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SIMULTANEOUS-MAXIMAL APPROXIMATION BY TAYLOR PARTIAL SUMS

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Abstract. In this work simultaneous-maximal approximation properties of Taylor partial sums in the canonical disks are investigated.

1. Introduction

Let D_R:=$\{z : |z| < R\}$, $R > 1$, $D := D_1$. By $A\left(\overline{D_R}\right)$ we denote the class of analytic in D_R and continuous on its closure $\overline{D_R}$ functions f equipped with the norm $\|f\|_{A\left(\overline{D_R}\right)} := \sup\limits_{z\in\overline{D_R}} |f(z)|$.

If $f \in A\left(\overline{D_R}\right)$, then by Taylor's theorem we have the series representation

$$f(z) = \sum_{k=0}^{\infty} a_k z^k, \quad |z| < R, \tag{1.1}$$

which converges in D_R and uniformly converges in every compact subset of D_R.

Here the Taylor coefficients a_k can be defined by the formulas

$$a_k = \frac{1}{2\pi i}\int_{|t|=R} \frac{f(t)}{t^{k+1}}dt, \ k = 0, 1, 2, ... \tag{1.2}$$

In [20, p.243] researched maximal convergence of partial sums of (1.1), namely for a given $f \in A\left(\overline{D_R}\right)$ estimated the error

$$|R_n(z; f)| = \left|f(z) - \sum_{k=0}^{n} a_k z^k\right| = \left|\sum_{k=n+1}^{\infty} a_k z^k\right|, \quad |z| \le 1$$

in the uniform norm on $\overline{D} \subset D_R$ and proved the following inequality in the special case:

$$|R_n(z; f)| \le E_n\left(f, \overline{D_R}\right)\frac{1}{R^n}\ln\frac{R+1}{R-1}, \ |z| \le 1,$$

where

$$E_n\left(f; \overline{D_R}\right) := \inf_{p_n\in W_n} \|f - p_n\|_{A\left(\overline{D_R}\right)}$$

is the best approximation number for f in the class W_n of algebraic polynomials p_n of degree at most n. This estimation characterizes maximal convergence property of Taylor's partial sums in uniform norm.

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Since the series (1.1) converges uniformly on compact subsets of D_R, for its derivatives we have

$$\begin{aligned} f'(z) &= \sum_{k=1}^{\infty} a_k k z^{k-1}, \quad z \in D_R, \\ f''(z) &= \sum_{k=2}^{\infty} a_k k(k-1) z^{k-2}, \quad z \in D_R \end{aligned}$$

and hence for mth ($m \in \mathbb{N}_0$:=$\{1, 2, 3, ...\}$) order derivative we have the series representation

$$f^{(m)}(z) = \sum_{k=m}^{\infty} a_k \frac{k!}{(k-m)!} z^{k-m}, \quad z \in D_R.$$

Now let $m \in \mathbb{N}_0$ and let

$$\begin{aligned} R_n\left(z; f^{(m)}\right) &: = f^{(m)}(z) - \sum_{k=m}^{n} a_k \frac{k!}{(k-m)!} z^{k-m} \\ &= \sum_{k=n+1}^{\infty} a_k \frac{k!}{(k-m)!} z^{k-m}, \quad |z| \leq 1 \end{aligned} \tag{1.3}$$

for a given $f \in A\left(\overline{D_R}\right)$.

In this work the simultaneous-maximal convergence property of Taylor's partial sums is investigated. Namely, we estimate the error $\left|R_n\left(z; f^{(m)}\right)\right|$ in the uniform norm on $\overline{D} \subset D_R$, with dependence of the parameters n, m, R and the best approximation number $E_n\left(f; \overline{D_R}\right)$.

This problem originates to P. K. Suetin, who in [20, p. 212] emphasized its correctness on the canonical domains, in particular on canonical disks. Note that in the mathematical literature there are many studies on maximal convergence of Taylor's or more general partial sums on canonical domains of the complex plane. Early results obtained in this area can be found in the books [22, Appendix], [6, Chapter 1] and [19, Chapter 2]. Relatively recent results regarding the maximal convergence and some other applications of Taylor and Faber series expansions in the complex plane are available in [20, Chapter 10] (see also the papers: [8]-[16], [2, 3, 7, 17, 18, 21] and the references given therein). There are also results only on simultaneous approximation of different approximation aggregates, which can be found for example in the books [4, Chapter7] and [1, Chapter 4]. However, the studies on both simultaneous and maximal convergence as far as we know are scarce.

Our new results are following:

Theorem 1.1. *Let $f \in A\left(\overline{D_R}\right)$, $R > 1$. Then for a given $m \in \mathbb{N}_0$ there exists a constant $C(R, m)$ such that for every natural numbers $n \geq m$ the inequality*

$$\left|R_n\left(z; f^{(m)}\right)\right| \leq \frac{C(R,m)(m+1)!nn!}{(n+1-m)!R^n(R-1)} E_n\left(f; \overline{D_R}\right), \quad |z| \leq 1$$

holds.

In particular, for $m = 1, 2$ we have:

Corollary 1.1. *Let* $f \in A\left(\overline{D_R}\right)$, $R > 1$. *Then there exists some constants* $C_1(R)$ *and* $C_2(R)$ *such that for every natural numbers* $n \geq 2$ *the inequalities*

$$\begin{aligned} \left|R_n\left(z; f'\right)\right| &\leq \frac{C_1(R)\, n}{R^n (R-1)} E_n\left(f; \overline{D_R}\right), \quad |z| \leq 1, \\ \left|R_n\left(z; f''\right)\right| &\leq \frac{C_2(R)\, n^2}{R^n (R-1)} E_n\left(f; \overline{D_R}\right), \quad |z| \leq 1 \end{aligned}$$

hold.

We also can estimate simultaneous maximal convergence of Taylor's partial sums in the Hardy space $H^p(D_R)$, $1 \leq p < \infty$, of analytic functions f in D_R with the finite norm:

$$\|f\|_{H^p(D_R)} := \sup_{0<r<R} \left\{ \int_{|z|=r} |f(z)|^p \, |dz| \right\}^{1/p} \leq c(f) < \infty.$$

Let $E_n^{(p)}(f; D_R) := \inf_{p_n \in W_n} \|f - p_n\|_{H^p(D_R)}$ be the best approximation number for f in $H^p(D_R)$, $1 \leq p < \infty$.

Theorem 1.2. *Let* $f \in H^p(D_R)$, $1 \leq p < \infty$, $R > 1$. *Then for a given* $m \in \mathbb{N}_0$ *there exists a constant* $C(R, m, p)$ *such that for every natural numbers* $n \geq m$ *the inequality*

$$\left|R_n\left(z; f^{(m)}\right)\right| \leq \frac{C(R, m, p)\,(m+1)!\,n\,n!}{(n+1-m)!\,R^n (R-1)} E_n^{(p)}(f; D_R), \quad |z| \leq 1$$

holds.

In the case of $m = 1, 2$ we have

Corollary 1.2. *Let* $f \in H^p(D_R)$, $1 \leq p < \infty$, $R > 1$. *Then there exists some constants* $C_3(R, p)$ *and* $C_4(R, p)$ *such that for every natural numbers* $n \geq 2$ *the inequalities*

$$\begin{aligned} \left|R_n\left(z; f'\right)\right| &\leq \frac{C_3(R, p)\, n}{R^n (R-1)} E_n^{(p)}(f; D_R), \\ \left|R_n\left(z; f''\right)\right| &\leq \frac{C_4(R, p)\, n^2}{R^n (R-1)} E_n^{(p)}(f; D_R) \end{aligned}$$

hold.

2. Proof of New Results

Let n is a natural number and $m \in \mathbb{N}_0$ with $m \leq n$ and let $P(n, m) = n!/(n-m)!$.

The following Lemma holds:

Lemma 2.1. *If $|u| < 1$, then for any natural number n and $m \in \mathbb{N}_0$ with $m \leq n$ the equality*

$$\begin{aligned}&\sum_{k=n+1}^{\infty} \frac{k!}{(k-m)!} u^{k-m}\\ =\ & \sum_{j=1}^{m} \binom{m-1}{j-1} P(n, m-j)\, u^{n+j-m} \left(\frac{(j-1)!n}{(1-u)^{j}} + \frac{j!}{(1-u)^{j+1}} \right)\end{aligned}$$

holds.

Proof. It is clear that

$$\begin{aligned}&\sum_{k=n+1}^{\infty} \frac{k!}{(k-m)!} u^{k-m}\\ =\ & \sum_{k=n+1}^{\infty} \left(u^{k}\right)^{(m)} = \left(\sum_{k=n+1}^{\infty} u^{k} \right)^{(m)} = \left(\frac{u^{n+1}}{1-u} \right)^{(m)}.\end{aligned}$$

Hence for the $1st$, $2nd$ and $3rd$ order derivatives of $\frac{u^{n+1}}{1-u}$ we have, respectively

$$\begin{aligned}&\left(\frac{u^{n+1}}{1-u} \right)'\\ =\ & \frac{(n+1)\, u^{n}(1-u)}{(1-u)^{2}} + \frac{u^{n+1}}{(1-u)^{2}} = u^{n} \left(\frac{n+1}{1-u} + \frac{u}{(1-u)^{2}} \right)\\ =\ & u^{n} \left(\frac{n}{1-u} + \frac{1}{1-u} + \frac{u}{(1-u)^{2}} \right) = u^{n} \left(\frac{n}{1-u} + \frac{1}{(1-u)^{2}} \right),\end{aligned}$$

$$\begin{aligned}&\left(\frac{u^{n+1}}{1-u} \right)''\\ =\ & \left(u^{n} \left(\frac{n}{1-u} + \frac{1}{(1-u)^{2}} \right) \right)'\\ =\ & nu^{n-1} \left(\frac{n}{1-u} + \frac{1}{(1-u)^{2}} \right) + u^{n} \left(\frac{n}{(1-u)^{2}} + \frac{2}{(1-u)^{3}} \right)\end{aligned}$$

and

$$\begin{aligned}&\left(\frac{u^{n+1}}{1-u} \right)'''\\ =\ & n(n-1)\, u^{n-2} \left(\frac{n}{1-u} + \frac{1}{(1-u)^{2}} \right)\\ & +2nu^{n-1} \left(\frac{n}{(1-u)^{2}} + \frac{2}{(1-u)^{3}} \right) + u^{n} \left(\frac{n}{(1-u)^{3}} + \frac{6}{(1-u)^{4}} \right)\\ =\ & n(n-1)\, u^{n-2} \left(\frac{n}{1-u} + \frac{1}{(1-u)^{2}} \right) + u^{n} \left(\frac{2n}{(1-u)^{3}} + \frac{6}{(1-u)^{4}} \right).\end{aligned}$$

Now applying simple computations and generalizing by induction these formulas we have that for any $m \in \mathbb{N}_0$

$$\sum_{k=n+1}^{\infty} \frac{k!}{(k-m)!} u^{k-m}$$
$$= \sum_{j=1}^{m} \binom{m-1}{j-1} P(n, m-j) u^{n+j-m} \left(\frac{(j-1)!n}{(1-u)^j} + \frac{j!}{(1-u)^{j+1}} \right).$$

□

Proof of Theorem 1.1. Taking into account the expressions (1.2) of Taylor coefficients a_k in (1.3) we have

$$R_n\left(z; f^{(m)}\right) = \frac{1}{2\pi i} \int_{|t|=R} f(t) \left[\sum_{k=n+1}^{\infty} \frac{k! z^{k-m}}{(k-m)! t^{k+1}} \right] dt, \qquad |z| \leq 1.$$

If $p_n(t)$ is the algebraic polynomial of best approximation for $f \in A\left(\overline{D_R}\right)$ in the uniform norm, then by ortogonality of the system $\left(z^k\right)$ on the circle $|t| = R$, this relation can be written as

$$R_n\left(z; f^{(m)}\right) = \frac{1}{2\pi} \int_{|t|=R} \left[f(t) - p_n(t)\right] \sum_{k=n+1}^{\infty} \frac{k! z^{k-m}}{(k-m)! t^{k+1}} dt, \quad |z| \leq 1$$

and hence

$$\left| R_n\left(z; f^{(m)}\right) \right| \leq \frac{1}{2\pi} \int_{|t|=R} |f(t) - p_n(t)| \left| \sum_{k=n+1}^{\infty} \frac{k! z^{k-m}}{(k-m)! t^{k+1}} \right| |dt| \tag{2.1}$$
$$\leq E_n\left(f; \overline{D_R}\right) \frac{1}{2\pi} \int_{|t|=R} \frac{1}{|t|^{m+1}} \left| \sum_{k=n+1}^{\infty} \frac{k! z^{k-m}}{(k-m)! t^{k-m}} \right| |dt|.$$

Since $|t| = R$ and $|z| \leq 1$, using the substitution $u = z/t$ in *Lemma 2.1* we have

$$\sum_{k=n+1}^{\infty} \frac{k! z^{k-m}}{(k-m)\, t^{k-m}} = \sum_{k=n+1}^{\infty} \frac{k!}{(k-m)!} u^{k-m}$$

$$= \sum_{j=1}^{m} \binom{m-1}{j-1} P(n, m-j) u^{n+j-m} \left(\frac{(j-1)!n}{(1-u)^j} + \frac{j!}{(1-u)^{j+1}} \right)$$

and then the integral on the right site of the inequality (2.1) can be estimated as

$$\frac{1}{2\pi} \int_{|t|=R} \frac{1}{|t|^{m+1}} \sum_{j=1}^{m} \binom{m-1}{j-1} P(n, m-j) \left| \frac{z}{t} \right|^{n+j-m} \tag{2.2}$$
$$\cdot \left[\frac{(j-1)!n\,|t|^j}{|t-z|^j} + \frac{j!\,|t|^{j+1}}{|t-z|^{j+1}} \right] |dt|$$

$$= \sum_{j=1}^{m} \binom{m-1}{j-1} P(n,m-j) \left[\frac{1}{2\pi} \int_{|t|=R} \frac{|z|^{n+j-m}}{|t|^{n+1}} \frac{(j-1)!n}{|t-z|^{j}} |dt| + \frac{1}{2\pi} \int_{|t|=R} \frac{|z|^{n+j-m}}{|t|^{n}} \frac{j!}{|t-z|^{j+1}} |dt| \right]$$

$$\leq \sum_{j=1}^{m} \binom{m-1}{j-1} P(n,m-j) \cdot \left[\frac{n.(j-1)!}{R^{n+1}2\pi} \int_{|t|=R} \frac{|dt|}{|t-z|^{j}} + \frac{j!}{R^{n}2\pi} \int_{|t|=R} \frac{|dt|}{|t-z|^{j+1}} \right].$$

Since $|t| = R$ and $|z| \leq 1$ and hence $|t-z| \geq R-1$, then from (2.2) and (2.1) we get

$$\left| R_n\left(z; f^{(m)}\right) \right| \leq E_n\left(f; \overline{D_R}\right) \sum_{j=1}^{m} \frac{(m-1)!}{(m-j)!(j-1)!} \frac{n!}{(n-m+j)!} \cdot \left[\frac{n(j-1)!}{R^{n}(R-1)^{j}} + \frac{j!}{R^{n-1}(R-1)^{j+1}} \right]$$

$$\leq E_n\left(f; \overline{D_R}\right) \left[\sum_{j=1}^{m} \frac{(m-1)!}{(m-j)!} \frac{nn!}{(n-m+j)!R^{n}(R-1)^{j}} + \sum_{j=1}^{m} \frac{(m-1)!}{(m-j)!} \frac{jn!}{(n-m+j)!R^{n-1}(R-1)^{j+1}} \right]$$

$$\leq E_n\left(f; \overline{D_R}\right) \left[\sum_{j=1}^{m} \frac{(m-1)!nn!}{(n-m+1)!R^{n}(R-1)^{j}} + \sum_{j=1}^{m} \frac{(m-1)!jn!}{(n-m+1)!R^{n-1}(R-1)^{j+1}} \right]$$

$$\leq E_n\left(f; \overline{D_R}\right) \left[\frac{(m-1)!nn!}{(n-m+1)!R^{n}} \sum_{j=1}^{m} \frac{1}{(R-1)^{j}} + \frac{(m-1)!n!}{(n-m+1)!R^{n}} \sum_{j=1}^{m} \frac{j}{(R-1)^{j}} \right]$$

$$\leq E_n\left(f; \overline{D_R}\right) \frac{2nm!n!}{(n-m+1)!R^{n}} \sum_{j=1}^{m} \frac{1}{(R-1)^{j}}$$

and then after simple computations we have the estimation

$$\left|R_n\left(z;f^{(m)}\right)\right| \leq E_n\left(f;\overline{D_R}\right)\frac{2\,(m+1)!nn!}{(n+1-m)!R^n}\begin{cases}\frac{1}{(R-1)^m}, & 1<R<2\\ \frac{1}{R-1}, & 2<R,\end{cases}$$

which implies the inequality

$$\left|R_n\left(z;f^{(m)}\right)\right| \leq \frac{C(R,m)\,(m+1)!nn!}{(n+1-m)!R^n}E_n\left(f;\overline{D_R}\right)\ ,\ \ |z|\leq 1.$$

□

Proof of Theorem 1.2. Since in the case of $f\in H^p\left(D_R\right),1\leq p<\infty,\ R>1$, the coefficients $a_k,\ k=0,1,2,...$, can be also defined by (1.2), as in the proof of *Theorem 1.1* we have

$$R_n\left(z;f^{(m)}\right)=\frac{1}{2\pi i}\int_{|t|=R}f\left(t\right)\left[\sum_{k=n+1}^{\infty}\frac{k!z^{k-m}}{(k-m)!t^{k+1}}\right]dt,\quad |z|\leq 1.$$

If $p=1$ and $p_n(t)$ is the best approximation algebraic polynomial of degree n for f in the norm of $H^1\left(D_R\right)$, then

$$\begin{aligned}&R_n\left(z;f^{(m)}\right)\\ =\ &\frac{1}{2\pi i}\int_{|t|=R}\left(f\left(t\right)-p_n\left(t\right)\right)\left[\sum_{k=n+1}^{\infty}\frac{k!z^{k-m}}{(k-m)!t^{k+1}}\right]dt,\quad |z|\leq 1.\end{aligned}\tag{2.3}$$

Setting $u=z/t$ in *Lemma 2.1* we have the following estimation for the sum standing on the right side of (2.3)

$$\begin{aligned}&\left|\sum_{k=n+1}^{\infty}\frac{k!z^{k-m}}{(k-m)!t^{k+1}}\right|\\ \leq\ &\frac{1}{|t|^{m+1}}\sum_{j=1}^{m}\binom{m-1}{j-1}P\left(n,m-j\right)\left|\frac{z}{t}\right|^{n+j-m}\\ &\cdot\left[\frac{(j-1)!n\left|t\right|^j}{\left|t-z\right|^j}+\frac{j!\left|t\right|^{j+1}}{\left|t-z\right|^{j+1}}\right]\end{aligned}$$

and then for $|z| \le 1$ and $|t| = R$ we have

$$\begin{aligned}
&\max_{|t|=R}\left|\sum_{k=n+1}^{\infty}\frac{k!z^{k-m}}{(k-m)!t^{k+1}}\right| \\
\le\ & \frac{1}{R^{m+1}}\sum_{j=1}^{m}\frac{(m-1)!}{(m-j)!\,(j-1)!}\frac{n!\,(j-1)!}{(n-m+j)!}\frac{n}{R^{n-m}\,(R-1)^j} \\
&+\frac{1}{R^{m+1}}\sum_{j=1}^{m}\frac{(m-1)!}{(m-j)!\,(j-1)!}\frac{n!}{(n-m+j)!}\frac{j!}{R^{n-m-1}\,(R-1)^{j+1}} \\
=\ & \frac{1}{R^{n+1}}\sum_{j=1}^{m}\frac{(m-1)!}{(m-j)!}\frac{n!}{(n-m+j)!}\frac{n}{(R-1)^j} \\
&+\frac{1}{R^{n}}\sum_{j=1}^{m}\frac{(m-1)!}{(m-j)!}\frac{n!}{(n-m+j)!}\frac{j}{(R-1)^{j+1}}.
\end{aligned}$$

Hence from (2.3) we have

$$\begin{aligned}
\left|R_n\left(z;f^{(m)}\right)\right| \le\ & \frac{E_n^{(1)}(f;D_R)}{2\pi R^{n+1}}\sum_{j=1}^{m}\frac{(m-1)!}{(m-j)!}\frac{n!}{(n-m+j)!}\frac{n}{(R-1)^j} \\
&+\frac{E_n^{(1)}(f;D_R)}{2\pi R^{n}}\sum_{j=1}^{m}\frac{(m-1)!}{(m-j)!}\frac{n!}{(n-m+j)!}\frac{j}{(R-1)^{j+1}}. \\
\le\ & \frac{m!n!nE_n^{(1)}(f;D_R)}{2\pi\,(n-m+1)!R^n}\begin{cases}\frac{1}{(R-1)^{m+1}}, & 1<R<2\\ \frac{1}{R-1}, & 2<R,\end{cases}
\end{aligned}$$

which implies the inequality

$$\left|R_n\left(z;f^{(m)}\right)\right| \le \frac{C(R,m)\,m!n!n}{(n+1-m)!R^n\,(R-1)}E_n^{(1)}(f;D_R), \quad |z|\le 1.$$

Now let $1<p<\infty$ and $p_n(t)$ is the best approximation algebraic polynomial of degree n for f in the norm of $H^p(D_R)$. As in the case of $p=1$ we have

$$R_n\left(z;f^{(m)}\right) = \frac{1}{2\pi i}\int_{|t|=R}(f(t)-p_n(t))\sum_{k=n+1}^{\infty}\frac{k!z^{k-m}}{(k-m)!t^{k+1}}dt, \quad |z|\le 1.$$

If $u = z/t$, then the equality proved in *Lemma 2.1* can be written as

$$\begin{aligned}
&\sum_{k=n+1}^{\infty}\frac{k!z^{k-m}}{(k-m)!t^{k+1}} \\
=\ & \frac{1}{t^{m+1}}\sum_{k=n+1}^{\infty}\frac{k!z^{k-m}}{(k-m)!t^{k-m}} \\
=\ & \frac{1}{t^{m+1}}\sum_{j=1}^{m}\binom{m-1}{j-1}P(n,m-j)\,u^{n+j-m}\left[\frac{(j-1)!n}{(1-u)^j}+\frac{j!}{(1-u)^{j+1}}\right]
\end{aligned}$$

and hence we have

$$\begin{aligned}
&\left|R_n\left(z;f^{(m)}\right)\right| \\
\leq\ & \frac{1}{2\pi}\int_{|t|=R}\frac{|f(t)-p_n(t)|}{|t|^{m+1}}\sum_{j=1}^{m}\begin{pmatrix} m-1 \\ j-1 \end{pmatrix}P(n,m-j)\left|\frac{z}{t}\right|^{n+j-m} \\
&\cdot\left[\frac{(j-1)!n\,|t|^j}{|t-z|^j}+\frac{j!\,|t|^{j+1}}{|t-z|^{j+1}}\right]|dt| \\
\leq\ & \frac{1}{2\pi}\int_{|t|=R}\frac{|f(t)-p_n(t)|}{|t|^{m+1}}\sum_{j=1}^{m}\frac{(m-1)!}{(m-j)!\,(j-1)!} \\
&\cdot\frac{n!}{(n-m+j)!}\frac{|z|^{n+j-m}}{|t|^{n+j-m}}\left[\frac{(j-1)!n\,|t|^j}{|t-z|^j}+\frac{j!\,|t|^{j+1}}{|t-z|^{j+1}}\right]|dt| \\
\leq\ & \frac{1}{2\pi}\sum_{j=1}^{m}\frac{(m-1)!nn!}{(m-j)!\,(n-m+j)!R^{n+1}}\int_{|t|=R}\frac{|f(t)-p_n(t)|\,|dt|}{|t-z|^j} \\
&+\frac{1}{2\pi}\sum_{j=1}^{m}\frac{(m-1)!n!j}{(m-j)!\,(n-m+j)!R^{n}}\int_{|t|=R}\frac{|f(t)-p_n(t)|\,|dt|}{|t-z|^{j+1}}
\end{aligned}$$

and by Holder's inequality

$$\begin{aligned}
&\left|R_n\left(z;f^{(m)}\right)\right| \\
\leq\ & \frac{1}{2\pi}\sum_{j=1}^{m}\frac{(m-1)!}{(m-j)!}\frac{n!}{(n-m+j)!}\frac{n}{R^{n+1}}E_n^{(p)}(f;D_R)\left[\int_{|t|=R}\frac{|dt|}{|t-z|^{jq}}\right]^{\frac{1}{q}} \\
&+\frac{1}{2\pi}\sum_{j=1}^{m}\frac{(m-1)!}{(m-j)!}\frac{n!}{(n-m+j)!}\frac{j}{R^{n}}E_n^{(p)}(f;D_R)\left[\int_{|t|=R}\frac{|dt|}{|t-z|^{(j+1)q}}\right]^{\frac{1}{q}}.
\end{aligned}$$

Evaluating the integrals standing on the right hand side appropriately, we obtain that

$$\left|R_n\left(z;f^{(m)}\right)\right| \leq E_n^{(p)}(f;D_R)\frac{mm!nn!}{(n+1-m)!R^{n+1}}\begin{cases} \frac{R^{1/q}}{(R-1)^{m+1}}, & 1<R<2 \\ \frac{R^{1/q}}{R-1}, & 2<R \end{cases},$$

which implies the inequality

$$\left|R_n\left(z;f^{(m)}\right)\right| \leq \frac{C(R,m,p)(m+1)!nn!}{(n+1-m)!R^n(R-1)}E_n^{(p)}(f;D_R), \quad |z|\leq 1.$$

□

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