Proceedings of the Institute of Mathematics and Mechanics,
National Academy of Sciences of Azerbaijan
Volume 48, Number 2, 2022, Pages 238–248
https://doi.org/10.30546/2409-4994.48.2.2022.238

OPTIMAL CONTROL PROBLEM WITH COEFFICIENTS FOR THE EQUATION OF VIBRATIONS OF A THIN PLATE WITH DISCONTINUOUS SOLUTION

HAMLET F. GULIYEV AND KHAYALA I. SEYFULLAYEVA

Abstract. Optimal control problem with coefficients for the equation of vibrations of a thin plate with discontinuous solution is considered in this work. Existence theorem for optimal pair is proved and necessary condition for optimality in the form of integral inequality is derived.

1. Introduction

Fourth order partial differential equations make an important part of mathematical physics. In practice, some real processes are described by fourth order partial differential equations. For example, equations of vibrations of a tuning fork [11], equations of elastic plate [1], equations of thin plate [5], circular plate equations [2], etc belong to this kind of equations. Therefore, the study of optimal control problems in the processes described by these equations is of great theoretical and practical significance.

Note that different optimal control problems for the vibrations of a thin plate have been considered in [3, 4, 10].

In case where the control function appears in the right-hand side of the equation, optimal control problem for second order hyperbolic equation and Petrovski-type correct system with discontinuous solution has been treated in [8].

Of particular interest are the optimal control problems where the control is included in the coefficients of the equation. Such type problems arising some essential difficulties related to their nonlinearity and incorrectness are under investigation [6].

In this work, we consider an optimal control problem using the coefficients of the equation of vibrations of a thin plate with discontinuous solution.

2. Problem statement

Let the controlled process be described by the equation

$$\frac{\partial^2 u}{\partial t^2} + \Delta^2 u - u^3 + \upsilon_1(x_1, x_2)\frac{\partial u}{\partial x_1} + \upsilon_2(x_1, x_2)\frac{\partial u}{\partial x_2} = 0, (x_1, x_2, t) \in Q, \qquad (2.1)$$

2010 *Mathematics Subject Classification.* 35L25, 49J20.
Key words and phrases. thin plate, equation of vibrations, optimal control, existence theorem, optimality condition.

with the initial conditions

$$u(x_1,x_2,0)=u_0(x_1,x_2), \frac{\partial u(x_1,x_2,0)}{\partial t}=u_1(x_1,x_2),(x_1,x_2)\in\Omega \qquad (2.2)$$

and the boundary conditions

$$\begin{gathered} u(0,x_2,t)=0, \tfrac{\partial u(0,x_2,t)}{\partial x_1}=0, u(x_1,0,t)=0, \\ \tfrac{\partial u(x_1,0,t)}{\partial x_2}=0,\ \ 0\le x_1\le a,\ \ 0\le t\le T, \\ u(a,x_2,t)=0, \tfrac{\partial u(a,x_2,t)}{\partial x_1}=0, u(x_1,b,t)=0, \\ \tfrac{\partial u(x_1,b,t)}{\partial x_2}=0,\ \ 0\le x_2\le b,\ \ 0\le t\le T, \end{gathered} \qquad (2.3)$$

where $Q=\Omega\times(0,T)$, $\Omega=(0,a)\times(0,b)$, T, a, b are the given positive numbers, $u_0(x_1,x_2)$, $u_1(x_1,x_2)$ are the given initial functions, $\upsilon_i(x_1,x_2)$ $(i=1,2)$ are the control functions, and Δ is a Laplace operator with respect to $x=(x_1,x_2)$.

Denote

$$V=\Big\{\upsilon(x_1,x_2):\ \upsilon=(\upsilon_1,\upsilon_2)\in(L_2(\Omega))^2, \\ |\upsilon_i(x_1,x_2)|\le M_i,\ \text{a.e. in}\ \Omega,\ i=1,2\Big\},$$

where M_i given positive number.

Define an admissible pair $\{\upsilon,u\}$ which satisfies the conditions (2.1)-(2.3) and

$$\upsilon(x_1,x_2)\in V, u(x_1,x_2,t)\in L_6(Q).$$

Assume that the set of admissible pairs is nonempty. (2.4)

Let

$$u_0(x_1,x_2)\in \overset{0}{W}{}_2^2(\Omega), u_1(x_1,x_2)\in L_2(\Omega), \qquad (2.5)$$

here $\overset{0}{W}{}_2^2(\Omega)$ is subspace $W_2^2(\Omega)$, whose elements on the boundary of Ω are equal to zero together with their first derivatives, and $W_2^2(\Omega)$ Hilbert space consisting of all elements of $L_2(\Omega)$, having generalized derivatives of the first and second orders from $L_2(\Omega)$ with the norm

$$\|u\|=\left\{\int\limits_\Omega\left[(u)^2+\sum_{i=1}^2\left(\frac{\partial u}{\partial x_i}\right)^2+\sum_{i,j=1}^2\left(\frac{\partial^2 u}{\partial x_i\partial x_j}\right)^2\right]dx\right\}^{1/2}.$$

Let's denote

$$U=\left\{u(x_1,x_2,t):\ \ u\in u(x_1,x_2,t)\in C\left([0;T];\overset{0}{W}{}_2^2(\Omega)\right),\right. \\ \left.\frac{\partial u(x_1,x_2,t)}{\partial t}\in C\left([0;T];L_2(\Omega)\right)\right\}.$$

For every admissible pair $\{\upsilon,u\}$, the satisfaction of conditions (2.1)-(2.3) is understood in the sense that $u(x_1,x_2,t)\in U$ and for every function $\eta(x_1,x_2,t)\in C^2(\bar Q)$, $\eta(x_1,x_2,T)=0$ the integral identity

$$\int_Q\left[-\frac{\partial u}{\partial t}\frac{\partial\eta}{\partial t}+\Delta u\Delta\eta-u^3\eta+\upsilon_1\frac{\partial u}{\partial x_1}\eta+\upsilon_2\frac{\partial u}{\partial x_2}\eta\right]dx_1dx_2dt-$$

$$-\int_{\Omega} u_1(x_1,x_2)\eta(x_1,x_2,0)dx_1dx_2 = 0$$

and the condition $u(x_1,x_2,0) = u_0(x_1,x_2)$ hold.

Define the functional

$$J(\upsilon,u) = \frac{1}{6}\|u-u_d\|^6_{L_6(Q)} + \frac{N}{2}\left(\|\upsilon_1\|^2_{L_2(\Omega)} + \|\upsilon_2\|^2_{L_2(\Omega)}\right), \tag{2.6}$$

where $u_d(x_1,x_2,t) \in L_6(Q)$, and $N>0$ is a given number.

Consider the following optimal control problem: find the minimum value of the functional (2.6), where $\{\upsilon,u\}$ is varying in the class of admissible pairs.

3. Existence of optimal pair

Theorem 3.1. *Let the conditions* (2.4), (2.5) *hold. Then there exists an optimal pair* $\{\upsilon^0,u^0\}$ *in the problem* (2.1)-(2.4), (2.6), *i.e.*

$$J(\upsilon^0,u^0) = \inf_{\{\upsilon,u\}} J(\upsilon,u).$$

Proof. Let $\{\upsilon^{(n)},\, u^{(n)}\} \in V\times U$ be a minimizing sequence, i.e.

$$\lim_{n\to\infty} J(\upsilon^{(n)},\, u^{(n)}) = \inf_{\{\upsilon,u\}} J(\upsilon,u). \tag{3.1}$$

Hence it follows

$$\left\|\upsilon_i^{(n)}\right\|_{L_2(\Omega)} \le const,\ i=1,2, \tag{3.2}$$

$$\left\|u^{(n)}\right\|_{L_6(Q)} \le const \tag{3.3}$$

and then by the relation (3.2) and definition of the class V we can derive from $\{\upsilon^{(n)}\}$ a subsequence, denoted again by $\{\upsilon^{(n)}\}$, such that

$$\upsilon_i^{(n)} \to \upsilon_i^0 \text{ in } L_2(\Omega) \text{ weakly as } n\to\infty,\ i=1,2. \tag{3.4}$$

From (3.3), $\upsilon^{(n)} \in V$ and from the equation

$$\frac{\partial^2 u^{(n)}}{\partial t^2} + \Delta^2 u^{(n)} + \upsilon_1^{(n)}\frac{\partial u^{(n)}}{\partial x_1} + \upsilon_2^{(n)}\frac{\partial u^{(n)}}{\partial x_2} = u^{(n)^3}$$

we obtain

$$\left\|u^{(n)}\right\|_U \le const, \left\|u^{(n)}\right\|_{W_2^{2,1}(Q)} \le const. \tag{3.5}$$

Hence it follows that the sequence $\{u^{(n)}\}$ belongs to a bounded set in the class U.

Then from embedding theorem of [8, p. 70], we obtain

$$u^{(n)} \to u^0 \text{ in } L_6(Q) \text{ strongly}, \tag{3.6}$$

$$\frac{\partial u^{(n)}}{\partial x_1} \to \frac{\partial u^0}{\partial x_1}, \frac{\partial u^{(n)}}{\partial x_2} \to \frac{\partial u^0}{\partial x_2} \text{ in } L_2(Q) \text{ strongly}; \tag{3.7}$$

besides,

$$\frac{\partial u^{(n)}}{\partial t} \to \frac{\partial u^0}{\partial t},\ \frac{\partial^2 u^{(n)}}{\partial x_1^2} \to \frac{\partial^2 u^0}{\partial x_1^2},$$

$$\frac{\partial^2 u^{(n)}}{\partial x_2^2} \to \frac{\partial^2 u^0}{\partial x_2^2}, \frac{\partial^2 u^{(n)}}{\partial x_1 \partial x_2} \to \frac{\partial^2 u^0}{\partial x_1 \partial x_2} \text{ in } L_2(Q) \text{ weakly.} \tag{3.8}$$

Let $\upsilon = \upsilon^{(n)}$, $u = u^{(n)}$ in the definition of the solution of the problem (2.1)-(2.3):

$$\int_Q \left[-\frac{\partial u^{(n)}}{\partial t} \frac{\partial \eta}{\partial t} + \Delta u^{(n)} \Delta \eta - {u^{(n)}}^3 \eta + \upsilon_1^{(n)} \frac{\partial u^{(n)}}{\partial x_1} \eta + \upsilon_2^{(n)} \frac{\partial u^{(n)}}{\partial x_2} \eta \right] dx_1 dx_2 dt -$$

$$- \int_\Omega u_1(x_1, x_2) \eta(x_1, x_2, 0) dx_1 dx_2 = 0. \tag{3.9}$$

Clearly,

$$\int_Q \upsilon_i^{(n)} \frac{\partial u^{(n)}}{\partial x_i} \eta dx_1 dx_2 dt = \int_Q \upsilon_i^{(n)} \frac{\partial u^0}{\partial x_i} \eta dx_1 dx_2 dt +$$

$$+ \int_Q \upsilon_i^{(n)} \left(\frac{\partial u^{(n)}}{\partial x_i} - \frac{\partial u^0}{\partial x_i} \right) \eta dx_1 dx_2 dt.$$

As $\eta \in C^2(\bar{Q})$, it follows that the function is bounded in Q. Therefore,

$$\left| \int_Q \upsilon_i^{(n)} \left(\frac{\partial u^{(n)}}{\partial x_i} - \frac{\partial u^0}{\partial x_i} \right) \eta dx_1 dx_2 dt \right| \le$$

$$\le C \left(\int_Q \left| \upsilon_i^{(n)} \right|^2 dx_1 dx_2 dt \right)^{\frac{1}{2}} \left(\int_Q \left(\frac{\partial u^{(n)}}{\partial x_i} - \frac{\partial u^0}{\partial x_i} \right)^2 dx_1 dx_2 dt \right)^{\frac{1}{2}}, \ i = 1, 2.$$

Hereinafter C will denote different constants independent of estimated quantities and admissible pairs.

By (3.4) and (3.7), we obtain

$$\lim_{n \to \infty} \int_Q \upsilon_i^{(n)} \frac{\partial u^{(n)}}{\partial x_i} \eta dx_1 dx_2 dt = \lim_{n \to \infty} \int_Q \upsilon_i^0 \frac{\partial u^0}{\partial x_i} \eta dx_1 dx_2 dt, \ i = 1, 2. \tag{3.10}$$

Then, taking into account the relations (3.6), (3.8) and (3.10) and passing to the limit in (3.8) as $n \to \infty$, we get

$$\int_Q \left[-\frac{\partial u^0}{\partial t} \frac{\partial \eta}{\partial t} + \Delta u^0 \Delta \eta - {u^0}^3 \eta + \upsilon_1^0 \frac{\partial u^0}{\partial x_1} \eta + \upsilon_2^0 \frac{\partial u^0}{\partial x_2} \eta \right] dx_1 dx_2 dt -$$

$$- \int_\Omega u_1(x_1, x_2) \eta(x_1, x_2, 0) dx_1 dx_2 = 0.$$

Therefore, $\{\upsilon^0, u^0\}$ is an admissible pair. As the functional $J(\upsilon, u)$ is continuous in $(L_2(\Omega))^2 \times L_6(Q)$, we have

$$\lim_{n \to \infty} J(\upsilon^{(n)}, u^{(n)}) = J(\upsilon^0, u^0). \tag{3.11}$$

Then it follows from (3.1) and (3.11) that

$$\inf_{\{\upsilon, u\}} J(\upsilon, u) = J(\upsilon^0, u^0).$$

Consequently, the pair $\{\upsilon^0, u^0\}$ gives the minimum value of the functional $J(\upsilon, u)$, i.e. $\{\upsilon^0, u^0\}$ is an optimal pair. □

Theorem 3.1 is proved.

4. Adaptive penalty method

Introduce adapted functional for the optimal pair $\{v^0, u^0\}$:

$$J_\varepsilon^a(v,u) = \frac{1}{6}\left\|u-u_d\right\|_{L_6(Q)}^6 + \frac{N}{2}\left(\left\|v_1\right\|_{L_2(\Omega)}^2 + \left\|v_2\right\|_{L_2(\Omega)}^2\right) +$$
$$+\frac{1}{2\varepsilon}\left\|\frac{\partial^2 u}{\partial t^2} + \Delta^2 u - u^3 + v_1\frac{\partial u}{\partial x_1} + v_2\frac{\partial u}{\partial x_2}\right\|_{L_2(Q)}^2 + \tag{4.1}$$
$$+\frac{1}{2}\left\|u-u^0\right\|_{L_2(Q)}^2 + \frac{1}{2}\left(\left\|v_1-v_1^0\right\|_{L_2(\Omega)}^2 + \left\|v_2-v_2^0\right\|_{L_2(\Omega)}^2\right)$$

where $\{v^0, u^0\}$ is a chosen optimal pair. Let's minimize this functional for

$$v(x_1,x_2)\in V, u(x_1,x_2,t)\in L_6(Q), \frac{\partial^2 u}{\partial t^2} + \Delta^2 u \in L_2(Q)$$

under the conditions

$$u(x_1,x_2,0) = u_0(x_1,x_2), \frac{\partial u(x_1,x_2,0)}{\partial t} = u_1(x_1,x_2),$$
$$u(0,x_2,t) = 0, \frac{\partial u(0,x_2,t)}{\partial x_1} = 0, u(x_1,0,t) = 0, \frac{\partial u(x_1,0,t)}{\partial x_2} = 0,$$
$$u(a,x_2,t) = 0, \frac{\partial u(a,x_2,t)}{\partial x_1} = 0, u(x_1,b,t) = 0, \frac{\partial u(x_1,b,t)}{\partial x_2} = 0.$$

Theorem 4.1. *For every fixed $\varepsilon > 0$, there exists a pair $\{v_\varepsilon, u_\varepsilon\}$ that gives the minimum value of the functional $J_\varepsilon^a(v,u)$, i.e.*

$$J_\varepsilon^a(v_\varepsilon, u_\varepsilon) = \inf J_\varepsilon^a(v,u). \tag{4.2}$$

The proof of Theorem 4.1 is similar to the proof of Theorem 3.1.

5. Convergence of adaptive penalty method

Theorem 5.1. *Let $\{v_\varepsilon, u_\varepsilon\}$ be some solution of the problem (4.2). Then for $\varepsilon \to 0$ we have*

$$v_{i\varepsilon} \to v_i^0 \text{ in } L_2(\Omega) \text{ strongly, } i = 1,2, \tag{5.1}$$
$$u_\varepsilon \to u^0 \text{ in } L_6(Q) \text{ strongly,} \tag{5.2}$$

where $\{v^0, u^0\}$ is a chosen optimal pair.

Proof. We have

$$J_\varepsilon^a(v_\varepsilon, u_\varepsilon) = \inf J_\varepsilon^a(v,u) \le J_\varepsilon^a(v^0,u^0) = J(v^0,u^0). \tag{5.3}$$

By definition of functional, we obtain

$$\left\|v_{i\varepsilon}\right\|_{L_2(\Omega)} + \left\|u_\varepsilon\right\|_{L_6(Q)} \le C, \; i = 1,2, \tag{5.4}$$

and also

$$\frac{\partial^2 u_\varepsilon}{\partial t^2} + \Delta^2 u_\varepsilon - u_\varepsilon^3 + v_{1\varepsilon}\frac{\partial u_\varepsilon}{\partial x_1} + v_{2\varepsilon}\frac{\partial u_\varepsilon}{\partial x_2} = \sqrt{\varepsilon} f_\varepsilon, \tag{5.5}$$

where $f_\varepsilon(x_1, x_2, t)$ is a function such that $\|f_\varepsilon\|_{L_2(Q)} \le C$,

$$u_\varepsilon(x_1, x_2, 0) = u_0(x_1, x_2), \frac{\partial u_\varepsilon(x_1, x_2, 0)}{\partial t} = u_1(x_1, x_2), \tag{5.6}$$

$$\begin{aligned} &u_\varepsilon(0, x_2, t) = 0, \tfrac{\partial u_\varepsilon(0,x_2,t)}{\partial x_1} = 0, u_\varepsilon(x_1, 0, t) = 0, \tfrac{\partial u_\varepsilon(x_1,0,t)}{\partial x_2} = 0, \\ &u_\varepsilon(a, x_2, t) = 0, \tfrac{\partial u_\varepsilon(a,x_2,t)}{\partial x_1} = 0, u_\varepsilon(x_1, b, t) = 0, \tfrac{\partial u_\varepsilon(x_1,b,t)}{\partial x_2} = 0. \end{aligned} \tag{5.7}$$

From (5.4), (5.5)-(5.7) and the relation $\upsilon_\varepsilon \in V$ it follows

$$\|u_\varepsilon\|_U \le C, \|u_\varepsilon\|_{W_2^{2,1}(Q)} \le C.$$

Consequently, we can derive from $\{\upsilon_\varepsilon, u_\varepsilon\}$ a subsequence, denoted again by $\{\upsilon_\varepsilon, u_\varepsilon\}$, such that

$$\upsilon_{i\varepsilon} \to \hat{\upsilon}_i \text{ in } L_2(\Omega) \text{ weakly as } \varepsilon \to 0 \text{ and } \hat{\upsilon} \in V, i = 1, 2,$$

$$u_\varepsilon \to \hat{u} \text{ in } W_2^{2,1}(Q) \text{weakly as } \varepsilon \to 0,$$

here $W_2^{2,1}(Q)$ is Hilbert space consisting of all elements of $L_2(Q)$ having generalized derivatives $\frac{\partial u}{\partial t}, \frac{\partial u}{\partial x_i}, \frac{\partial^2 u}{\partial x_i \partial x_j}$ from $L_2(Q)$ with the norm

$$\|u\| = \left\{ \int\limits_Q \left[(u)^2 + \sum_{i=1}^{2} \left(\frac{\partial u}{\partial x_i} \right)^2 + \sum_{i,j=1}^{2} \left(\frac{\partial^2 u}{\partial x_i \partial x_j} \right)^2 + \left(\frac{\partial u}{\partial t} \right)^2 \right] dxdt \right\}^{1/2}.$$

Besides, by result from [8, p. 70]

$$u_\varepsilon \to \hat{u} \text{ in } L_6(Q) \text{ strongly.}$$

Then, in the sense of generalized solution of the problem (2.1)-(2.3), the following relations hold:

$$\frac{\partial^2 \hat{u}}{\partial t^2} + \Delta^2 \hat{u} - \hat{u}^3 + \hat{\upsilon}_1 \frac{\partial \hat{u}}{\partial x_1} + \hat{\upsilon}_2 \frac{\partial \hat{u}}{\partial x_2} = 0,$$

$$\hat{u}(x_1, x_2, 0) = u_0(x_1, x_2), \frac{\partial \hat{u}(x_1, x_2, 0)}{\partial t} = u_1(x_1, x_2),$$

$$\hat{u}(0, x_2, t) = 0, \tfrac{\partial \hat{u}(0,x_2,t)}{\partial x_1} = 0, \hat{u}(x_1, 0, t) = 0, \tfrac{\partial \hat{u}(x_1,0,t)}{\partial x_2} = 0,$$

$$\hat{u}(a, x_2, t) = 0, \tfrac{\partial \hat{u}(a,x_2,t)}{\partial x_1} = 0, \hat{u}(x_1, b, t) = 0, \tfrac{\partial \hat{u}(x_1,b,t)}{\partial x_2} = 0.$$

So, the inequality

$$J_\varepsilon^a(\upsilon_\varepsilon, u_\varepsilon) \ge J(\upsilon_\varepsilon, u_\varepsilon) + \frac{1}{2} \left\|\hat{u} - u^0\right\|_{L_2(Q)}^2 + \frac{1}{2} \left(\left\|\hat{\upsilon}_1 - \upsilon_1^0\right\|_{L_2(\Omega)}^2 + \left\|\hat{\upsilon}_2 - \upsilon_2^0\right\|_{L_2(\Omega)}^2 \right)$$

leads to

$$\underline{\lim}_{\varepsilon \to 0} J_\varepsilon^a(\upsilon_\varepsilon, u_\varepsilon) \ge J(\hat{\upsilon}, \hat{u}) + \frac{1}{2} \left\|\hat{u} - u^0\right\|_{L_2(Q)}^2 + \frac{1}{2} \left(\left\|\hat{\upsilon}_1 - \upsilon_1^0\right\|_{L_2(\Omega)}^2 + \left\|\hat{\upsilon}_2 - \upsilon_2^0\right\|_{L_2(\Omega)}^2 \right).$$

And, as by (5.3) we have

$$\overline{\lim}_{\varepsilon \to 0} J_\varepsilon^a(\upsilon_\varepsilon, u_\varepsilon) \le J(\upsilon^0, u^0),$$

it follows that

$$J(\hat{\upsilon}, \hat{u}) \leq J(\upsilon^0, u^0).$$

Therefore,

$$J(\hat{\upsilon}, \hat{u}) = J(\upsilon^0, u^0).$$

Then

$$\frac{1}{2}\left\|\hat{u}-u^0\right\|^2_{L_2(Q)} + \frac{1}{2}\left(\left\|\hat{\upsilon}_1-\upsilon_1^0\right\|^2_{L_2(\Omega)} + \left\|\hat{\upsilon}_2-\upsilon_2^0\right\|^2_{L_2(\Omega)}\right) = 0,$$

so $\hat{\upsilon} = \upsilon^0$, $\hat{u} = u^0$, and consequently, we obtain a convergence without extracting a subsequence (because the limit is unique). So we get the validity of the relation (5.2).

By (5.3),

$$J(\upsilon^0, u^0) \geq J_\varepsilon^a(\upsilon_\varepsilon, u_\varepsilon) \geq J(\upsilon_\varepsilon, u_\varepsilon)$$

and

$$\varliminf_{\varepsilon\to 0} J(\upsilon_\varepsilon, u_\varepsilon) \geq J(\upsilon^0, u^0).$$

Then

$$J(\upsilon^0, u^0) \geq \varlimsup_{\varepsilon\to 0} J_\varepsilon^a(\upsilon_\varepsilon, u_\varepsilon) \geq \varliminf_{\varepsilon\to 0} J(\upsilon_\varepsilon, u_\varepsilon) \geq J(\upsilon^0, u^0).$$

Therefore,

$$J(\upsilon_\varepsilon, u_\varepsilon) \to J(\upsilon^0, u^0).$$

Hence, by definitions of the functional $J(\upsilon, u)$ we obtain the relation (5.1). □

Theorem 5.1 is proved.

6. Optimality system for penalty problem

As a class of admissible controls we consider

$$V = \Big\{\upsilon(x_1,x_2): \ \upsilon = (\upsilon_1,\upsilon_2) \in \left(W_2^1(\Omega)\right)^2, \ |\upsilon_i(x_1,x_2)| \leq$$
$$\leq M_i, \ \left|\frac{\partial \upsilon_i(x_1,x_2)}{\partial x_j}\right| \leq M_{ij} \text{ a.e. in } \Omega, \ i,j=1,2\Big\},$$

where M_i, M_j are the given positive numbers.

Now let's find the necessary conditions for $\{\upsilon_\varepsilon, u_\varepsilon\}$ to be the solution of the problem (4.2):

$$\frac{d}{d\lambda} J_\varepsilon^a(\upsilon_\varepsilon, u_\varepsilon + \lambda\xi)\Big|_{\lambda=0} = 0 \ \ \forall \xi \in C^2(\bar{Q}),$$

$$\xi(x_1,x_2,0) = 0, \frac{\partial \xi(x_1,x_2,0)}{\partial t} = 0,$$

$$\xi(0,x_2,t) = 0, \frac{\partial \xi(0,x_2,t)}{\partial x_1} = 0, \xi(x_1,0,t) = 0, \frac{\partial \xi(x_1,0,t)}{\partial x_2} = 0,$$

$$\xi(a,x_2,t) = 0, \frac{\partial \xi(a,x_2,t)}{\partial x_1} = 0, \xi(x_1,b,t) = 0, \frac{\partial \xi(x_1,b,t)}{\partial x_2} = 0$$

and

$$\frac{d}{d\lambda} J_\varepsilon^a(\upsilon_\varepsilon + \lambda(\upsilon - \upsilon_\varepsilon), u_\varepsilon)\Big|_{\lambda=0} \geq 0 \ \ \forall \upsilon \in V, \upsilon_\varepsilon \in V. \tag{6.1}$$

For this aim, let's calculate the derivative of the functional

$$J_\varepsilon^a(\upsilon_\varepsilon, u_\varepsilon+\lambda\xi) = \frac{1}{6}\left\|u_\varepsilon+\lambda\xi-u_d\right\|_{L_6(Q)}^6 + \frac{N}{2}\left(\left\|\upsilon_{1\varepsilon}\right\|_{L_2(\Omega)}^2 + \left\|\upsilon_{2\varepsilon}\right\|_{L_2(\Omega)}^2\right) +$$
$$+\frac{1}{2\varepsilon}\left\|\frac{\partial^2(u_\varepsilon+\lambda\xi)}{\partial t^2} + \Delta^2(u_\varepsilon+\lambda\xi) - (u_\varepsilon+\lambda\xi)^3 +\right.$$
$$\left. +\upsilon_{1\varepsilon}\frac{\partial(u_\varepsilon+\lambda\xi)}{\partial x_1} + \upsilon_{2\varepsilon}\frac{\partial(u_\varepsilon+\lambda\xi)}{\partial x_2}\right\|_{L_2(Q)}^2 +$$
$$+\frac{1}{2}\left\|u_\varepsilon+\lambda\xi-u^0\right\|_{L_2(Q)}^2 + \frac{1}{2}\left(\left\|\upsilon_{1\varepsilon}-\upsilon_1^0\right\|_{L_2(\Omega)}^2 + \left\|\upsilon_{2\varepsilon}-\upsilon_2^0\right\|_{L_2(\Omega)}^2\right)$$

with respect to λ and substitute $\lambda = 0$:

$$\left.\frac{d}{d\lambda}J_\varepsilon^a(\upsilon_\varepsilon, u_\varepsilon+\lambda\xi)\right|_{\lambda=0} = \int_Q (u_\varepsilon-u_d)^5\,\xi dx_1dx_2dt +$$
$$+\frac{1}{\varepsilon}\int_Q\left(\frac{\partial^2 u_\varepsilon}{\partial t^2} + \Delta^2 u_\varepsilon - u_\varepsilon^3 + \upsilon_{1\varepsilon}\frac{\partial u_\varepsilon}{\partial x_1} + \upsilon_{2\varepsilon}\frac{\partial u_\varepsilon}{\partial x_2}\right)\times$$
$$\times\left(\frac{\partial^2\xi}{\partial t^2} + \Delta^2\xi - 3u_\varepsilon^2\xi + \upsilon_{1\varepsilon}\frac{\partial\xi}{\partial x_1} + \upsilon_{2\varepsilon}\frac{\partial\xi}{\partial x_2}\right)dx_1dx_2dt +$$
$$+\int_Q\left(u_\varepsilon-u^0\right)\xi dx_1dx_2dt. \tag{6.2}$$

Denote

$$\psi_\varepsilon = -\frac{1}{\varepsilon}\left(\frac{\partial^2 u_\varepsilon}{\partial t^2} + \Delta^2 u_\varepsilon - u_\varepsilon^3 + \upsilon_{1\varepsilon}\frac{\partial u_\varepsilon}{\partial x_1} + \upsilon_{2\varepsilon}\frac{\partial u_\varepsilon}{\partial x_2}\right). \tag{6.3}$$

Then from (6.2) we obtain

$$-\int_Q \psi_\varepsilon\left(\frac{\partial^2\xi}{\partial t^2} + \Delta^2\xi - 3u_\varepsilon^2\xi + \upsilon_{1\varepsilon}\frac{\partial\xi}{\partial x_1} + \upsilon_{2\varepsilon}\frac{\partial\xi}{\partial x_2}\right)dx_1dx_2dt +$$
$$+\int_Q (u_\varepsilon-u_d)^5\xi dx_1dx_2dt + \int_Q (u_\varepsilon-u^0)\xi dx_1dx_2dt = 0. \tag{6.4}$$

The equation (6.4) means that $\psi_\varepsilon(x_1,x_2,t)$ is a weak solution of the following problem:

$$\frac{\partial^2\psi_\varepsilon}{\partial t^2} + \Delta^2\psi_\varepsilon - 3u_\varepsilon^2\psi_\varepsilon - \frac{\partial}{\partial x_1}(\upsilon_{1\varepsilon}\psi_\varepsilon) - \frac{\partial}{\partial x_2}(\upsilon_{2\varepsilon}\psi_\varepsilon) = (u_\varepsilon-u_d)^5 + (u_\varepsilon-u^0), \tag{6.5}$$

$$\psi_\varepsilon(x_1,x_2,T) = 0, \frac{\partial\psi_\varepsilon(x_1,x_2,T)}{\partial t} = 0, \tag{6.6}$$

$$\begin{gathered}\psi_\varepsilon(0,x_2,t) = 0, \tfrac{\partial\psi_\varepsilon(0,x_2,t)}{\partial x_1} = 0, \psi_\varepsilon(x_1,0,t) = 0, \tfrac{\partial\psi_\varepsilon(x_1,0,t)}{\partial x_2} = 0,\\ \psi_\varepsilon(a,x_2,t) = 0, \tfrac{\partial\psi_\varepsilon(a,x_2,t)}{\partial x_1} = 0,\ \psi_\varepsilon(x_1,b,t) = 0, \tfrac{\partial\psi_\varepsilon(x_1,b,t)}{\partial x_2} = 0\end{gathered} \tag{6.7}$$

By the definition of the class V, it follows that the solution $\psi_\varepsilon(x_1,x_2,t)$ of the boundary value problem (6.5)-(6.7) belongs to the class [6, p. 214]

$$Y = \left\{\psi:\ \psi\in C\left([0;T];W_2^{\frac{4}{3}}(\Omega)\right),\ \ \frac{\partial\psi_\varepsilon}{\partial t}\in C\left([0;T];W_2^{-\frac{2}{3}}(\Omega)\right)\right\},$$

here $\overset{\circ}{W}{}_2^{4/3}(\Omega) = V^{\frac{2}{3}}$, where $V^{\theta} = D(A^{\theta/2}), 0 \le \theta \le 1$, $D(A) = W_2^4(\Omega) \cap \overset{\circ}{W}{}_2^{2}(\Omega)$, $A = \Delta^2$, here $W_2^4(\Omega)$ is the Hilbert space consisting of all elements of $L_2(\Omega)$, having generalized derivatives up to order 4 inclusive from $L_2(\Omega)$, (see [9], pp. 167, 214) and $W_2^{-\frac{2}{3}}(\Omega)$ dual space to the space $\overset{\circ}{W}{}_2^{2/3}(\Omega)$, i.e. $W_2^{-\frac{2}{3}}(\Omega) = \left(\overset{\circ}{W}{}_2^{2/3}(\Omega)\right)'$. The right-hand side $(u_\varepsilon - u_d)^5 + (u_\varepsilon - u^0)$ of the equation (6.5) belongs to $C\left([0,T]\,; L^{6/5}(Q)\right)$ and $\upsilon \in V$. Then, by Theorem 2.2 of [6, p. 163-164], for $\psi_\varepsilon(x_1, x_2, t)$ we have

$$\|\psi_\varepsilon\|_Y \le C.$$

We can pass to the limit in the problem (6.5)-(6.7) as $\varepsilon \to 0$, and the limit function $\psi(x_1, x_2, t)$ will be a weak solution of the following adjoint problem:

$$\frac{\partial^2 \psi}{\partial t^2} + \Delta^2 \psi - 3u^2\psi - \frac{\partial}{\partial x_1}(\upsilon_1 \psi) - \frac{\partial}{\partial x_2}(\upsilon_2 \psi) = (u - u_d)^5,$$

$$\psi(x_1, x_2, T) = 0, \frac{\partial \psi(x_1, x_2, T)}{\partial t} = 0,$$

$$\psi(0, x_2, t) = 0, \tfrac{\partial \psi(0, x_2, t)}{\partial x_1} = 0, \psi(x_1, 0, t) = 0, \tfrac{\partial \psi(x_1, 0, t)}{\partial x_2} = 0,$$

$$\psi(a, x_2, t) = 0, \tfrac{\partial \psi(a, x_2, t)}{\partial x_1} = 0, \ \psi(x_1, b, t) = 0, \tfrac{\partial \psi(x_1, b, t)}{\partial x_2} = 0.$$

Now let's simplify the condition (6.1). For this, let's calculate the derivative of the functional

$$J_\varepsilon^a(\upsilon_\varepsilon + \lambda(\upsilon - \upsilon_\varepsilon), u_\varepsilon) = \frac{1}{6}\|u_\varepsilon - u_d\|_{L_6(Q)}^6 +$$
$$+\frac{N}{2}\left(\|\upsilon_{1\varepsilon} + \lambda(\upsilon_1 - \upsilon_{1\varepsilon})\|_{L_2(\Omega)}^2 + \|\upsilon_{2\varepsilon} + \lambda(\upsilon_2 - \upsilon_{2\varepsilon})\|_{L_2(\Omega)}^2\right) +$$
$$+\frac{1}{2\varepsilon}\left\|\frac{\partial^2 u_\varepsilon}{\partial t^2} + \Delta^2 u_\varepsilon - u_\varepsilon^3 + (\upsilon_{1\varepsilon} + \lambda(\upsilon_1 - \upsilon_{1\varepsilon}))\frac{\partial u_\varepsilon}{\partial x_1} + \right.$$
$$\left. + (\upsilon_{2\varepsilon} + \lambda(\upsilon_2 - \upsilon_{2\varepsilon}))\frac{\partial u_\varepsilon}{\partial x_2}\right\|_{L_2(Q)}^2 + \frac{1}{2}\left\|u_\varepsilon - u^0\right\|_{L_2(Q)}^2 + \frac{1}{2}(\|\upsilon_{1\varepsilon} + \lambda(\upsilon_1 - \upsilon_{1\varepsilon}) -$$
$$-\upsilon_1^0\|_{L_2(Q)}^2 + \|\upsilon_{2\varepsilon} + \lambda(\upsilon_2 - \upsilon_{2\varepsilon}) - \upsilon_2^0\|_{L_2(Q)}^2)$$

with respect to λ and substitute $\lambda = 0$:

$$\frac{d}{d\lambda}J_\varepsilon^a(\upsilon_\varepsilon + \lambda(\upsilon - \upsilon_\varepsilon), u_\varepsilon)\Big|_{\lambda=0} = N\int_\Omega (\upsilon_{1\varepsilon}\cdot(\upsilon_1 - \upsilon_{1\varepsilon}) + \upsilon_{2\varepsilon}\cdot(\upsilon_2 - \upsilon_{2\varepsilon}))\,dx_1dx_2 +$$
$$+\frac{1}{\varepsilon}\int_Q\left(\frac{\partial^2 u_\varepsilon}{\partial t^2} + \Delta^2 u_\varepsilon - {u_\varepsilon}^3 + \upsilon_{1\varepsilon}\frac{\partial u_\varepsilon}{\partial x_1} + \upsilon_{2\varepsilon}\frac{\partial u_\varepsilon}{\partial x_2}\right)\times$$
$$\times\left((\upsilon_1 - \upsilon_{1\varepsilon})\frac{\partial u_\varepsilon}{\partial x_1} + (\upsilon_2 - \upsilon_{2\varepsilon})\frac{\partial u_\varepsilon}{\partial x_2}\right)dx_1dx_2dt +$$
$$+\int_\Omega\left((\upsilon_{1\varepsilon} - \upsilon_1^0)(\upsilon_1 - \upsilon_{1\varepsilon}) + (\upsilon_{2\varepsilon} - \upsilon_2^0)(\upsilon_2 - \upsilon_{2\varepsilon})\right)dx_1dx_2.$$

Given the designation (6.3), we obtain the inequality

$$\int_Q\left[\left(N\upsilon_{1\varepsilon} - \psi_\varepsilon\frac{\partial u_\varepsilon}{\partial x_1} + (\upsilon_{1\varepsilon} - \upsilon_1^0)\right)(\upsilon_1 - \upsilon_{1\varepsilon}) + \left(N\upsilon_{2\varepsilon} - \psi_\varepsilon\frac{\partial u_\varepsilon}{\partial x_2} + \right.\right.$$

$$+(\upsilon_{2\varepsilon}-\upsilon_2^0)\big)\,(\upsilon_2-\upsilon_{2\varepsilon})\big]\,dx_1dx_2dt\geq 0\ \ \forall\upsilon\in V. \tag{6.8}$$

Here we pass to the limit as $\varepsilon\to 0$. In view of $\psi_\varepsilon(x_1,x_2,t)\in Y$, by the embedding theorem of [8, p. 70] we have

$$\psi_\varepsilon\to\psi \text{ in } L_2(Q) \text{ strongly},\ \psi\in Y, \tag{6.9}$$

$$\frac{\partial u_\varepsilon}{\partial x_i}\to\frac{\partial u^0}{\partial x_i} \text{ in } L_2(Q) \text{ strongly},\ u^0\in U,\ i=1,2 \tag{6.10}$$

and by definition of the class V we obtain

$$\upsilon_{i\varepsilon}\to\upsilon_i^0 \text{ in } L_2(\Omega) \text{ strongly},\ \upsilon^0\in V,\ i=1,2. \tag{6.11}$$

Taking into account the relations (6.9), (6.10) and (6.11), we pass to the limit in (6.8) and obtain

$$\int_Q\left[\left(N\upsilon_1^0-\psi\frac{\partial u^0}{\partial x_1}\right)(\upsilon_1-\upsilon_1^0)+\left(N\upsilon_2^0-\psi\frac{\partial u^0}{\partial x_2}\right)(\upsilon_2-\upsilon_2^0)\right]dx_1dx_2dt\geq 0\ \forall\upsilon\in V. \tag{6.12}$$

Thus, the following theorem is proved.

Theorem 6.1. *Under the given conditions on the data of the problem* (2.1)-(2.6), *for the optimal pair* $\{\upsilon^0,u^0\}$ *there exists a triple* $\{\upsilon^0,u^0,\psi\}$ *such that*

$$\frac{\partial^2u^0}{\partial t^2}+\Delta^2u^0-u^{0^3}+\upsilon_1^0\frac{\partial u^0}{\partial x_1}+\upsilon_2^0\frac{\partial u^0}{\partial x_2}=0,$$

$$\frac{\partial^2\psi}{\partial t^2}+\Delta^2\psi-3u^{0^2}\psi-\frac{\partial}{\partial x_1}\left(\upsilon_1^0\psi\right)-\frac{\partial}{\partial x_2}\left(\upsilon_2^0\psi\right)=(u^0-u_d)^5,$$

$$u^0(x_1,x_2,0)=u_0(x_1,x_2),\frac{\partial u^0(x_1,x_2,0)}{\partial t}=u_1(x_1,x_2),$$

$$\psi(x_1,x_2,T)=0,\frac{\partial\psi(x_1,x_2,T)}{\partial t}=0,$$

$$\psi(0,x_2,t)=0,\tfrac{\partial\psi(0,x_2,t)}{\partial x_1}=0,\psi(x_1,0,t)=0,\tfrac{\partial\psi(x_1,0,t)}{\partial x_2}=0,$$

$$\psi(a,x_2,t)=0,\tfrac{\partial\psi(a,x_2,t)}{\partial x_1}=0,\ \psi(x_1,b,t)=0,\tfrac{\partial\psi(x_1,b,t)}{\partial x_2}=0,$$

$u^0\in U$, $\psi\in Y$ *and the inequality* (6.12) *holds.*

References

[1] J.L. Arman, *Applications of the theory of optimal control of distributed-parameter systems to structural optimization.* Moscow, Mir, 1977. (in Russian)

[2] B.M.Budak , A. A.Samarski, A.N. Tikhonov, *Collection of problems in mathematical physics.* M.: Nauka, 1972. (in Russian)

[3] T.Daqun, L.W. Robert, S.K. Agrawal, Optimal shape control of composite thin plates with piezoelectric actuators. *Journal of Intelligent Material Systems and Structures.* **9** (1998) 458-467.

[4] V.S.Deineka, Optimal control of the dynamic state of a thin compound plate. *Cybernetics and Systems Analysis.* **42** (2006) no.4, 151-175.

[5] V.Komkov, *Optimal control theory for damping the vibrations of simple elastic systems.* Moscow, Mir, 1975. (in Russian)

[6] J.L.Lions, *Control of distributed singular systems.* Moscow, Nauka, 1987. (in Russian)

[7] J.L.Lions, *Optimal control for the systems described by the partial differential equations.* Moscow, Mir, 1972. (in Russian)

[8] J.L. Lions, Some *Methods of Solving Non-Linear Boundary Value Problems. Moscow, Mir,* 1972. (in Russian)

[9] J.L. Lions, E. Magenes *Non-homogeneous boundary value problems and applications.* Moscow, Mir, 1971. (in Russian)

[10] M.C.Ray, J.N.Reddy, Optimal control of thin circular cylindrical laminated composite shells using active constrained layer damping treatment, *Smart Materials and Structures.* **13** (2004) no.1.

[11] A.N.Tikhonov, A.A. Samarski *Equations of mathematical physics.* M.: Nauka, 1972. (in Russian)

[12] F.P. Vasilyev, A.Z. Ishmukhametov, M.M.Potapov, *Generalized method of moments in optimal control problems.* M., MGU, 1989. (in Russian)

Hamlet F. Guliyev
Baku State University, Baku, Azerbaijan
E-mail address: `hamletquliyev51@gmail.com`

Khayala I. Seyfullayeva
Sumgayit State University, Sumgayit, Azerbaijan
E-mail address: `xeyaleseyfullayeva82@gmail.com`

Received: June 26, 2022; Revised: October 12, 2022; Accepted: October 17, 2022