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ON SOLVABILITY CONDITIONS OF BOUNDARY VALUE PROBLEMS FOR A CLASS OF OPERATOR-DIFFERENTIAL EQUATIONS OF THE THIRD ORDER IN SOBOLEV TYPE SPACES

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Abstract. In the work, we consider issues of well-posed and unique solvability of all possible boundary value problems for one class of operator-differential equations of the third order with a multiple characteristic in Sobolev type spaces of the third order on the semi-axis. In addition, we find exact values of the norms of intermediate derivatives operators in Sobolev-type spaces and establish their connection with solvability conditions. Note that the solvability conditions obtained are expressed only by the operator coefficients of the equation, which makes it easy to check them in the application.

1. Introduction

Beginning in the 1970s, interest in the study of solvability of boundary value problems for operator-differential equations increased (see, for example, [11], [12], [13], [18], [24]). In the last 10 years in this direction, we note the works [7], [8], [20], [21], in which operator-differential equations of the second order are widely studied. There are also plenty of works dedicated to high-order operator-differential equations. Here we should note the work of A.A. Shkalikov [25], in which he carried out a detailed and deep analysis of these issues against the background of the theory of self-adjoint and normal operators.

In [3], they study a class of operator-differential equations of the third order, which includes equations with real and real multiple characteristics. Note that such equations are widely used in modeling problems in mechanics and engineering, in particular, in filtration problems [9], in problems of the dynamics of arches and rings [23], etc.

In [4], [5], [6], [10], [15], they carry out a systematic study of fourth-order operator-differential equations with a multiple characteristic. At the same time, the main attention is paid to the issues of well-posed and unique solvability of operator-differential equations and of spectral problems of polynomial operator pencils in Sobolev type spaces, which are the symbols of these equations. Considerable attention in spectral problems is given to the study of the completeness

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of part of the eigenvectors and associated vectors. Equations of this kind are encountered in stability problems for plates made of plastic material [26].

Despite the considerable number of journal publications (see, for example, [1], [2], [19]) dedicated to various aspects of the theory of operator-differential equations of an odd order, there are comparatively few works that study widely operator-differential equations of the third order with a multiple characteristic (see, for example, [3]).

In this paper, we consider on the semi-axis $\mathbb{R}_+ = [0, +\infty)$ three types of boundary value problems for one class of operator-differential equations of the third order with a multiple characteristic and establish exact sufficient solvability conditions for them in terms of the operator coefficients of the equation under study. As noted above, one of these boundary value problems was studied in [3].

2. Problem statement

Let H be a separable Hilbert space with inner product (x, y), $x, y \in H$, A be a self-adjoint positive-definite operator in H ($A = A^* \geq cE$, $c > 0$, E is a unit operator), and H_γ ($\gamma \geq 0$) is the scale of Hilbert spaces generated by the operator A, i.e.

$$H_\gamma = D(A^\gamma), \quad (x, y)_\gamma = (A^\gamma x, A^\gamma y), \quad x, y \in D(A^\gamma).$$

The domain of definition of the operator A^γ is denoted by $D(A^\gamma)$. When $\gamma = 0$ we assume that $H_0 = H$, $(x, y)_0 = (x, y)$, $x, y \in H$. Here the operator A^γ is determined from the spectral expansion of the operator A, i.e.

$$A^\gamma = \int_c^{+\infty} \sigma^\gamma dE_\sigma, \quad \gamma \geq 0,$$

where E_σ is a decomposition of the unit of the operator A.

As is known, if $\omega < 0$ and $A = A^* \geq cE$, $c > 0$, then $e^{\omega t A}$ is a strongly continuous semigroup of bounded operators generated by the operator ωA on $\mathbb{R}_+$ (see [16]), i.e.

$$e^{\omega t A} = \int_c^{+\infty} e^{\sigma \omega t} dE_\sigma.$$

Denote by $L_2(\mathbb{R}_+; H)$ the Hilbert space of all vector functions $u(t)$ defined on $\mathbb{R}_+$ with values in H and the norm

$$\|u\|_{L_2(\mathbb{R}_+;H)} = \left(\int_0^{+\infty} \|u(t)\|_H^2 \, dt \right)^{1/2}$$

(see [16]).

We define the following set:

$$W_2^3(\mathbb{R}_+; H) = \left\{ u(t) : \ \frac{d^3 u(t)}{dt^3} \in L_2(\mathbb{R}_+; H), \ A^3 u(t) \in L_2(\mathbb{R}_+; H) \right\}$$

(for more details see [17, Ch. 1]). Here and below, derivatives are understood in the sense of the theory of distributions (see [17]). $W_2^3(\mathbb{R}_+; H)$ becomes a Hilbert space with respect to the norm

$$\|u\|_{W_2^3(\mathbb{R}_+;H)} = \left(\int_0^{+\infty} \left(\left\| \frac{d^3 u(t)}{dt^3} \right\|_H^2 + \left\| A^3 u(t) \right\|_H^2 \right) dt \right)^{1/2}.$$

Consider in space $W_2^3(\mathbb{R}_+;H)$ the following subspaces

$$\overset{o}{W}{}_2^3(\mathbb{R}_+;H;l,m) = \left\{ u(t): \ u(t) \in W_2^3(\mathbb{R}_+;H), \ \frac{d^l u(0)}{dt^l} = \frac{d^m u(0)}{dt^m} = 0 \right\},$$

where l are m are fixed integers such that $l < m$, and they can take one the following values: $l = 0, 1$; $m = 1, 2$.

We note one property of the space $W_2^3(\mathbb{R}_+;H)$, which is important in obtaining the main results of this paper: if $u(t) \in W_2^3(\mathbb{R}_+;H)$, then

$$\left\| A^{3-j} u^{(j)} \right\|_{L_2(\mathbb{R}_+;H)} \le c_j(\mathbb{R}_+) \left\| u \right\|_{W_2^3(\mathbb{R}_+;H)}, \quad j = \overline{0,3}$$

(theorem on intermediate derivatives (see [17, Ch. 1])). Here $c_j(\mathbb{R}_+)$, $j = \overline{0,3}$, are constants not depending on the function $u(t)$.

Let us proceed to the formulation of the problems under study.

In space H, consider the following operator-differential equation of the third order, the main part of which has a multiple characteristic:

$$\left(-\frac{d}{dt} + A\right)\left(\frac{d}{dt} + A\right)^2 u(t) + A_1 \frac{d^2 u(t)}{dt^2} + A_2 \frac{du(t)}{dt} = f(t), \quad t \in \mathbb{R}_+, \tag{2.1}$$

where $f(t) \in L_2(\mathbb{R}_+;H)$, A is a self-adjoint positive-definite operator, and A_1, A_2 are linear, generally speaking, unbounded operators in H. Assuming that $u(t) \in W_2^3(\mathbb{R}_+;H)$, we add to Eq. (2.1) the boundary conditions at zero of the form

$$\frac{d^l u(0)}{dt^l} = \frac{d^m u(0)}{dt^m} = 0, \tag{2.2}$$

where l and m are fixed integers such that $l < m$, and they can take one of the following values: $l = 0, 1$; $m = 1, 2$.

Definition 2.1. If the function $u(t) \in W_2^3(\mathbb{R}_+;H)$ satisfies Eq. (2.1) almost everywhere in $\mathbb{R}_+$, and the boundary conditions (2.2) are satisfied in the sense of

$$\lim_{t \to 0} \left\| A^{5/2-l} \frac{d^l u(t)}{dt^l} \right\|_H = 0, \quad \lim_{t \to 0} \left\| A^{5/2-m} \frac{d^m u(t)}{dt^m} \right\|_H = 0,$$

then $u(t)$ will be called a *regular solution* of a boundary value problem of the form (2.1) and (2.2).

In the present paper, we derive conditions for the existence and uniqueness of a regular solution of boundary value problems of the form (2.1) and (2.2). These conditions are expressed only in terms of the operator coefficients of Eq. (2.1). We also indicate the connection of these conditions with finding the exact values of the norms of intermediate derivatives operators in subspaces $\overset{o}{W}{}_2^3(\mathbb{R}_+;H;l,m)$ with respect to the norm of the operator generated by the main part of Eq. (2.1).

3. On boundedness of the operator $P^{(l,m)}$

Denote, respectively, by $P_0^{(l,m)}$, $P_1^{(l,m)}$, and $P^{(l,m)}$ operators acting from space $\overset{o}{W}{}_2^3(\mathbb{R}_+;H;l,m)$ to space $L_2(\mathbb{R}_+;H)$ as follows:

$$P_0^{(l,m)} u(t) \equiv \left(-\frac{d}{dt} + A\right)\left(\frac{d}{dt} + A\right)^2 u(t), u(t) \in \overset{o}{W}{}_2^3(\mathbb{R}_+;H;l,m),$$

$$P_1^{(l,m)} u(t) \equiv A_1 \frac{d^2 u(t)}{dt^2} + A_2 \frac{du(t)}{dt},\ u(t) \in \overset{o}{W}{}_2^3(\mathbb{R}_+; H; l, m),$$

$$P^{(l,m)} u(t) \equiv P_0^{(l,m)} u(t) + P_1^{(l,m)} u(t), u(t) \in \overset{o}{W}{}_2^3(\mathbb{R}_+; H; l, m).$$

The following two lemmas hold.

Lemma 3.1. *Let A be a self-adjoint positive-definite operator in H. Then the operator $P_0^{(l,m)}$ acts boundedly from space $\overset{o}{W}{}_2^3(\mathbb{R}_+; H; l, m)$ to space $L_2(\mathbb{R}_+; H)$.*

Proof. Since at $u(t) \in \overset{o}{W}{}_2^3(\mathbb{R}_+; H; l, m)$

$$\left\| P_0^{(l,m)} u \right\|_{L_2(\mathbb{R}_+;H)} = \left\| \left(-\frac{d}{dt} + A \right) \left(\frac{d}{dt} + A \right)^2 u \right\|_{L_2(\mathbb{R}_+;H)} =$$

$$= \left\| -\frac{d^3 u}{dt^3} - A \frac{d^2 u}{dt^2} + A^2 \frac{du}{dt} + A^3 u \right\|_{L_2(\mathbb{R}_+;H)} \leq$$

$$\leq \left\| \frac{d^3 u}{dt^3} \right\|_{L_2(\mathbb{R}_+;H)} + \left\| A \frac{d^2 u}{dt^2} \right\|_{L_2(\mathbb{R}_+;H)} + \left\| A^2 \frac{du}{dt} \right\|_{L_2(\mathbb{R}_+;H)} + \left\| A^3 u \right\|_{L_2(\mathbb{R}_+;H)},$$

then, taking into account the theorem on intermediate derivatives [24, Ch. 1], we obtain

$$\left\| P_0^{(l,m)} u \right\|_{L_2(\mathbb{R}_+;H)} \leq const \left\| u \right\|_{W_2^3(\mathbb{R}_+;H)}.$$

The lemma has been proven. □

Lemma 3.2. *Let A be a self-adjoint positive-definite operator, and $A_j A^{-j}$, $j = 1, 2$, are bounded operators in H. Then the operator $P_1^{(l,m)}$ acts boundedly from space $\overset{o}{W}{}_2^3(\mathbb{R}_+; H; l, m)$ to space $L_2(\mathbb{R}_+; H)$.*

Proof. Taking into account that $u(t) \in \overset{o}{W}{}_2^3(\mathbb{R}_+; H; l, m)$, as well as the theorem on intermediate derivatives [17, Ch. 1], we have:

$$\left\| P_1^{(l,m)} u \right\|_{L_2(\mathbb{R}_+;H)} \leq \left\| A_1 A^{-1} \right\|_{H \to H} \left\| A \frac{d^2 u}{dt^2} \right\|_{L_2(\mathbb{R}_+;H)} +$$

$$+ \left\| A_2 A^{-2} \right\|_{H \to H} \left\| A^2 \frac{du}{dt^s} \right\|_{L_2(\mathbb{R}_+;H)} \leq const \left\| u \right\|_{W_2^3(\mathbb{R}_+;H)}.$$

The lemma has been proven. □

Lemmas 3.1 and 3.2 imply the following theorem.

Theorem 3.1. *Let the conditions of Lemma 3.2 be satisfied. Then the operator $P^{(l,m)}$ acts boundedly from space $\overset{o}{W}{}_2^3(\mathbb{R}_+; H; l, m)$ to space $L_2(\mathbb{R}_+; H)$.*

4. On isomorphism of the operator $P^{(l,m)}$

If $\xi \in H_{5/2}$, then $e^{-tA}\xi \in W_2^3(\mathbb{R}_+; H)$, but if $\eta \in H_{3/2}$, then $tAe^{-tA}\eta \in W_2^3(\mathbb{R}_+; H)$ (see, for example, [14]).

Obviously, the homogeneous equation $P_0^{(l,m)}u(t) = 0$ has only a trivial solution from $\overset{o}{W}{}_2^3(\mathbb{R}_+; H; l, m)$.

Theorem 4.1. *Let A be a self-adjoint positive-definite operator in H. Then in case $A_j = 0$, $j = 1, 2$, each of the boundary value problems of the form* (2.1) *and* (2.2) *has a unique regular solution for any $f(t) \in L_2(\mathbb{R}_+; H)$.*

The **proof** of the theorem follows from the fact that in the case $A_j = 0$, $j = 1, 2$, the regular solution of the boundary value problem of the form (2.1) and (2.2) is given by one of the following formulas

in case $l = 0$, $m = 1$:

$$u(t) = \int_0^{+\infty} G(t-s)\, f(s)\, ds - \frac{1}{4}(E + 2tA)\int_0^{+\infty} e^{-(t+s)A}\left(A^{-2}f(s)\right) ds,$$

in case $l = 0$, $m = 2$:

$$u(t) = \int_0^{+\infty} G(t-s)\, f(s)\, ds - \frac{1}{4}\int_0^{+\infty} e^{-(t+s)A}\left(A^{-2}f(s)\right) ds,$$

in case $l = 1$, $m = 2$:

$$u(t) = \int_0^{+\infty} G(t-s)\, f(s)\, ds + \frac{1}{4}(3E + 2tA)\int_0^{+\infty} e^{-(t+s)A}\left(A^{-2}f(s)\right) ds,$$

where

$$G(t-s) = \frac{1}{4}\begin{cases} (E + 2(t-s)A)\, e^{-(t-s)A}A^{-2}, & t-s > 0, \\ e^{(t-s)A}A^{-2}, & t-s < 0. \end{cases}$$

From Theorem 4.1, along with Lemma 3.1, the next theorem follows.

Theorem 4.2. *The operator $P_0^{(l,m)}$ performs an isomorphism from the space $\overset{o}{W}{}_2^3(\mathbb{R}_+; H; l, m)$ onto the space $L_2(\mathbb{R}_+; H)$.*

Corollary 4.1. $\left\| P_0^{(l,m)}u \right\|_{L_2(\mathbb{R}_+;H)}$ *is the norm in space $\overset{o}{W}{}_2^3(\mathbb{R}_+; H; l, m)$, equivalent to the original norm $\|u\|_{W_2^3(\mathbb{R}_+;H)}$.*

Since, as is well known, the intermediate derivatives operators

$$A^{3-j}\frac{d^j u}{dt^j} : \overset{o}{W}{}_2^3(\mathbb{R}_+; H; l, m) \to L_2(\mathbb{R}_+; H), \quad j = 1, 2,$$

are continuous (see [17, Ch. 1]), then, by virtue of the corollary, the norms of these operators can be estimated in terms of the norm $\left\| P_0^{(l,m)}u \right\|_{L_2(\mathbb{R}_+;H)}$.

First, we give a conditional theorem on solvability for each of the boundary value problems of the form (2.1) and (2.2).

Theorem 4.3. *Let A be a self-adjoint positive-definite operator, $A_j A^{-j}$, $j = 1, 2$, are bounded operators in H, and there holds the inequality* $\sum\limits_{j=1}^{2} N_{3-j}^{(l,m)} \left\| A_j A^{-j} \right\|_{H \to H} < 1$, *where*

$$N_j^{(l,m)} = \sup_{0 \neq u \in \overset{o}{W}{}_2^3(\mathbb{R}_+;H;l,m)} \frac{\left\| A^{3-j} \frac{d^j u}{dt^j} \right\|_{L_2(\mathbb{R}_+;H)}}{\left\| P_0^{(l,m)} u \right\|_{L_2(\mathbb{R}_+;H)}}, \quad j = 1, 2.$$

Then each of the boundary value problems of the form (2.1) *and* (2.2) *has a unique regular solution for any $f(t) \in L_2(\mathbb{R}_+; H)$.*

Proof. Considering all three cases together, we represent the boundary value problem (2.1) and (2.2) as an operator equation

$$P_0^{(l,m)} u(t) + P_1^{(l,m)} u(t) = f(t),$$

where $f(t) \in L_2(\mathbb{R}_+; H)$, $u(t) \in \overset{o}{W}{}_2^3(\mathbb{R}_+; H; l, m)$. Since, by Theorem 4.2, the operator $P_0^{(l,m)}$ has a bounded inverse $P_0^{(l,m)^{-1}}$ acting from $L_2(\mathbb{R}_+; H)$ on $\overset{o}{W}{}_2^3(\mathbb{R}_+; H; l, m)$, after the replacement $u(t) = P_0^{(l,m)^{-1}} v(t)$ we obtain in the space $L_2(\mathbb{R}_+; H)$ the equation

$$\left(E + P_1^{(l,m)} P_0^{(l,m)^{-1}} \right) v(t) = f(t).$$

Under the conditions of the theorem, we have:

$$\left\| P_1^{(l,m)} P_0^{(l,m)^{-1}} v \right\|_{L_2(\mathbb{R}_+;H)} = \left\| P_1^{(l,m)} u \right\|_{L_2(\mathbb{R}_+;H)} \leq$$

$$\leq \left\| A_1 A^{-1} \right\|_{H \to H} \left\| A \frac{d^2 u}{dt^2} \right\|_{L_2(\mathbb{R}_+;H)} + \left\| A_2 A^{-2} \right\|_{H \to H} \left\| A^2 \frac{du}{dt} \right\|_{L_2(\mathbb{R}_+;H)} \leq$$

$$\leq \sum_{j=1}^{2} N_{3-j}^{(l,m)} \left\| A_j A^{-j} \right\|_{H \to H} \left\| P_0^{(l,m)} u \right\|_{L_2(\mathbb{R}_+;H)} =$$

$$= \sum_{j=1}^{2} N_{3-j}^{(l,m)} \left\| A_j A^{-j} \right\|_{H \to H} \|v\|_{L_2(\mathbb{R}_+;H)}.$$

Therefore, under the fulfillment of the condition $\sum\limits_{j=1}^{2} N_{3-j}^{(l,m)} \left\| A_j A^{-j} \right\|_{H \to H} < 1$, the operator $E + P_1^{(l,m)} P_0^{(l,m)^{-1}}$ is invertible and $u(t)$ can be found by the formula

$$u(t) = P_0^{(l,m)^{-1}} \left(E + P_1^{(l,m)} P_0^{(l,m)^{-1}} \right)^{-1} f(t),$$

at that

$$\|u\|_{W_2^3(\mathbb{R}_+;H)} \leq \left\| P_0^{(l,m)^{-1}} \right\|_{L_2(\mathbb{R}_+;H) \to W_2^3(\mathbb{R}_+;H)} \times$$

$$\times \left\| \left(E + P_1^{(l,m)} P_0^{(l,m)^{-1}} \right)^{-1} \right\|_{L_2(\mathbb{R}_+;H) \to L_2(\mathbb{R}_+;H)} \|f\|_{L_2(\mathbb{R}_+;H)} \leq$$

$$\leq const \, \|f\|_{L_2(\mathbb{R}_+;H)}.$$

The theorem has been proven. □

Corollary 4.2. *Under the conditions of Theorem 4.3, the operator $P^{(l,m)}$ performs an isomorphism from the space $\overset{o}{W}{}_2^3(\mathbb{R}_+; H; l, m)$ onto the space $L_2(\mathbb{R}_+; H)$.*

5. On estimation of the norms of intermediate derivatives operators and exact conditions for the solvability of boundary value problems

Theorem 4.3 naturally gives rise to the problem of exact values or estimates of the quantities $N_j^{(l,m)}$, $j = 1, 2$, which is extremely important for indicating a wider class of operator-differential equations of the form (2.1), for which each of the considered boundary value problems is solvable.

Note that in [3], in the case $l = 0$, $m = 1$, the following quantities were calculated for $N_j^{(0,1)}$, $j = 1, 2$:

$$N_1^{(0,1)} = \frac{2}{3\sqrt{3}}, N_2^{(0,1)} = \frac{1}{\sqrt{2}\left(\sqrt{5}+1\right)^{1/2}}.$$

Using from [3] the procedure for calculating the quantities $N_j^{(0,1)}$, $j = 1, 2$, while factorizing the same operator pencils

$$P_j(\lambda; \beta; A) = \left((i\lambda)^2 E + A^2\right)^3 - \beta (i\lambda)^{2j} A^{6-2j}, \quad j = 1, 2,$$

that depend on the parameter $\beta \in \left[0, \frac{27}{4}\right)$, where E is a unit operator, i is the imaginary unit, and carrying out similar reasoning, taking into account the specifics of each of the boundary value problems of the form (2.1) and (2.2), we obtain the following statement.

Theorem 5.1. *The following equalities hold*

$$N_1^{(0,2)} = N_2^{(0,2)} = \frac{2}{3\sqrt{3}}, N_1^{(1,2)} = \frac{1}{\sqrt{2}\left(\sqrt{5}+1\right)^{1/2}}, N_2^{(1,2)} = \frac{2}{3\sqrt{3}}.$$

Remark 5.1. For details on calculating the norms of intermediate derivatives operators, see [22].

The obtained results allow us to formulate, using the operator coefficients of Eq. (2.1), exact conditions for the existence and uniqueness of a regular solution to each of the boundary value problems of the form (2.1) and (2.2).

Theorem 5.2. *Let A be a self-adjoint positive-definite operator, $A_j A^{-j}$, $j = 1, 2$, are bounded operators in H, and the following inequality holds for $l = 0$, $m = 1$:*

$$\frac{1}{\sqrt{2}\left(\sqrt{5}+1\right)^{1/2}} \left\|A_1 A^{-1}\right\|_{H\to H} + \frac{2}{3\sqrt{3}} \left\|A_2 A^{-2}\right\|_{H\to H} < 1;$$

in case $l = 0$, $m = 2$, the following inequality holds:

$$\frac{2}{3\sqrt{3}} \left(\left\|A_1 A^{-1}\right\|_{H\to H} + \left\|A_2 A^{-2}\right\|_{H\to H}\right) < 1;$$

and in case $l=1$, $m=2$, the following inequality holds:

$$\frac{2}{3\sqrt{3}}\left\|A_1A^{-1}\right\|_{H\to H}+\frac{1}{\sqrt{2}\left(\sqrt{5}+1\right)^{1/2}}\left\|A_2A^{-2}\right\|_{H\to H}<1.$$

Then each of the boundary value problems of the form (2.1) *and* (2.2) *has a unique regular solution for any $f(t)\in L_2(\mathbb{R}_+;H)$.*

We note that the obtained conditions for the solvability of each of the boundary value problems of the form (2.1) and (2.2) are easily verified in applications due to their being expressed in terms of the operator coefficients of Eq. (2.1).

6. Appendix

Let us illustrate the results obtained by the example of initial-boundary value problems for partial differential equations. On the half-strip $\mathbb{R}_+\times[0,\pi]$, we consider the equation:

$$\left(-\frac{\partial}{\partial t}-\frac{\partial^2}{\partial x^2}\right)\left(\frac{\partial}{\partial t}-\frac{\partial^2}{\partial x^2}\right)^2u\left(t,x\right)+$$

$$+p\left(x\right)\frac{\partial^4u\left(t,x\right)}{\partial x^2\partial t^2}+q\left(x\right)\frac{\partial^5u\left(t,x\right)}{\partial x^4\partial t}=f\left(t,x\right) \tag{6.1}$$

under the following conditions:

$$\frac{\partial^l u\left(0,x\right)}{\partial t^l}=0,\frac{\partial^m u\left(0,x\right)}{\partial t^m}=0, \tag{6.2}$$

$$\frac{\partial^{2k}u\left(t,0\right)}{\partial x^{2k}}=\frac{\partial^{2k}u\left(t,\pi\right)}{\partial x^{2k}}=0, k=0,1,2, \tag{6.3}$$

where $p\left(x\right)$, $q\left(x\right)$ are functions bounded on the segment $[0,\pi]$, $f\left(t,x\right)\in L_2\left(\mathbb{R}_+;L_2\left[0;\pi\right]\right)$; l and m are fixed integers such that $l<m$, and they can take one of the following values: $l=0,1$; $m=1,2$. Note that problems of the form (6.1)-(6.3) are special cases of boundary value problems of the form (2.1) and (2.2). Indeed, here $H=L_2\left[0,\pi\right]$, $A_1=p\left(x\right)\frac{\partial^2}{\partial x^2}$, $A_2=q\left(x\right)\frac{\partial^4}{\partial x^4}$. The operator A is defined in $L_2\left[0,\pi\right]$ by the equality $Au=-\frac{d^2u}{dx^2}$ with the conditions $u\left(0\right)=u\left(\pi\right)=0$. Taking Theorem 5.2 into account, we obtain that each of the problems (6.1)-(6.3) has a unique solution from the space $W^{3,6}_{t,x,2}\left(\mathbb{R}_+;L_2\left[0;\pi\right]\right)$ if the the following conditions hold in case $l=0$, $m=1$:

$$\frac{1}{\sqrt{2}\left(\sqrt{5}+1\right)^{1/2}}\sup_{0\le x\le\pi}\left|p\left(x\right)\right|+\frac{2}{3\sqrt{3}}\sup_{0\le x\le\pi}\left|q\left(x\right)\right|<1;$$

in case $l=0$, $m=2$, if the following conditions hold:

$$\frac{2}{3\sqrt{3}}\left(\sup_{0\le x\le\pi}\left|p\left(x\right)\right|+\sup_{0\le x\le\pi}\left|q\left(x\right)\right|\right)<1;$$

and, in case $l=1$, $m=2$, if the following conditions hold:

$$\frac{2}{3\sqrt{3}}\sup_{0\le x\le\pi}\left|p\left(x\right)\right|+\frac{1}{\sqrt{2}\left(\sqrt{5}+1\right)^{1/2}}\sup_{0\le x\le\pi}\left|q\left(x\right)\right|<1.$$

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