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ON SOME TSALLIS RELATIVE OPERATOR ENTROPY PROPERTIES RELATED TO HELLINGER METRICS

ABDENBI EL HILALI, MOHAMED CHERGUI, AND BOUAZZA EL WAHBI

Abstract. The current paper is centered on investigating some properties for the Tsallis relative operator entropy acting on positive definite matrices with respect to some versions of Hellinger distance. Particularly, distances between the Tsallis relative entropy and some means of two symmetric positive definite matrices are estimated.

1. Introduction

In this paper, $\mathbb{M}_n$ stands for the algebra of $n \times n$ matrices over $\mathbb{R}$ and $\mathbb{P}_n$ for the cone of all symmetric positive definite matrices. I will denote the identity matrix in $\mathbb{M}_n$.
On $\mathbb{P}_n$, a partial order can be defined by setting for any matrices $A, B \in \mathbb{P}_n$,

$$A \leq B \Longleftrightarrow B - A \geq 0, \text{ i.e. } B - A \text{ is a positive semi-definite matrix.}$$

In [9, 10], Fujji and Kamei introduced the relative operator entropy defined for two positive definite matrices A and B by

$$S(A \mid B) = A^{\frac{1}{2}} \log\left(A^{\frac{-1}{2}} B A^{\frac{-1}{2}}\right) A^{\frac{1}{2}}, \tag{1.1}$$

as a generalization of the following operator entropy [15]

$$S(A \mid I) = -A \log A.$$

In [19], Yanagi et al extend (1.1) by defining the so called Tsallis relative operator entropy for symmetric positive definite matrices as follows

$$T_p(A|B) = \frac{A \sharp_p B - A}{p}, \text{ for } p \in [-1, 1] \setminus \{0\}, \tag{1.2}$$

where $A \sharp_p B := A^{\frac{1}{2}} \left(A^{\frac{-1}{2}} B A^{\frac{-1}{2}}\right)^p A^{\frac{1}{2}}$ for all $p \in \mathbb{R}$ is the p-weighted geometric mean of A and B. If $p = 1/2$, we denote simply $A \sharp B$, known in the literature by Pusz-Woronwicz's geometric mean[18]. Throughout this paper, we stand $C := A^{\frac{-1}{2}} B A^{\frac{-1}{2}}$.
The generalization can be perceived by recalling the following result

$$\lim_{p \to 0} T_p(A|B) = S(A \mid B).$$

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Many important results concerning these operator entropies have been carried out in few last decades. We refer the interested reader for instance to [5, 9, 12, 15, 16, 19] and the references therein.

In this paper, we restrict ourselves to recall the following inequalities [8, 9, 11, 19] that will be needed in the sequel. For $A, B \in \mathbb{P}_n$ and $p \in (0,1]$, we have

$$A - AB^{-1}A \leq S(A|B) \leq T_p(A|B) \leq B - A. \tag{1.3}$$

$$A \leq T_p(A|B), \text{ for all } B \geq eA. \tag{1.4}$$

$$T_p(A \mid B) \leq T_q(A \mid B), \ \forall\, 0 < p \leq q \leq 1. \tag{1.5}$$

Inspired by some new concepts defined in [6], the authors undertook very recently a geometric study in [4, 7] for relative operator entropy and for Tsallis operator entropy with respect to Riemaniann and log-determinant metrics.

In the current work, we continue investigating further geometric properties for these operators. For this, we recall the following three versions of Hellinger distance, defined for any two positive definite matrices A and B as follows [1],

$$d_1(A,B) = \sqrt{trA + trB - 2tr\big(A^{\frac{1}{2}}B^{\frac{1}{2}}\big)}, \tag{1.6}$$

$$d_2(A,B) = \sqrt{trA + trB - 2tr\big(A^{\frac{1}{2}}BA^{\frac{1}{2}}\big)^{\frac{1}{2}}} = \sqrt{trA + trB - 2tr\big(AB\big)^{\frac{1}{2}}}, \tag{1.7}$$

and

$$d_3(A,B) = \sqrt{tr(A+B) - 2.tr(A\sharp B)}. \tag{1.8}$$

$tr(A)$ refers to the trace of the matrix A. Some of its properties that will be needed in this work are listed above [17, 20, 21]. For any matrices A, B and C from $\mathbb{M}_n$, we have

- Cyclicity of the trace: $tr(AB) = tr(BA)$.
- Invariance of trace under cyclic permutations: $tr(ABC) = tr(CAB) = tr(BCA)$.

We have also the following inequalities [2]

$$tr(A\sharp B) \leq tr\big(A^{\frac{1}{2}}B^{\frac{1}{2}}\big) \leq tr(AB)^{\frac{1}{2}}, \tag{1.9}$$

which allow to establish the following comparison

$$d_2(A,B) \leq d_1(A,B) \leq d_3(A,B). \tag{1.10}$$

The authors proved in [6] that the distances d_2, called also the Bures-Wasserstein metric [3], and d_1 are equivalent. More precisely, for two positive definite matrices A and B, we have

$$d_2(A,B) \leq d_1(A,B) \leq \sqrt{2}\, d_2(A,B). \tag{1.11}$$

In virtue of their practical aspect in many areas, the recalled metrics are intensively invested in developing new concepts and establishing important properties. It is worthy to mention that d_3 is not a distance because it does not satisfy the triangle inequality. However d_3^2 is a divergence, hence it serves as good distance measure. For more details about these last points, the reader is invited to see [1, 3, 6, 13].

The present work focuses on investigating some properties for the Tsallis relative operator entropy recalled in (1.2) with respect to the Hellinger metric versions d_i.

The remainder of the paper is organized as follows. In section 2, we state some results concerning monotonicity. Section 3 is devoted to determine a localization of $T_p(A \mid B)$. In section 4, we present some estimations of distance between Tsallis relative operator entropy and some operator means. At the end of the paper, some open problems are presented as well.

2. Monotonicity property

In the current section, we aim to study the monotonicity of the map $p \longmapsto d_i\big(A, T_p(A \mid B)\big)$. We begin by stating some preliminary tools which will be needed.

Lemma 2.1. *Let x be a strictly positive number. The function $\theta : p \mapsto \dfrac{x^p - 1}{p}$ is increasing on $(0, 1]$ and we have*

$$\lim_{p \to 0^+} \frac{x^p - 1}{p} = \log x.$$

Proof. It can be deduced by the use of some classical tools of real analysis. □

The following result due to Petz concerns the monotonicity of the trace functions [17, Proposition 1].

Lemma 2.2. *Let $A, B \in \mathbb{M}_n$ be two symmetric matrices and $f : \mathbb{R} \to \mathbb{R}$ be an increasing continuous function. We have,*

$$A \leq B \Longrightarrow tr[f(A)] \leq tr[f(B)].$$

Lemma 2.2 leads straightforward to the following useful result.

Lemma 2.3. *Let A, B and X be three matrices from $\mathbb{P}_n$. We have*

$$A \leq B \Longrightarrow tr(X\,A) \leq tr(X\,B).$$

The following lemma [8] provides a comparison between distances with respect to Hellinger metric d_1.

Lemma 2.4. *Let A, B and D be three matrices from $\mathbb{P}_n$ such that $A \leq B \leq D$. We have*

$$d_1(A, B) \leq d_1(A, D).$$

Now, we are in a position to state our main results.

Theorem 2.1. *Let A and B be two positive definite matrices such that $B \geq eA$. We have*

$$d_1\big(A, T_p(A \mid B)\big) \leq d_1\big(A, T_q(A \mid B)\big), \tag{2.1}$$

for all $0 < p \leq q \leq 1$.

Proof. Using the inequalities (1.4) and (1.5), we obtain

$$A \leq T_p(A \mid B) \leq T_q(A \mid B),\ \forall\, 0 < p \leq q \leq 1.$$

Applying the Lemma 2.4, we get the desired result (2.1). □

The following Lemma provides a comparison between distances with respect to Bures-Wasserstein distance for a convenient positive definite matrices.

Lemma 2.5. *Let A, B and D be three matrices from $\mathbb{P}_n$ such that $A^{-1} \leq B \leq D$. We have*

$$d_2(A, B) \leq d_2(A, D). \tag{2.2}$$

Proof. If $A^{-1} \leq B \leq D$ then $2I \leq \big(A^{\frac{1}{2}}BA^{\frac{1}{2}}\big)^{\frac{1}{2}} + \big(A^{\frac{1}{2}}DA^{\frac{1}{2}}\big)^{\frac{1}{2}}$.
Employing Lemma 2.3, we have

$$2.tr\left[\big(A^{\frac{1}{2}}DA^{\frac{1}{2}}\big)^{\frac{1}{2}} - \big(A^{\frac{1}{2}}BA^{\frac{1}{2}}\big)^{\frac{1}{2}}\right] \leq$$
$$tr\left[\Big(\big(A^{\frac{1}{2}}DA^{\frac{1}{2}}\big)^{\frac{1}{2}} + \big(A^{\frac{1}{2}}BA^{\frac{1}{2}}\big)^{\frac{1}{2}}\Big)\Big(\big(A^{\frac{1}{2}}DA^{\frac{1}{2}}\big)^{\frac{1}{2}} - \big(A^{\frac{1}{2}}BA^{\frac{1}{2}}\big)^{\frac{1}{2}}\Big)\right].$$

By the invariance of trace under cyclic permutations, we get

$$\begin{aligned} 2.tr\left[\big(A^{\frac{1}{2}}DA^{\frac{1}{2}}\big)^{\frac{1}{2}} - \big(A^{\frac{1}{2}}BA^{\frac{1}{2}}\big)^{\frac{1}{2}}\right] &\leq tr\Big(A^{\frac{1}{2}}DA^{\frac{1}{2}} - A^{\frac{1}{2}}BA^{\frac{1}{2}}\Big) \\ &\leq tr\big(A(D-B)\big) \\ &\leq tr(D-B). \end{aligned}$$

Which is equivalent to the inequality (2.2). □

Theorem 2.2. *Let A and B be two positive definite matrices such that $A \leq I$ and $B \geq A^{\frac{1}{2}}\exp(A^{-2})A^{\frac{1}{2}}$.*
For all $p, q \in (0\,,\,1]$ with $p \leq q$, we have the following inequality:

$$d_2\big(A, T_p(A \mid B)\big) \leq d_2\big(A, T_q(A \mid B)\big). \tag{2.3}$$

Proof. If $B \geq A^{\frac{1}{2}}\exp(A^{-2})A^{\frac{1}{2}}$, then $C \geq \exp(A^{-2})$.
Thanks to the monotonicity of the logarithm function on $\mathbb{P}_n$, we can write

$$\log C \geq A^{-2}.$$

So,

$$T_p(A \mid B) \geq S(A \mid B) \geq A^{-1} \text{ for all } p \in (0\,,\,1].$$

By the inequality (1.5), we have

$$T_q(A \mid B) \geq T_p(A \mid B) \geq A^{-1} \text{ for all } 0 < p \leq q \leq 1.$$

Using Lemma 2.5, the proof of (2.3) is ended. □

Remark 2.1. When the conditions $eA \leq B$ and $B \geq A^{\frac{1}{2}}\exp(A^{-2})A^{\frac{1}{2}}$ are not satisfied in Theorems 2.1 and 2.2 respectively, the inequalities (2.1) and (2.3) may no longer remain valid. The following counterexample shows this situation.

Consider the next two positive matrices

$$A = \begin{pmatrix} 0.7 & 0 & 0 \\ 0 & 0.2 & 0.05 \\ 0 & 0.05 & 0.1 \end{pmatrix} \text{ and } B = \begin{pmatrix} 0.84 & 0 & 0 \\ 0 & 0.24 & 0.06 \\ 0 & 0.06 & 0.12 \end{pmatrix}.$$

Let us note that $A \leq I$ and $B = 1.2\,A < A^{\frac{1}{2}}\exp(A^{-2})A^{\frac{1}{2}}$.
Calculations with Matlab software give

- $d_1(A, T_{0.25}(A|B)) = 0.5681 > d_1(A, T_{0.5}(A|B)) = 0.5631.$
- $d_2(A, T_{0.15}(A|B)) = 0.5701 > d_2(A, T_{0.2}(A|B)) = 0.5691.$

3. A localization of $T_p(A \mid B)$

The ongoing section is devoted to determining the position of $T_p(A \mid B)$ with respect to the sphere centered at A and with radius $d_i(A, B)$. Our first result reads as follows.

Theorem 3.1. *Let $A, B \in \mathbb{P}_n$ such that $B \geq eA$ and $p \in (0\,,1]$. The following inequality holds*

$$d_1\big(A, T_p(A \mid B)\big) \leq d_1(A, B). \tag{3.1}$$

Proof. Using the right side of inequality (1.3) combined with the inequality (1.4), we get

$$A \leq T_p(A \mid B) \leq B.$$

Applying Lemma 2.4, we deduce the desired inequality (3.1). □

Theorem 3.2. *Let $A, B \in \mathbb{P}_n$ such that $A \leq I$ and $B \geq A^{\frac{1}{2}} \exp(A^{-2}) A^{\frac{1}{2}}$. We have*

$$d_2\big(A, T_p(A \mid B)\big) \leq d_2(A, B), \tag{3.2}$$

for all $p \in (0\,,\,1]$.

Proof. By the condition $B \geq A^{\frac{1}{2}} \exp(A^{-2}) A^{\frac{1}{2}}$, we have

$$A^{-1} \leq T_p(A \mid B), \text{ for all } p \in (0\,,\,1].$$

This combined with the right side of the inequality (1.3) gives

$$A^{-1} \leq T_p(A \mid B) \leq B, \text{ for all } 0 < p \leq 1.$$

Using Lemma 2.2, we deduce the inequality (3.2). □

To provide similar result for d_3, we need the following lemma.

Lemma 3.1. *The real function $u : x \longmapsto u(x) = x - 4x^{\frac{1}{2}} + 2(\log x)^{\frac{1}{2}} + 2$ is positive on $[\delta_1\,,\,\infty)$, where δ_1 denotes the unique real number such that $u(\delta_1) = 0$ and $1.95 < \delta_1 < 1.96$.*

Proof. For $x > e$, we have $u(x) = \big(x^{\frac{1}{2}} - 2\big)^2 + 2\big((\log x)^{\frac{1}{2}} - 1\big) > 0$.
Let $1 < x \leq e$. We have

$$\begin{aligned} u'(x) &= \frac{x\sqrt{x \log x} - 2x\sqrt{\log x} + \sqrt{x}}{x\sqrt{x \log x}} \\ &= \frac{(x\sqrt{\log x} - 2\sqrt{x \log x} + 1)}{x\sqrt{\log x}} \\ &= \frac{(x \log x - 2\sqrt{x \log x} + 1)}{x\sqrt{\log x}} \geq 0. \end{aligned}$$

The last inequality is obtained by virtue of the fact $0 < \log x \leq \sqrt{\log x}$.
So, the function u is strictly increasing on $[1\,,\,e]$.
Since u is continuous on $(1\,,\,e]$, it establishes a bijection from $(1\,,\,e]$ onto $\big(u(1^+), u(e)\big] = \big(-1, (\sqrt{e} - 2)^2\big]$.
Thus, there exists a unique number δ_1 verifying $u(\delta_1) = 0$.
Since $u(1.95)\,u(1.96) < 0$, then $1.95 < \delta_1 < 1.96$. With simple manipulations the proof is achieved. □

Theorem 3.3. *Let A and B be two positive definite matrices such that $B \geq \delta_1 A$ and $p \in \big(0, \frac{1}{2}\big]$. We have the following inequality*

$$d_3\big(A, T_p(A \mid B)\big) \leq d_3(A, B), \tag{3.3}$$

where δ_1 is the fixed real number defined in Lemma 3.1.

Proof. If $B \geq \delta_1 A$ then $C \geq \delta_1 I$.
Using successively Lemma 2.1 and Lemma 3.1, we get the following inequalities

$$\begin{aligned} \frac{C^p - I}{p} - 2\Big(\frac{C^p - I}{p}\Big)^{\frac{1}{2}} &\leq 2\big(C^{\frac{1}{2}} - I\big) - 2\big(\log C\big)^{\frac{1}{2}} \\ &\leq C - 2C^{\frac{1}{2}}. \end{aligned}$$

Employing Lemma 2.3, we obtain

$$tr\bigg(A^{\frac{1}{2}}\Big(\frac{C^p - I}{p}\Big)A^{\frac{1}{2}} - 2A^{\frac{1}{2}}\Big(\frac{C^p - I}{p}\Big)^{\frac{1}{2}}A^{\frac{1}{2}}\bigg) \leq tr\bigg(A^{\frac{1}{2}}CA^{\frac{1}{2}} - 2A^{\frac{1}{2}}C^{\frac{1}{2}}A^{\frac{1}{2}}\bigg).$$

Whence, the inequality (3.3) is obtained. □

Remark 3.1. If the condition $B \geq \delta_1 A$ is not satisfied, the inequalities (3.1),(3.2) and (3.3) may not hold. In fact, the following counterexample confirms this.

Let us consider the following two positive matrices

$$A = \begin{pmatrix} 0.45 & 0.1 & 0 \\ 0.1 & 0.25 & 0 \\ 0 & 0 & 0.3 \end{pmatrix} \text{ and } B = \begin{pmatrix} 0.72 & 0.16 & 0 \\ 0.16 & 0.40 & 0 \\ 0 & 0 & 0.48 \end{pmatrix}.$$

We have $A < I$ and $B = 1.6\,A < A^{\frac{1}{2}} \exp(A^{-2})A^{\frac{1}{2}}$. Calculations give

- $d_1(A, T_{0.1}(A|B)) = 0.3063 > d_1(A, B) = 0.2649$.
- $d_2(A, T_{0.4}(A|B)) = 0.2809 > d_2(A, B) = 0.2649$.
- $d_3(A, T_{0.25}(A|B)) = 0.2938 > d_3(A, B) = 0.2649$.

4. Distances involving $T_p(A \mid B)$ and some operator means

In this section, we are interested in estimating distances involving $T_p(A \mid B)$ and geometric and harmonic mean operators.

4.1. Estimating $d_i\big(A\sharp B, T_p(A \mid B)\big)$. To state our first result, we need the following lemma.

Lemma 4.1. *We have the following two assertions:*
- *The real function defined by $v(x) = \frac{1}{4}x - \frac{8}{5}x^{\frac{5}{8}} + \frac{8}{5}$ is strictly increasing on $[123, \infty)$ and there exists a unique δ_2 satisfying $v(\delta_2) = 0$ and $123,39 < \delta_2 < 123,4$.*
- *The function $g(x) = \frac{1}{8}x - \frac{8}{5}(x^{\frac{5}{8}} - 1)$ is strictly increasing on $[861, \infty)$ and there exists a unique δ_3 such that $g(\delta_3) = 0$ and $861,96 < \delta_3 < 861,97$.*

Proof. The proof is a routine exercise of real analysis and hence it is omitted. □

Theorem 4.1. *Let A and B be two positive definite matrices and $p \in (0, 1]$. If one the following two conditions holds*

i) $p \in \big[0\,,\,\frac{1}{4}\big]$ *and* $B \geq 16A$

ii) $p \in \left[\frac{1}{2}\,,\,\frac{5}{8}\right]$ and $B \geq \delta_2 A$,

then we have

$$d_1\big(A\sharp B, T_p(A \mid B)\big) \leq \frac{1}{2} d_1(A,B), \tag{4.1}$$

where δ_2 refers to the real number defined in the first assertion in Lemma 4.1.

Proof. For all $x \geq 0$, we have

$$4(x^{\frac{1}{4}} - 1) \leq x^{\frac{1}{2}}. \tag{4.2}$$

$i)$ Let us suppose $B \geq 16A$, that is $C \geq 16I$.
Substituting x by C in the inequality (4.2), we get

$$4(C^{\frac{1}{4}} - I) \leq C^{\frac{1}{2}} \leq \frac{1}{4}C.$$

So,

$$T_{\frac{1}{4}}(A \mid B) \leq A\sharp B \leq \frac{1}{4}B.$$

Thanks to the inequalities (1.4) and (1.5), we obtain for all $p \in \left[0\,,\,\frac{1}{4}\right]$

$$\frac{1}{4}A \leq T_p(A \mid B) \leq T_{\frac{1}{4}}(A \mid B) \leq A\sharp B \leq \frac{1}{4}B.$$

These inequalities combined to the monotonicity of the square root function, it holds

$$\frac{1}{2}A^{\frac{1}{2}} \leq T_p^{\frac{1}{2}}(A \mid B) \leq (A\sharp B)^{\frac{1}{2}} \leq \frac{1}{2}B^{\frac{1}{2}}.$$

So,

$$(A\sharp B)^{\frac{1}{2}} - T_p^{\frac{1}{2}}(A \mid B) \leq \frac{1}{2}\big(B^{\frac{1}{2}} - A^{\frac{1}{2}}\big).$$

Using Lemma 2.2 with the square function, we obtain

$$tr\bigg(\big((A\sharp B)^{\frac{1}{2}} - T_p^{\frac{1}{2}}(A \mid B)\big)^2\bigg) \leq \frac{1}{4} tr\bigg(\big(B^{\frac{1}{2}} - A^{\frac{1}{2}}\big)^2\bigg),$$

which leads to (4.1).
$ii)$ If $B \geq \delta_2 A$ then $C \geq \delta_2 I$. Thus,

$$\frac{1}{4}I \leq C^{\frac{1}{2}} \leq 2(C^{\frac{1}{2}} - I).$$

Hence,

$$\frac{1}{4}A \leq A\sharp B \leq T_{\frac{1}{2}}(A \mid B).$$

Combining the inequality (1.5) and Lemma 4.1, we have for all $p \in [\frac{1}{2}, \frac{5}{8}]$

$$\frac{1}{4}A \leq A\sharp B \leq T_{\frac{1}{2}}(A \mid B) \leq T_p(A \mid B) \leq T_{\frac{5}{8}}(A \mid B) \leq \frac{1}{4}A^{\frac{1}{2}}CA^{\frac{1}{2}}.$$

So, we deduce

$$T_p^{\frac{1}{2}}(A \mid B) + \frac{1}{2}A^{\frac{1}{2}} \leq (A\sharp B)^{\frac{1}{2}} + \frac{1}{2}B^{\frac{1}{2}},$$

which can be rephrased as follows

$$T_p^{\frac{1}{2}}(A \mid B) - (A\sharp B)^{\frac{1}{2}} \leq \frac{1}{2}B^{\frac{1}{2}} - \frac{1}{2}A^{\frac{1}{2}}.$$

Applying Lemma 2.2 with the square function, we obtain

$$tr\Big(\big(T_p^{\frac{1}{2}}(A \mid B) - (A\sharp B)^{\frac{1}{2}}\big)^2\Big) \leq \frac{1}{4}tr\Big(\big(B^{\frac{1}{2}} - A^{\frac{1}{2}}\big)^2\Big).$$

Whence, the inequality (4.1) is deduced. □

The analog of inequality (4.1) with d_2 reads as follows.

Theorem 4.2. *Let A and B be two positive definite matrices and $p \in (0,1]$. If one the following two conditions holds*

i) $p \in \left[0\,,\,\frac{1}{4}\right]$ and $B \geq 64A$

ii) $p \in \left[\frac{1}{2}\,,\,\frac{5}{8}\right]$ and $B \geq \delta_3 A$,

we get the inequality

$$d_2(A\sharp B, T_p(A|B)) \leq \frac{1}{2}d_2(A,B), \tag{4.3}$$

where δ_3 stands for to the real number defined in the second assertion of Lemma 4.1.

Proof. • If $B \geq 64A$, then $C \geq 64I$. Thus, we get the following inequalities

$$4(C^{\frac{1}{4}} - I) \leq C^{\frac{1}{2}} \leq \frac{1}{8}C$$

and

$$T_{\frac{1}{4}}(A|B) \leq A\sharp B \leq \frac{1}{8}B.$$

These combined with the inequalities (1.4) and (1.5), enable us to write for all $p \in \left[0\,,\,\frac{1}{4}\right]$

$$\frac{1}{8}A \leq T_p(A|B) \leq T_{\frac{1}{4}}(A|B) \leq A\sharp B \leq \frac{1}{8}B.$$

Using the monotonicity of the square root function, we have

$$\frac{1}{2\sqrt{2}}A^{\frac{1}{2}} \leq T_p^{\frac{1}{2}}(A|B) \leq (A\sharp B)^{\frac{1}{2}} \leq \frac{1}{2\sqrt{2}}B^{\frac{1}{2}}.$$

So,

$$(A\sharp B)^{\frac{1}{2}} - T_p^{\frac{1}{2}}(A|B) \leq \frac{1}{2\sqrt{2}}\big(B^{\frac{1}{2}} - A^{\frac{1}{2}}\big).$$

Applying Lemma 2.2 with the square function, we obtain

$$tr\Big(\big((A\sharp B)^{\frac{1}{2}} - T_p^{\frac{1}{2}}(A|B)\big)^2\Big) \leq \frac{1}{8}tr\Big(\big(B^{\frac{1}{2}} - A^{\frac{1}{2}}\big)^2\Big).$$

• If $B \geq \delta_3 A$ then $C \geq \delta_3 I$. Thus,

$$\frac{1}{8}I \leq C^{\frac{1}{2}} \leq 2(C^{\frac{1}{2}} - I),$$

which gives

$$\frac{1}{8}A \leq A\sharp B \leq T_{\frac{1}{2}}(A|B).$$

Substituting x with C in the second assertion of Lemma 4.1, we get

$$\frac{1}{8}C \geq \frac{8}{5}(C^{\frac{5}{8}} - I) \text{ for any } \frac{1}{2} \leq p \leq \frac{5}{8}.$$

Thus,

$$T_{\frac{5}{8}}(A \mid B) \le \frac{1}{8}B.$$

By using the inequality (1.5), we deduce the following chain of inequalities

$$\frac{1}{8}A \le A\sharp B \le T_{\frac{1}{2}}(A|B) \le T_p(A|B) \le T_{\frac{5}{8}}(A|B) \le \frac{1}{8}B,$$

it yields

$$T_p^{\frac{1}{2}}(A|B) + \frac{1}{2\sqrt{2}}A^{\frac{1}{2}} \le (A\sharp B)^{\frac{1}{2}} + \frac{1}{2\sqrt{2}}B^{\frac{1}{2}},$$

that is,

$$T_p^{\frac{1}{2}}(A|B) - (A\sharp B)^{\frac{1}{2}} \le \frac{1}{2\sqrt{2}}\big(B^{\frac{1}{2}} - A^{\frac{1}{2}}\big).$$

By Lemma 2.2, we obtain

$$tr\Big(\big(T_p^{\frac{1}{2}}(A|B) - \big(A\sharp B\big)^{\frac{1}{2}}\big)^2\Big) \le \frac{1}{8}tr\Big(\big(B^{\frac{1}{2}} - A^{\frac{1}{2}}\big)^2\Big).$$

To achieve the proof, we apply the inequality (1.11) and we get

$$d_2(A\sharp B, T_p(A|B)) \le d_1(A\sharp B, T_p(A|B)) \le \frac{1}{2\sqrt{2}}d_1(A,B) \le \frac{1}{2}d_2(A,B). \qquad \square$$

For establishing the analog of inequality (4.1) involving d_3, the following lemma will be useful.

Lemma 4.2. *The following two assertions hold.*
- *The function defined on* $[7,\infty)$ *by* $w(x) = \frac{1}{4}x - \frac{3}{2}x^{\frac{1}{2}} + \log x + \frac{1}{4}$ *is strictly increasing and there exists a unique number* δ_4 *such that* $w(\delta_4) = 0$ *and* $7.20 < \delta_4 < 7.21$.
- *The function defined on* $[59,\infty)$ *by* $t(x) = 37 + 5x + 10x^{\frac{1}{2}} - 32x^{\frac{5}{8}}$ *is strictly increasing and there exists a unique number* δ_5 *such that* $t(\delta_5) = 0$ *and* $59,29 < \delta_5 < 59,30$.

Proof. The proof is straightforward, then it is omitted. $\square$

Theorem 4.3. *Let* $A, B \in \mathbb{P}_n$ *and* $p \in (0,1]$*. If one of the following conditions is satisfied*

i) $p \in \big(0, \frac{1}{4}\big]$ *and* $B \ge \delta_4 A$

ii) $p \in \big[\frac{1}{2}, \frac{5}{8}\big]$ *and* $B \ge \delta_5 A$,

the following inequality holds

$$d_3(A\sharp B, T_p(A|B)) \le \frac{1}{2}d_3(A,B). \tag{4.4}$$

δ_4 *and* δ_5 *stand for the numbers defined in Lemma 4.2.*

Proof. • Let us take $B \ge \delta_4 A$ and $p \in \big(0, \frac{1}{4}\big]$. Using Lemma 2.1, we have

$$\frac{C^p - I}{p} \le 4(C^{\frac{1}{4}} - I) \le C^{\frac{1}{2}},$$

or equivalently

$$T_p(A|B) \le T_{\frac{1}{4}}(A|B) \le A\sharp B.$$

Hence,

$$-2(A\sharp B)\sharp T_p(A|B) \leq -2T_p(A|B). \tag{4.5}$$

Thus, to prove the inequality (4.4), it suffices to establish the following inequality:

$$tr\big(A\sharp B - T_p(A|B)\big) \leq \frac{1}{4}tr\big(A + B - 2A\sharp B\big).$$

According to the first statement in Lemma 4.2, we have

$$\frac{3}{2}C^{\frac{1}{2}} - \log C \leq \frac{1}{4}(I + C),$$

which can be rephrased as follows

$$C^{\frac{1}{2}} - \log C \leq \frac{1}{4}\big(I + C - 2C^{\frac{1}{2}}\big).$$

Employing Lemma 2.3 and the inequality (1.3), we obtain

$$tr\big(A\sharp B - T_p(A|B)\big) \leq tr\big(A\sharp B - S(A \mid B)\big) \leq \frac{1}{4}tr\big(A + B - 2A\sharp B\big). \tag{4.6}$$

From (4.5) and (4.6), we deduce

$$tr\big(A\sharp B+T_p(A|B)-2(A\sharp B)\sharp T_p(A|B)\big) \leq tr\big(A\sharp B-T_p(A|B)\big) \leq \frac{1}{4}tr\big(A+B-2A\sharp B\big).$$

- Now, let us take $p \in \big[\frac{1}{2}, \frac{5}{8}\big]$ and $B \geq \delta_5\, A$. So,

$$C^{\frac{1}{2}} \leq 2\big(C^{\frac{1}{2}} - I\big) \leq \frac{C^p - I}{p}.$$

That is

$$A\sharp B \leq T_{\frac{1}{2}}(A|B) \leq T_p(A|B),$$

which implies

$$-2(A\sharp B)\sharp T_p(A|B) \leq -2(A\sharp B).$$

Thus, to get the inequality (4.4) it suffices to prove the following inequality

$$tr\big(T_p(A|B) - A\sharp B\big) \leq \frac{1}{4}tr\big(A + B - 2A\sharp B\big).$$

According to the second statement in Lemma 4.2, we get

$$0 \leq 5C + 10C^{\frac{1}{2}} - 32C^{\frac{5}{8}} + 37I,$$

which is equivalent to

$$\frac{8}{5}\big(C^{\frac{8}{5}} - I\big) - C^{\frac{1}{2}} \leq \frac{1}{4}\big(I + C - 2C^{\frac{1}{2}}\big).$$

By Lemma 2.3, we get

$$tr\big(T_p(A|B) - A\sharp B\big) \leq \frac{1}{4}tr\big(A + B - 2A\sharp B\big).$$

Consequently, we get

$$tr\big(A\sharp B+T_p(A|B)-2(A\sharp B)\sharp T_p(A|B)\big) \leq tr\big(A\sharp B-T_p(A|B)\big) \leq \frac{1}{4}tr\big(A+B-2A\sharp B\big).$$

Therefore, the proof of the theorem is ended. $\square$

Remark 4.1. The Theorems 4.1, 4.2 and 4.3 can be interpreted as follows: for some convenient matrices A and B from $\mathbb{P}_n$ and a scalar p, the Tsallis entropy $T_p(A|B)$ lies inside the sphere centered at the geometric mean $A\sharp B$ with radius equal to the half of the distance $d_i(A, B)$.

Remark 4.2. The conditions in Theorems 4.1, 4.2 and 4.3 are essential to get the inequalities (4.1), (4.3) and (4.4) as the following example shows.

Example 4.1. Considering the positive matrix $A = \begin{pmatrix} 3 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix}$, we have the following values

B	p	$d_1(A\sharp B, T_p(A\|B))$	$d_2(A\sharp B, T_p(A\|B))$	$d_3(A\sharp B, T_p(A\|B))$	$\frac{1}{2}\, d_i(A, B)$
$2A$	0.62	0.6336	0.6336	0.6336	0.5037
$2.5A$	0.01	0.73	0.73	0.73	0.7117
$16A$	0.9	3.7133	3.7133	3.7133	3.6742
$64A$	0.87	8.8879	8.8879	8.8879	8.5732
$88A$	0.89	11.3602	11.3602	11.3602	10.2644
$124A$	0.86	12.6485	12.6485	12.6485	12.4134
$862A$	0.88	37.7651	37.7651	37.7651	34.7336

TABLE 1. Some numerical examples

4.2. Distances involving harmonic operator mean.

We recall that the harmonic mean of two symmetric positive definite matrices A and B is defined by

$$\mathcal{H}(A, B) = 2\left(A^{-1} + B^{-1}\right)^{-1}.$$

To point out our results, we need the following lemma which is simple to prove.

Lemma 4.3. *i) For the function defined on* $[46, \infty)$ *by* $s(x) = \frac{1}{4}x - 2(x^{\frac{1}{2}} - 1)$, *there exists a unique real number* δ_6 *satisfying* $s(\delta_6) = 0$ *such that* $46.62 < \delta_6 < 46.63$. *Moreover, for any* $x \geq \delta_6$, $s(x) \geq 0$.
ii) For the function defined on $[222, \infty)$ *by* $t(x) = \frac{1}{8}x - 2(x^{\frac{1}{2}} - 1)$, *there exists a unique real number* δ_7 *satisfying* $t(\delta_7) = 0$ *such that* $222.85 < \delta_7 < 222.86$. *Moreover, for any* $x \geq \delta_7$, $t(x) \geq 0$.
iii) For the function defined on $[61, \infty)$ *by* $m(x) = \frac{1}{8}x - \frac{5}{4}x^{\frac{1}{2}} + \left(1 + x^{-1}\right)^{-1} + \frac{9}{8}$, *there exists a unique real number* δ_8 *satisfying* $m(\delta_8) = 0$ *such that* $61.63 < \delta_8 < 61.64$. *Moreover, for any* $x \geq \delta_8$, $m(x) \geq 0$.

Theorem 4.4. *Let* A *and* B *be two positive definite matrices such that* $B \geq \delta_6 A$. *The following inequality*

$$d_1\big(\mathcal{H}(A, B), T_p(A \mid B)\big) \leq \frac{1}{2} d_1(A, B) \tag{4.7}$$

holds for every $p \in \left(0, \frac{1}{2}\right]$. δ_6 *stands for the fixed real number defined in Lemma 4.3.*

Proof. Let $B \geq \delta_6\, A$ and $p \in \big(0, \frac{1}{2}\big]$. So, $C \geq \delta_6 I$.
Using the first statement in Lemma 4.3 and (1.5), we get

$$2(C^{\frac{1}{2}} - I) \leq \frac{1}{4}C \implies T_{\frac{1}{2}}(A \mid B) \leq \frac{1}{4}B$$
$$\implies T_p(A \mid B) \leq \frac{1}{4}B.$$

So,

$$T_p^{\frac{1}{2}}(A \mid B) \leq \frac{1}{2}B^{\frac{1}{2}}. \tag{4.8}$$

We can easily check that for any $x \geq \delta_6$, we have $\log x - 2\big(1 + x^{-1}\big)^{-1} \geq 0$.
Substituting x by C, we get

$$2\big(I + C^{-1}\big)^{-1} \leq \log C.$$

So,

$$\mathcal{H}(A, B) \leq S(A \mid B),$$

which implies by the use of the right side of the inequality (1.3),

$$\mathcal{H}(A, B) \leq T_p(A \mid B). \tag{4.9}$$

Thus,

$$\mathcal{H}^{\frac{1}{2}}(A, B) \leq T_p^{\frac{1}{2}}(A \mid B). \tag{4.10}$$

Since, $A \leq B$ then $A = \mathcal{H}(A, A) \leq \mathcal{H}(A, B)$.
So,

$$\frac{1}{2}A^{\frac{1}{2}} \leq \mathcal{H}^{\frac{1}{2}}(A, B). \tag{4.11}$$

Using the inequalities (4.8) and (4.11), we obtain

$$\frac{1}{2}A^{\frac{1}{2}} + T_p^{\frac{1}{2}}(A \mid B) \leq \mathcal{H}^{\frac{1}{2}}(A, B) + \frac{1}{2}B^{\frac{1}{2}}.$$

Employing Lemma 2.2, we can state

$$tr\left(\big(T_p^{\frac{1}{2}}(A \mid B) - \mathcal{H}^{\frac{1}{2}}(A, B)\big)^2\right) \leq \frac{1}{4}tr\left(\big(B^{\frac{1}{2}} - A^{\frac{1}{2}}\big)^2\right),$$

which is equivalent to the desired inequality (4.7). □

Theorem 4.5. *Let $A, B \in \mathbb{P}_n$ such that $B \geq \delta_7 A$ and $p \in \big(0, \frac{1}{2}\big]$. We have*

$$d_2\big(\mathcal{H}(A, B), T_p(A \mid B)\big) \leq \frac{1}{2}d_2(A, B), \tag{4.12}$$

where δ_7 is the real number defined in Lemma 4.3.

Proof. According to the conditions of the theorem, and by the second statement of Lemma 4.3, we have

$$2(C^{\frac{1}{2}} - I) \leq \frac{1}{8}C \implies T_{\frac{1}{2}}(A \mid B) \leq \frac{1}{8}B$$
$$\implies T_p(A \mid B) \leq \frac{1}{8}B$$
$$\implies T_p^{\frac{1}{2}}(A \mid B) \leq \frac{1}{2\sqrt{2}}B^{\frac{1}{2}}.$$

Since $\delta_7 > \delta_6$, then by the inequality (4.9) we have

$$\mathcal{H}(A,B) \leq T_p(A \mid B).$$

Using the following inequality

$$\frac{1}{2\sqrt{2}}A^{\frac{1}{2}} \leq \mathcal{H}^{\frac{1}{2}}(A,B),$$

we get,

$$T_p^{\frac{1}{2}}(A \mid B) + \frac{1}{2\sqrt{2}}A^{\frac{1}{2}} \leq \frac{1}{2\sqrt{2}}B^{\frac{1}{2}} + \mathcal{H}^{\frac{1}{2}}(A,B),$$

which leads by using Lemma 2.2

$$tr\Big(\big(T_p^{\frac{1}{2}}(A \mid B) - \mathcal{H}^{\frac{1}{2}}(A,B)\big)^2\Big) \leq \frac{1}{8}tr\Big(\big(B^{\frac{1}{2}} - A^{\frac{1}{2}}\big)^2\Big).$$

Applying the inequality (1.11), we obtain

$$\begin{aligned} d_2\big(\mathcal{H}(A,B), T_p(A \mid B)\big) &\leq d_1\big(\mathcal{H}(A,B), T_p(A \mid B)\big) \\ &\leq \frac{1}{2\sqrt{2}}d_1(A,B) \\ &\leq \frac{1}{2}d_2(A,B). \end{aligned}$$

Thus, the proof of (4.12) is achieved. □

Theorem 4.6. *Let A and B be two positive definite matrices such that $B \geq \delta_8 A$ and $p \in \big(0, \frac{1}{2}\big]$. we have*

$$d_3\big(\mathcal{H}(A,B), T_p(A \mid B)\big) \leq \frac{1}{2}d_3(A,B), \tag{4.13}$$

where δ_8 is the real number defined in Lemma 4.3.

Proof. If $B \geq \delta_8 A$, then $C \geq \delta_8 I$. So, by the third assertion in Lemma 4.3 it holds

$$2\big(C^{\frac{1}{2}} - I\big) - 2\big(I + C^{-1}\big)^{-1} \leq \frac{1}{4}\big(I + C - 2C^{\frac{1}{2}}\big).$$

Multiplying left and right the both sides of this last inequality by $A^{\frac{1}{2}}$, we obtain

$$T_{\frac{1}{2}}(A \mid B) - \mathcal{H}(A,B) \leq \frac{1}{4}\big(A + B - 2A\sharp B\big).$$

So, for all $p \in \big(0, \frac{1}{2}\big]$,

$$T_p(A \mid B) - \mathcal{H}(A,B) \leq \frac{1}{4}\big(A + B - 2A\sharp B\big). \tag{4.14}$$

On the other part, by virtue of the inequality (4.9), we have

$$\mathcal{H}(A,B) \leq \mathcal{H}(A,B)\sharp T_p(A \mid B) \leq T_p(A \mid B).$$

Consequently,

$$-2\mathcal{H}(A,B)\sharp T_p(A \mid B) \leq -2\mathcal{H}(A,B). \tag{4.15}$$

Summing the inequalities (1.5), (4.14) and (4.15), we conclude

$$tr\big(\mathcal{H}(A,B) + T_p(A \mid B) - 2\mathcal{H}(A,B)\sharp T_p(A \mid B)\big) \leq \frac{1}{4}tr\big(A + B - 2A\sharp B\big),$$

which is equivalent to the inequality (4.13). □

Remark 4.3. If any condition in one of Theorems 4.4, 4.5 and 4.6 lacks, the related inequalities are not necessary true. Many counterexamples can be provided to confirm this statement.

5. Conclusion and open problems

Our findings can be summarized as follows. For the three versions d_i of Hellinger metric for positive matrices, we pointed out in this paper that under convenient assumptions, we have the following results.

i) The map $p \longmapsto d_i(A, T_p(A|B)$ is increasing on $(0, 1]$ for $i = 1$ or $i = 2$.
ii) $T_p(A|B)$ lies into the sphere centered at A with radius equal to $d_i(A, B)$.
iii) $T_p(A|B)$ lies inside both spheres centered respectively at the geometric mean $A\sharp B$ and the harmonic mean $\mathcal{H}(A, B)$ with radius equal to the half of the distance $d_i(A, B)$.

At the end of this work, we share some open problems for future investigations:
Problem 1: Prove or disprove that the conditions in the theorems stated in this paper are necessary.
Problem 2: Let m be a mean operator in the sense of Kubo-Ando theory [14]. Prove or disprove that

$$d_i\big(m(A, B); T_p(A|B)\big) \leq \frac{1}{2}\, d_i(A, B).$$

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Abdenbi El Hilali
Department of Mathematics, Science Faculty, LAGA-Lab Ibn Tofail University, Kenitra, Morocco.
E-mail address: abdenbielhilali1976@gmail.com

Mohamed Chergui
Department of Mathematics, CRMEF, EREAM Team, LaREAMI-Lab, Kénitra, Morocco.
E-mail address: chergui_m@yahoo.fr

Bouazza El Wahbi
Department of Mathematics, Science Faculty, LAGA-Lab Ibn Tofail University, Kenitra, Morocco.
E-mail address: bouazza.elwahbi@uit.ac.ma

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