+2)\left\{(2n+1)f_1 - 3f_2 + f_3 + \frac{n}{2}\kappa^2\right\}$. This relation together with (3.22) give

$$\Lambda = (n+2)\left\{(2n+1)f_1 - 3f_2 + f_3 + (n+2)H^2\right\} \tag{3.23}$$

$$= (n+2)\left\{(2n+1)f_1 - 3f_2 + f_3 + (n+2)\kappa^2\right\}$$

$$= (n+2)K_{\mathcal{M}},$$

which implies that $\rho = n(n+2)K_{\mathcal{M}}$, as desired. ∎

Theorem 1.6 of [7] state that an minimal immersed nontrivial almost Ricci soliton $\mathcal{M}^n$ in S^{n+1} with $\rho \geq n(n \geq 2)$ and the norm of the second fundamental form obtain its maximum, then S^n must be isometric. Now, with help of Theorem 3.3, we can state a generalization of in the following.

Corollary 3.1. *Let the data (d, ψ, Λ, k) be a complete k-ANES on hypersurface $\mathcal{M}^n$ of GSS-forms $\overline{\mathcal{M}}^{2n+1}$ with functions f_1, f_2, f_3 . If $\lambda \geq (n+2)H^2$, then $\mathcal{M}^n$ is isometric to $\mathcal{S}^n$.*

From Theorem (3.3) (1) which entails the following corollary.

Corollary 3.2. *If the data (d, ψ, λ, k) be a complete k-ANES on hypersurface $\mathcal{M}^n$ of GSS-forms $\overline{\mathcal{M}}^{2n+1}$ with functions f_1, f_2, f_3, then $\mathcal{M}^n$ admits the steady k-ANES.*

Corollary 3.3. *Let (d, ψ, Λ, k) be a complete k-ANES on hypersurface $\mathcal{M}^n$ of GSS-forms $\overline{\mathcal{M}}^{2n+1}$ of functions f_1, f_2, f_3. Consider that $\rho \geq n(n+2)$, the norm of the second fundamental form obtain its maximum and $\lambda \geq (n+2)H^2$. Then $\mathcal{M}^n$ is isometric to $\mathcal{S}^n$.*

Proof. From Simon's formula [31], we obtain

$$\Delta \|B\|^2 = \|\nabla B\|^2 + (2n - \|B\|^2)\|B\|^2 \geq 0. \tag{3.24}$$

Also, the immersion is minimal with $\rho \geq m(m-2)$, therefore from (3.6) we arrive at

$$2\frac{\|B\|^2}{(n+2)} = n - \frac{(n+2)}{2}\rho \leq 2n+1.$$

From Hopf's strong maximum principle and equation (3.24), we find that $\nabla B = 0$ on $\overline{\mathcal{M}}^{n+1}$. Thus from Proposition 1 of [27] we conclude that $\mathcal{M}^n$ is compact and from Theorem 3.3 the result follows. □

Theorem 3.4. *Let the data (d, ψ, Λ, k) be a complete k-ANES on hypersurface $\mathcal{M}^n$ of GSS-forms $\overline{\mathcal{M}}^{2n+1}$ with functions f_1, f_2, f_3 is bounded from above and its potential function $\psi : \mathcal{M}^n \longrightarrow \mathcal{R}$ is non-negative and $\psi \in \mathcal{L}^s(\mathcal{M})$ for some $s > 1$. If*

1. *$\Lambda \geq (2n+1)(n+2)f_1 - 3(n+2)f_2 + 2(n+2)f_3 + \frac{n(n+2)}{2}H^2$ is totally geodesic with $\Lambda = (2n+1)(n+2)f_1 - 3(n+2)f_2 + 2(n+2)f_3$, and the scalar curvature $\rho = 2n(2n+1)f_1 + 6n(n+2)f_2 - 4n(n+2)f_3$.*
2. *$\Lambda \geq (n+2)\left\{(2n+1)f_1 - 3f_2 + f_3 + \frac{n}{2}H^2\right\}$, $\mathcal{M}^n$ is totally umbilical. Particularly, the scalar curvature $\rho = n(n+2)\left\{(2n+1)f_1 - 3f_2 + f_3\right\}K_{\mathcal{M}}$ is constant, where $K_{\mathcal{M}} = \frac{2\Lambda}{(n+2)}$ is the sectional curvature of $\mathcal{M}^n$.*

Proof. The hypothesis on Λ and equation (3.20) give

$$\mathcal{L}_r\psi = n[\Lambda - (2n+1)(n+2)f_1 - 3(n+2)f_2 + 2(n+2)f_3 + \frac{n(n+2)}{2}H^2]\|B\|^2$$
$$\geq 0. \tag{3.25}$$

As P_k is bounded from above, therefore $\omega\Delta\psi \geq \mathcal{L}_k\psi \geq 0$ for a positive constant ω. Using Lemma 3.2, we conclude that ψ is constant. Therefore $\mathcal{L}_n\psi = 0$, and equation (3.25) shows that $\mathcal{M}^n$ is totally geodesic,

$$\Lambda = (2n+1)(n+2)f_1 - 3(n+2)f_2 + 2(n+2)f_3$$

and the scalar curvature

$$\rho = (2n+1)(n+2)f_1 - 3(n+2)f_2 + 2(n+2)f_3$$

proving assertion (1). Assertion (2) can be obtained by following process of the proof of Theorem 3.3. □

4. Some Applications

As an application of Theorem (3.1), we obtain the following results for the Sasakian space form, Kenmotsu space form and cosymplectic space form the following values of f_1, f_2, f_3

(i) A Sasakian space form is the generalized Sasakian space form with $f_1 = \frac{c+3}{4}$ and $f_2 = f_3 = \frac{c-1}{4}$.

(ii) A Kenmotsu space form is the generalized Sasakian space form with $f_1 = \frac{c-3}{4}$ and $f_2 = f_3 = \frac{c+1}{4}$.

(iii) A cosymplectic space form is the generalized Sasakian space form with $f_1 = f_2 = f_3 = \frac{c}{4}$.

Theorem 4.1. *Let (d, ψ, Λ, k) denote a complete k-ANES on hypersurface $\mathcal{M}^n$ of Sasakian space forms $\overline{\mathcal{M}}^{2n+1}$ with constant sectional curvature c with bounded B and potential function $\psi : \mathcal{M}^n \longrightarrow \mathcal{R}$ such that $|\nabla\psi| \in \mathcal{L}^1(\mathcal{M})$. If*

(1) $\frac{c+3}{4} \leq 0$, $\frac{c-1}{4} \leq 0$ *and* $\Lambda > 0$, *then $\mathcal{M}^n$ can not be minimal,*
(2) $\frac{c+3}{4} < 0$, $\frac{c-1}{4} < 0$ *and* $\lambda \geq 0$, *then $\mathcal{M}^n$ can not be minimal.*
(3) $\frac{c+3}{4} = 0$, $\frac{c-1}{4} = 0$, $\lambda \geq 0$ *and $\mathcal{M}^n$ is minimal, then $\mathcal{M}^n$ is isometric to the $\mathbb{R}^n$.*

Theorem 4.2. *Let (d, ψ, Λ, k) denote a complete k-ANES on hypersurface $\mathcal{M}^n$ of Kenmotsu space forms $\overline{\mathcal{M}}^{2n+1}$ with constant sectional curvature c with bounded B and potential function $\psi : \mathcal{M}^n \longrightarrow \mathcal{R}$ such that $|\nabla\psi| \in \mathcal{L}^1(\mathcal{M})$. If*

(1) $\frac{c-3}{4} \leq 0$, $\frac{c+1}{4} \leq 0$ *and* $\Lambda > 0$, *then $\mathcal{M}^n$ can not be minimal,*
(2) $\frac{c-3}{4} < 0$, $\frac{c+1}{4} < 0$ *and* $\lambda \geq 0$, *then $\mathcal{M}^n$ can not be minimal.*
(3) $\frac{c-3}{4} = 0$, $\frac{c+1}{4} = 0$, $\lambda \geq 0$ *and $\mathcal{M}^n$ is minimal, then $\mathcal{M}^n$ is isometric to the $\mathbb{R}^n$.*

Theorem 4.3. *Let (d, ψ, Λ, k) denote a complete k-ANES on hypersurface $\mathcal{M}^n$ of cosymplectic space form $\overline{\mathcal{M}}^{2n+1}$ with constant sectional curvature c with bounded B and potential function $\psi : \mathcal{M}^n \longrightarrow \mathcal{R}$ such that $|\nabla\psi| \in \mathcal{L}^1(\mathcal{M})$. If*

(1) $\frac{c}{4} \leq 0$ *and* $\Lambda > 0$, *then $\mathcal{M}^n$ can not be minimal,*
(2) $\frac{c}{4} < 0$ *and* $\lambda \geq 0$, *then $\mathcal{M}^n$ can not be minimal.*
(3) $\frac{c}{4} = 0$ *and* $\lambda \geq 0$ *and $\mathcal{M}^n$ is minimal, then $\mathcal{M}^n$ is isometric to the $\mathbb{R}^n$.*

This also generalizes Theorem 3.2 for others spaces as follows: Next, we have:

Theorem 4.4. *Let (d, ψ, λ, k) denote a complete k-ANES on hypersurface $\mathcal{M}^n$ of Sasakian space forms $\overline{\mathcal{M}}^{2n+1}$ of sectional curvature $K_{\mathcal{M}}$, P_k is bounded from above (in the sense of quadratic forms) and potential function $\psi : \mathcal{M}^n \longrightarrow \mathcal{R}$ is non-negative such that $\psi \in \mathcal{L}^s(\mathcal{M})$ for some $s > 1$. If*

(1) $K_{\mathcal{M}} \leq 0$ *and* $\Lambda > 0$, *then* $\mathcal{M}^n$ *can not be minimal,*
(2) $K_{\mathcal{M}} < 0$ *and* $\Lambda \geq 0$, *then* $\mathcal{M}^n$ *can not be minimal,*
(3) $K_{\mathcal{M}} \leq 0$, $\Lambda \geq 0$ *and* $\mathcal{M}^n$ *is minimal, then* $\mathcal{M}^n$ *is flat and totally geodesic.*

Theorem 4.5. *Let (d, ψ, λ, k) denote a complete k-ANES on hypersurface $\mathcal{M}^n$ of Kenmotsu space forms $\overline{\mathcal{M}}^{2n+1}$ of sectional curvature $K_{\mathcal{M}}$, P_k is bounded from above (in the sense of quadratic forms) and potential function $\psi : \mathcal{M}^n \longrightarrow \mathcal{R}$ is non-negative such that $\psi \in \mathcal{L}^s(\mathcal{M})$ for some $s > 1$. If*

(1) $K_{\mathcal{M}} \leq 0$ *and* $\Lambda > 0$, *then* $\mathcal{M}^n$ *can not be minimal,*
(2) $K_{\mathcal{M}} < 0$ *and* $\Lambda \geq 0$, *then* $\mathcal{M}^n$ *can not be minimal,*
(3) $K_{\mathcal{M}} \leq 0$, $\Lambda \geq 0$ *and* $\mathcal{M}^n$ *is minimal, then* $\mathcal{M}^n$ *is flat and totally geodesic.*

Theorem 4.6. *Let (d, ψ, λ, k) denote a complete k-ANES on hypersurface $\mathcal{M}^n$ of cosymplectic space form $\overline{\mathcal{M}}^{2n+1}$ of sectional curvature $K_{\mathcal{M}}$, P_k is bounded from above (in the sense of quadratic forms) and potential function $\psi : \mathcal{M}^n \longrightarrow \mathcal{R}$ is non-negative such that $\psi \in \mathcal{L}^s(\mathcal{M})$ for some $s > 1$. If*

(1) $K_{\mathcal{M}} \leq 0$ *and* $\Lambda > 0$, *then* $\mathcal{M}^n$ *can not be minimal,*
(2) $K_{\mathcal{M}} < 0$ *and* $\Lambda \geq 0$, *then* $\mathcal{M}^n$ *can not be minimal,*
(3) $K_{\mathcal{M}} \leq 0$, $\Lambda \geq 0$ *and* $\mathcal{M}^n$ *is minimal, then* $\mathcal{M}^n$ *is flat and totally geodesic.*

5. Compact gradient r-Newton-Einstein soliton

In this segment, our main results based on some triviality results when the gradient r-Newton-Einstein soliton on GSS-forms $\overline{\mathcal{M}}^{2n+1}$ is compact and Λ is a constant. In addition useful consequences are given in the following lemmas.

Lemma 5.1 ([30])**.** *If $\overline{\mathcal{M}}$ is compact without boundary or if $\overline{\mathcal{M}}$ is non compact and ψ has compact support then*

(i) $\displaystyle\int_{\overline{\mathcal{M}}} L_r(\psi) d\overline{\mathcal{M}} = 0,$

(ii) $\displaystyle\int_{\overline{\mathcal{M}}} \psi L_r(\psi) d\overline{\mathcal{M}} = -\int_{\overline{\mathcal{M}}} \langle P_r \nabla\psi, \nabla\psi \rangle.$

For our purpose, it also will be appropriate to deal with the so-called traceless second fundamental form of the hypersurface of GSS-forms $\overline{\mathcal{M}}^{2n+1}$, which is given by $\Phi = A - HI$. Observe that tr $\Phi = 0$ and $|\Phi|^2 = \mathrm{tr}(\Phi^2) = |A|^2 - nH^2 \geq 0$, with equality if and only if $\overline{\mathcal{M}}^{2n+1}$ is totally umbilical.

To conclude this section we recall the following Lemma due to Yau and corresponds to Theorem 3 of [37].

Lemma 5.2. *Let u be a non-negative smooth subharmonic function on a complete Riemannian manifold M^n. If $u \in L^p(\mathcal{M})$, for some $p > 1$, then u is constant.*

Here we use the notation $L^p(\mathcal{M}) = \{u : \mathcal{M}^n \to \mathbb{R}\ ;\ \int_{\mathcal{M}} |u|^p d\mathcal{M} < +\infty\}$ for each $p \geq 1$.

Theorem 5.1. *Let $\mathcal{M}^n$ be a compact gradient r-Newton-Einstein soliton immersed into a GSS-forms $\overline{\mathcal{M}}^{2n+1}$ of functions f_1, f_2, f_3 with constant sectional curvature c, such that P_r is bounded from above or from below (in the sense of quadratic forms). If holds any one of the following*

i) *either $\frac{n}{2} > -1$ and the scalar curvature is $\rho \geq 0$ and $\Lambda \geq 0$ or $\rho \leq 0$ and $\Lambda \leq 0$ or;*
ii) *$\frac{n}{2} < -1$ and the scalar curvature is $\rho \geq 0$ and $\Lambda \leq 0$ or $\rho \leq 0$ and $\Lambda \geq 0$ or,*
iii) *the scalar curvature, either $\rho \geq \frac{2n\Lambda}{n+2}$ or $\rho \leq \frac{2n\Lambda}{n+2}$,*

then $\mathcal{M}$ most be constant scalar curvature and trivial.

Proof. From Lemma 5.1 and estructural equation we obtain

$$0 = \int_{\mathcal{M}} L_r\psi = \int_{\mathcal{M}} [\Lambda n - (\frac{n}{2} + 1)\rho].$$

Hence, if holds (i), (ii) we obtain $\rho = \Lambda = 0$ and from estructural equation we get $L_r\psi = 0$. Since P_r is bounded from above or from below (in the sense of quadratic forms), there is a positive constant $C > 0$ such that

$$0 = L_r\psi \leq C\Delta\psi, \text{ or } 0 = L_r\psi \geq -C\Delta\psi,$$

respectively. So, ψ is a subharmonic function. Since $\mathcal{M}$ is compact we conclude from Hopf's theorem that ψ is a constant function. Therefore $\mathcal{M}$ is trivial.

Finally the item (iii) follows identically to (i) and (ii). □

Theorem 5.2. *Let $\mathcal{M}^n$ be a compact gradient r-Newton-Einstein soliton immersed into a GSS-forms $\overline{\mathcal{M}}^{2n+1}$ of functions f_1, f_2, f_3 with constant sectional curvature c, such that P_r is bounded from above or from below (in the sense of quadratic forms) and $\frac{n}{2} \neq -1$. If M has constant scalar curvature, then $\mathcal{M}^n$ is trivial.*

Proof. From Lemma 5.1 and estructural equation we have

$$\int_{\mathcal{M}} |n\Lambda - (\frac{n}{2} + 1)\rho|^2 = \int_{\mathcal{M}} (n\Lambda - (\frac{n}{2} + 1)\rho)L_r\psi = (n\Lambda - (\frac{n}{2} + 1)\rho) \int_{\mathcal{M}} L_r\psi = 0.$$

Hence, we obtain $\rho = \frac{2n\Lambda}{n+2}$ and $L_r\psi = 0$. Using that P_r is bounded from above or from below (in the sense of quadratic forms) we can proceeding as in the proof of Theorem 5.1 to conclude that $\mathcal{M}^n$ is trivial. □

In the next result we proved Schur's type inequality. We proved the following.

Theorem 5.3. *Let $\mathcal{M}^n$ be a compact gradient r-Newton-Einstein soliton immersed into a GSS-forms $\overline{\mathcal{M}}^{2n+1}$ of functions f_1, f_2, f_3 with constant sectional curvature c, such that P_r is bounded from below (in the sense of quadratic forms) and $\frac{n}{2} > -1$. Then*

$$\int_{\mathcal{M}} |\rho - \overline{\rho}|^2 \leq \frac{nC}{(n-2)(n+2)} \|\overset{\circ}{\mathrm{Ric}}\|_{L^2} \|\nabla^2\psi - \frac{\Delta\psi}{n} g\|_{L^2}. \tag{5.1}$$

Proof. We recall the contracted second Bianchi identity tells us that

$$\mathrm{divRic} + \frac{1}{2}\nabla\rho = 0,$$

and hence that

$$\text{div}\ \overset{\circ}{\text{Ric}} = -\frac{n-2}{2n}\nabla\rho.$$

Since $\mathcal{M}$ is compact we get using our assumption on P_r that

$$\begin{aligned}
\int_{\mathcal{M}} |n\Lambda - (\frac{n}{2}+1)\rho|^2 &= \int_{\mathcal{M}} (n\Lambda - (\frac{n}{2}+1)\rho)L_r\psi = \int_{\mathcal{M}} (n\Lambda - (\frac{n}{2}+1)\rho)\text{div}(P_r\nabla\psi) \\
&= -(\frac{n}{2}+1)\int_{\mathcal{M}} \langle\nabla\rho, P_r\nabla\psi\rangle \leq C(\frac{n}{2}+1)\int_{\mathcal{M}} \langle\nabla\rho, \nabla\psi\rangle \\
&= -\frac{nC(n+2)}{n-2}\int_{\mathcal{M}} \langle \text{div}\ \overset{\circ}{\text{Ric}}, \nabla\psi\rangle \\
&= \frac{nC(n+2)}{n-2}\int_{\mathcal{M}} \langle \overset{\circ}{\text{Ric}}, \nabla^2\psi\rangle \\
&= \frac{nC(n+2)}{n-2}\int_{\mathcal{M}} \langle \overset{\circ}{\text{Ric}}, \nabla^2\psi - \frac{\Delta\psi}{n}g\rangle \\
&\leq \frac{nC(n+2)}{n-2}\ \|\overset{\circ}{\text{Ric}}\|_{L^2}\ \|\nabla^2\psi - \frac{\Delta\psi}{n}g\|_{L^2},
\end{aligned}$$

where we used that $\langle\overset{\circ}{\text{Ric}}, g\rangle = 0$. Since $\mathcal{M}$ is compact we have

$$n\Lambda = (\frac{n}{2}+1)\overline{\rho},$$

where $\overline{\rho}$ stands for the average of ρ. Therefore,

$$(\frac{n}{2}+1)^2\int_{\mathcal{M}} |\rho - \overline{\rho}|^2 = \int_{\mathcal{M}} |n\Lambda + (\frac{n}{2}+1)\rho|^2,$$

i.e.,

$$(\frac{n}{2}+1)^2\int_{\mathcal{M}} |\rho - \overline{\rho}|^2 \leq \frac{nC(n+2)}{n-2}\ \|\overset{\circ}{\text{Ric}}\|_{L^2}\ \|\nabla^2\psi - \frac{\Delta\psi}{n}g\|_{L^2},$$

i.e.,

$$\int_{\mathcal{M}} |\rho - \overline{\rho}|^2 \leq \frac{nC}{(n-2)(n+2)}\ \|\overset{\circ}{\text{Ric}}\|_{L^2}\ \|\nabla^2\psi - \frac{\Delta\psi}{n}g\|_{L^2}. \tag{5.2}$$

This completes the proof. □

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