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OPTIMAL CONTROL OF SEMILINEAR HIGHER-ORDER DIFFERENTIAL INCLUSIONS

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Abstract. The paper investigates an optimal control problem described by higher-order differential inclusions (DFIs). In terms of the Euler-Lagrange type adjoint DFIs and Hamiltonian, a sufficient optimality condition for higher-order DFIs is derived. At the same time, when constructing the Euler-Lagrange type adjoint DFI, without using traditional approaches to constructing an adjoint operator and a discrete-approximate method, the new method of adjoint DFI of Mahmudov for "higher-order problems" is used. It is shown also that the adjoint DFI for the first order DFI coincides with the classical Euler-Lagrange inclusion, and the optimality conditions coincide with the results of Rockafellar on the Mayer problem with first order DFIs. Thus, the obtained results are universal in the sense that sufficient optimality conditions can be formulated for a DFI of any order. At the end of the paper, problems with a high-order polyhedral DFIs and higher-order linear optimal control problems are considered, the optimality conditions of which are transformed into the Pontryagin maximum principle. Also, for high-order polyhedral optimization, from the point of view of abstract economics, non-negative adjoint variables can be interpreted as the price of a resource.

1. Introduction

The paper concerns with the unseparated Mayer problem of the higher-order ordinary differential inclusions:

$$\text{minimize } f(x(0), x(T)), \tag{1.1}$$

$$\text{(HD)} \qquad \frac{d^s x(t)}{dt^s} \in \sum_{i=1}^{s-1} A_i x^{(i)}(t) + F(x(t), t), \qquad \text{a.e. } t \in [0, T], \tag{1.2}$$

$$x^{(i)}(0) = x_i^0, i = 1, \ldots, s-1 \tag{1.3}$$

where $F(\cdot, t) : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is a set-valued mapping [4, 7, 12, 18, 26], $f : \mathbb{R}^{2n} \to \mathbb{R}^1$ is a proper function, s is an arbitrary fixed natural number, T is an arbitrary positive real number. We label this problem as (HD). It is required to find a feasible trajectory (arc)$x(t), t \in [0, T]$ of the ordinary differential inclusion (DFI) (1.2) that satisfies the initial condition (1.3), and minimizes the so-called

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Mayer functional $f(x(0), x(T))$. Let us refine the definition of the concept of a solution of problem for s-th order DFIs (1.2)-(1.3); suppose $AC^i([0,T], \mathbb{R}^n)$ is the space of i-times differentiable functions $x(\cdot) \in AC^{s-1}([0,T]) \cap W_{1,s}([0,T])$, where $W_{1,s}([0,T])$ is a Banach space of absolutely continuous functions, up to order $s-1, x^{(s)}(\cdot) \in L^1([0,T], \mathbb{R}^n)$. Then $x(\cdot)$ is a feasible solution of a problem (1.2)-(1.3) if it satisfies almost everywhere (a.e.) the s-th order DFI (1.2) and condition (1.3), where as usual, $L^1([0,T], \mathbb{R}^n)$ is the Banach space of integrable (in the Lebesgue sense) functions $v(\cdot) : [0,T] \to \mathbb{R}^n$ endowed with the norm $\|v(\cdot)\|_1 = \int_0^T |v(t)| dt$.

It should be noted that for many applications type problems (HD) are important; attract more and more attention in connection with the development of feedback control systems and dynamic systems described by higher-order differential equations with a discontinuous righthand side. For example, in the case of state constraints Clarke and Wolenski [11] give an excellent introduction to this problem with first order DFIs and describes several applications. Such problems often arise not only in problems of automatic control, mechanics, control science and economics, designing optimal or stabilizing feedback, but also in aerospace engineering, anti-vibration.

Obviously, for convexity of the problem (HD) we will assume that the set-valued mapping $F(\cdot, t)$ is convex and f is proper convex function. In fact, from further presentations it will be clear that the convexity of the problem (HD) is assumed for the sake of simplicity of the results obtained and, definition of LAM, through the Hamiltonian function makes it possible to generalize these results to the nonconvex case.

In a certain sense, the problem (HD) is an essential generalization of the Loewen and Rockefeller problem [20] with $s = 1$, where under a number of stringent conditions, the necessary optimality conditions are derived; Nevertheless, our sufficient optimality conditions contain more convenient forms of the transversality condition and the associated inclusions of Mahmudov of the Euler-Lagrange type. Our results allow us to simplify the proof of the maximum principle and obtain a new adjoint Mahmudov's [24] inclusion, which is a generalization of the Euler-Lagrange inclusion to the case of higher-order optimal control problems. In addition, it is interesting to note that the results of Theorem 3.1 of this paper and Theorem 4.3 of Loewen and Rockafellar [20], and their transversality conditions for $s = 1$, coincide. Moreover, the simplicity of the locally adjoint mapping (LAM) approach and the method of the "cone of tangent directions" instead of the normal cone simplifies the derivation and formulation of optimality conditions.

Notice that a significant part of the studies related to ordinary differential equations/inclusions are contained in the following works [2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 13, 14, 15, 17, 18, 19, 25, 36, 38]. In the paper of Mordukhovich [36] the Bolza problem for DFIs with general restrictions at the end is considered. First, a finite difference method is developed for the problem posed and a discrete approximate problem is constructed that ensures strong convergence of optimal solutions. Second, this direct method is used to obtain the necessary optimality conditions in the refined Euler-Lagrange form without the standard convexity assumptions, which is satisfied without any relaxation. In the works of Mardanov et al. [31, 32, 34] by introducing the new concept of a convexity and relative interior of a set a new discrete analogue of Pontryagin's maximum principle is obtained. The paper [27] deals with the optimal control problem described by second order differential inclusions. Based on the infimal convolution concept of convex functions, dual problems for differential inclusions are constructed and the results of duality are proved. In this case, it turns out that Euler-Lagrange type inclusions are "duality relations" for both primary and dual problems. The paper [23] considers a Bolza problem of optimal control theory with a varying time interval given by convex, nonconvex functional-differential inclusions. The main goal is to derive sufficient optimality conditions for neutral functional-differential inclusions, which contain time delays in both state and velocity variables. Both state and endpoint constraints are involved. Presence of constraint conditions implies jump conditions for conjugate variable which are typical for such problems.

Thibault et al. [3, 4] and Marco, Murillo [29] proved an existence theorem for an absolutely continuous solution for second/higher -order DFIs. Cubiotti and Yao [13] presented a new proof of a classical result by Bressan on the Cauchy problem for first-order DFIs with null initial condition. This approach allows to prove the result directly for k-th order DFIs, under weaker regularity assumptions on the involved set-valued map. In the paper [2] is proved a result which is the counterpart of the above for quasi monotone set-valued maps, based on the concept of single-directional property. Sufficient conditions for the controllability of second-order DFIs in Banach spaces with nonlocal conditions were established in [6]. They rely on the fixed-point theorem for condensing Martelli mappings. Strong duality, stating that the optimal values of the primal convex problem and its dual Lagrangian problem are equal (i.e. zero duality gap) and the dual problem reaches its maximum, is the cornerstone of convex optimization. In particular, it plays an important role in the numerical solution, as well as in the application of convex semidefinite optimization. Strong duality requires a specification known as constraint qualification (CQ). There are several CQs in the literature that are sufficient for strong duality. In [7] it is shown existence and uniqueness of the generated trajectories as well as their weak asymptotic convergence to a zero of the operators. The work [18] presents new necessary and sufficient CQs for strong duality in a convex semidefinite optimization. The work [28] discusses the problem of optimal control theory given by second-order sweeping processes with discrete and DFIs. The use of difference operators of the first and second -order connects the second order sweeping processes with the discrete-approximate problem. Based on this, optimality conditions for discrete-approximate inclusions and transversality conditions are obtained. The establishment of adjoint inclusions of the Euler-Lagrange type is based on the existence of equivalence relations for LAMs. The paper [25] is devoted to the duality of the Mayer problem for k-th order viable DFIs with endpoint constraints, where k is an arbitrary natural number. Using locally adjoint mappings in the form of Euler-Lagrange type inclusions and transversality conditions, sufficient optimality conditions are obtained. It is noteworthy that the EulerLagrange type inclusions for both primary and dual problems are "duality relations".

Semilinear differential/discrete inclusions are attracting more and more attention due to the development of control theory, for which the reader can refer to

[1, 11, 16, 35] and their references. The paper [1] discusses the controllability problem for damped second-order impulsive neutral functional-differential systems in Banach spaces. Sufficient conditions for controllability results are derived using the Sadovskii fixed-point theorem combined with a non-compact condition on the cosine family of operators. An earlier article [11] considers an optimal control problem in which the dynamic equation and cost function depend on the recent past of the trajectory. It is shown that for a given optimal solution there exists an associated arc of bounded variation that satisfies the associated Hamiltonian inclusion. From this result one can easily derive the well-known smooth versions of the Pontryagin maximum principle for hereditary problems. The paper [35] deals with optimal control problems for dynamical systems governed by constrained functional-DFIs of neutral type. Such control systems contain time delays not only in state variables but also in velocity variables, which make them essentially more complicated than delay-differential inclusions. The main goal is to derive necessary optimality conditions of both Euler-Lagrange and Hamiltonian types.

In principle, from the point of view of universality, more interesting results were obtained on the so-called "higher-order problems", in recent decades by Mahmudov [24, 25, 28], since they include useful forms of the Weierstrass-Pontryagin condition and related Euler-Lagrange type adjoint inclusions. Higher-order optimality conditions are in fact the main optimality tool for high-order problems commonly encountered in practice. It suffices to recall that due to the absence of higher-order optimality conditions, it was impossible to construct even an adjoint equation for the well-known time-optimal problem given by a simple secondorder differential equation $x''(t) = u, u \in U = [-1, 1]$ (see, for example [37]). Recall that, as a rule, this problem is reduced with the help of additional variables to the system consisting of two first-order equations, and further research is carried out in a two-dimensional phase space, the main reason for which is that the Weierstrass-Pontryagin principle is valid precisely only for first-order controlled systems.

Note that in order to obtain the optimality condition for the Mayer problem (HD) described by ordinary high-order delayed DFIs with constraints at the initial points, one can use the traditional discrete-approximate method [30, 33], where the problem (HD) is replaced by the following s-th order discrete-approximate problem:
minimize $f(x(0), x(T))$ subject to $\Delta^s x(t) \in \sum_{i=1}^{s-1} A_i \Delta^i x(t) + F(x(t), t), t = 0, h, \dots, T - sh; \Delta^i x(0) = x_i^0, i = 1, \dots, s-1$, where $\Delta^i (i = 0, \dots, s)$ is a i-th order difference operator, h is a discrete step on the t-axis and $x(t) \equiv x_h(t)$ is a grid function on a uniform grid on $[0, T]$. Thus, the approximation method makes it possible to construct an adjoint DFI and thereby establish necessary and sufficient conditions for a rather complicated discrete-approximation problem of s-th order. Then by passing to the limit in necessary and sufficient conditions of this problem as $h \to 0$, we can derive the optimality condition for the Mayer problem (HD) described by high-order DFIs. However, due to the complexity of the resulting higher-order difference derivative, the approach to constructing these optimality conditions in this way is omitted. Instead, when considering the high-order adjoint inclusion of Mahmudov [25], the formal construction of the

adjoint inclusion for the stated problem is used.
Thus, the novelty of our problem lies in the consideration of a combination of high-order DFIs for controlled systems. Note that the problem posed does not lose its novelty even in the case of first-order DFIs.

The obtained results can be organized in the following order:
In Section 2, for the convenience of the readers, all definitions, basic facts and concepts from the book of Mahmudov [21] are given.
In Section 3, a sufficient optimality condition for the problem (HD) with semilinear s-th order DFI is proved. The construction of conjugate DFIs is based on some auxiliary propositions and the Euler-Lagrange type of conjugate DFIs, obtained in [25]. In addition, the so-called transversality conditions associated with the endpoints of the "adjoint" trajectory are also formulated. It turns out that, the classical adjoint Euler-Lagrange inclusion follows from the existing optimality conditions (Remarks 3.2 and 3.3). Further, it is also possible to obtain optimality conditions in the form of a Hamiltonian function for the problem posed by Loewen and Rockafellar [20], which means that there is actually no "gap" between the necessary and sufficient conditions.
In Section 4 are given some applications of problem (HD). At the beginning of the section, some sufficient optimality conditions are obtained for a high-order semilinear optimal control problem (LHR) in the form of Ponrtyagin's maximum principle [37]. At the end of the section, we consider a polyhedral optimization problem and give some interpretation of it related to abstract economics.

2. Necessary Facts, Preliminary Information

All definitions and concepts that we come across can be found in Mahmudov's book [21]. Suppose that $G : \mathbb{R}^{ns} \rightrightarrows \mathbb{R}^n$ is a set-valued mapping from ns-dimensional Euclidean space $\mathbb{R}^{ns}$ into the family of subsets of $\mathbb{R}^n$, $\langle x, v\rangle$ be an inner product of x and v. G is convex closed if its graph gph $G = \{(z, v) : v \in G(z)\}, z = (x, x_1, \ldots, x_{s-1})$ is a convex closed set in $\mathbb{R}^{n(s+1)}$. Let's give important definitions of Hamiltonian function and argmaximum set for a set-valued mapping G, which we will often see in the paper:

$$H_G(z, v^*) = \sup_v \{\langle v, v^*\rangle : v \in G(z)\},$$
$$G_A(z; v^*) = \{v \in G(z) : \langle v, v^*\rangle = H_G(z, v^*)\}, v^* \in \mathbb{R}^n.$$

If $G(z) = \varnothing$ in order to ensure the concavity of the Hamiltonian function, we put $H_G(z, v^*) = -\infty$. For such a mapping G, the cone of tangent directions [21, p.61] at the point $(z, v) \in \operatorname{gph} G$ is defined as follows

$$K_G(z, v) \equiv K_{\text{gphG}}(z, v) = \text{ cone } [\operatorname{gph} G - (z, v)] = \{(\bar{z}, \bar{v}) :$$
$$\bar{z} = \gamma(z_1 - z), \bar{v} = \gamma(v_1 - v), \gamma > 0\}, \forall (x_1, v_1) \in \operatorname{gph} G.$$

The main objects of our study are the so-called LAMs, which are natural extensions of the adjoint operator to the classical derivatives of smooth mappings. A set-valued mapping $G^*(\cdot, z, v) : \mathbb{R}^n \rightrightarrows \mathbb{R}^{ns}$ defined by

$$G^*(v^*; (z, v)) = \{z^* : (z^*, -v^*) \in K_G^*(z, v)\}$$

is called the LAM to G at a point $(z,v) \in \operatorname{gph} G$, where $K_G^*(z,v)$ is the dual cone. Note that, using the definition of the cone of tangent vectors in the non-convex case, the LAM for nonconvex set-valued mappings is determined by the same formula [21, p.129].
In terms of Hamiltonian mappings, a "dual" mapping defined by

$$G^*\left(v^*;(z,v)\right) := \{z^* : H_G\left(z_1,v^*\right) - H_G\left(z,v^*\right) \\ \leq \langle z^*, z_1 - z\rangle, \forall z_1 \in \mathbb{R}^{ns}\}, v \in G\left(z;v^*\right)$$

is called the LAM to "nonconvex" mapping G at a point $(z,v) \in \operatorname{gph} G$. Obviously, in the convex case $H_G\left(\cdot,v^*\right)$ is concave and the latter definition of LAM coincide with the previous definition of LAM. Similarly to the definition of the Weierstrass excess function, for all fixed z^*, v^* let us denote $\mathfrak{R}\left(z,z_1,z^*,v^*\right) = H_G\left(z_1,v^*\right) - H_G\left(z,v^*\right) - \langle z^*, z_1 - z\rangle$. It follows that in particular, if $H_G\left(\cdot,v^*\right)$ is concave, then the function $\mathfrak{R}\left(\cdot,\cdot,z^*,v^*\right)$ is a nonpositive, i.e., $\mathfrak{R}\left(z,z_1,z^*,v^*\right) \leq 0$. The geometric meaning of this is that, for each z_1, the graph of the function $H_G\left(\cdot,v^*\right)$ lies below its tangent plane $H_G\left(z,v^*\right) + \langle z^*, z_1 - z\rangle$ at the point z, which can be interpreted as a local concavity property of the Hamilton function.

A function $f = f(x,y)$ is called a proper function if it does not assume the value $-\infty$ and is not identically equal to $+\infty$. Obviously, f is proper if and only if $\operatorname{dom} f \neq \varnothing$ and $f(x,y)$ is finite for $(x,y) \in \operatorname{dom} f = \{(x,y) : f(x,y) < +\infty\}$.

3. Sufficient Condition of Optimality for a Problem (HD) with s-th Order DFI

In this section, we formulate a sufficient optimality condition in the form of an adjoint EulerLagrange type inclusion for the problem under consideration. Due to the fact that the construction of a Euler-Lagrange-type inclusion, as well as the transversality conditions, are complicated by the accompanying discrete and discrete-approximation problems [21], we omit them and use the generalized adjoint DFI [25] together with the following auxiliary propositions.

Proposition 3.1. *Let $G(z) = \sum_{i=1}^{s-1} A_i x_i + F(x)$, where $A_i, i = 1,\ldots,s-1$ be $n \times n$ matrix and F be a convex set-valued mapping $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$. Then the following formulas are true:*
(i) $H_G\left(z,v^*\right) = \sum_{i=1}^{s-1} \langle A_i x_i, v^*\rangle + H_F\left(x,v^*\right)$,
(ii) $G_A\left(z;v^*\right) = F_A\left(x;v^*\right), v^* \in \mathbb{R}^n$.

Proof. In fact, by definition of argmaximum set we obtain

$$H_G\left(z,v^*\right) = \sup_v \left\{\langle v,v^*\rangle : v \in G(z)\right\} = \sup_{v_1} \left\{\left\langle \sum_{i=1}^{s-1} A_j x_j + v_1, v^*\right\rangle : v_1 \in F(x)\right\}$$
$$= \sum_{i=1}^{s-1} \langle A_i x_i, v^*\rangle + \sup_{v_1} \left\{\langle v_1, v^*\rangle : v_1 \in F(x)\right\} = \sum_{i=1}^{s-1} \langle A_i x_i, v^*\rangle + H_F\left(x,v^*\right).$$

Then considering this formula and definition of argmaximum set again, we have

$$G_A(z;v^*) = \left\{ v \in \sum_{i=1}^{s-1} A_i x_i + F(x) : \langle v, v^* \rangle = \sum_{i=1}^{s-1} \langle A_i x_i, v^* \rangle + H_F(x, v^*) \right\}$$
$$= \left\{ v \in \sum_{i=1}^{s-1} A_i x_i + F(x) : \left\langle v - \sum_{i=1}^{s-1} A_i x_i, v^* \right\rangle = H_F(x, v^*) \right\}$$
$$= \{ v_1 \in F(x) : \langle v_1, v^* \rangle = H_F(x, v^*) \} = F_A(x; v^*).$$

□

Proposition 3.2. *The LAM G^* and the LAM F^* for $v \in F_A(x;v^*)$ are related by the following formula*

$$G^*(v^*;(z,v)) = \left\{ \left(x^*, A_1^* v^*, \ldots, A_{s-1}^* v^* \right) : x^* \in F^*(v^*;(x,v)) \right\}.$$

Proof. By condition (i) of Proposition 3.1 we have

$$H_G(z, v^*) = \sum_{j=1}^{s-1} \langle A_j x_j, v^* \rangle + H_F(x, v^*).$$

Obviously, dom $\sum_{i=1}^{s-1} \langle A_i x_i, v^* \rangle = \bigcap_{i=1}^{s-1} \operatorname{dom} \langle A_i x_i, v^* \rangle = \mathbb{R}^{n(s-1)}$ and for $x \in$ ridom $H_F(\cdot, v^*)$ these functions are continuous. Then, according to well-known theorems [23] of convex analysis

$$\partial_z H_G(z, v^*) = \left\{ \left(x^*, A_1^* v^*, \ldots, A_{s-1}^* v^* \right) : x^* \in \partial_x H_F(x, v^*) \right\}. \tag{3.1}$$

On the other hand, by Theorem 2.1[23] $G^*(v^*;(z,v)) = \partial_z H_G(z, v^*)$, if $v \in G_A(z;v^*)$ and since $G^*(v^*;(z,v)) = \partial_z H_G(z,v^*)$, $F^*(v^*;(x,v)) = \partial_x H_F(x,v^*)$ we have from (3.1) the needed result. Here by condition (ii) of Proposition 3.1 $G_A(z;v^*) = F_A(x;v^*)$ and both the LAM G^* and F^* are nonempty. □

To construct adjoint DFIs, we return to Mahmudov's adjoint inclusion [30] and Proposition 3.2. Thus, the reminded generalized inclusion consists of the following:

$$\left((-1)^s x^{*(s)}(t) + \frac{d\varphi_{s-1}^*}{dt}(t), \varphi_{s-1}^*(t) + \frac{d\varphi_{s-2}^*}{dt}, \ldots, \varphi_2^*(t) + \frac{d\varphi_1^*}{dt}, \varphi_1^*(t) \right) \tag{3.2}$$
$$\in G^*\left(x^*(t); \left(\tilde{x}(t), \tilde{x}'(t), \ldots, \tilde{x}^{(s)}(t) \right), t \right),$$

where $G(\cdot, t) : \mathbb{R}^{ns} \rightrightarrows \mathbb{R}^n$ and auxiliary functions $\varphi_i^*(\cdot), i = 1, \ldots s-1$ arise due to the presence of $\tilde{x}'(\cdot), \ldots, \tilde{x}^{(s-1)}(\cdot)$. Then, taking into account the structure of LAM G^* in Proposition 3.2, in terms of variables (3.2), we obtain that

$$(-1)^s x^{*(s)}(t) + \frac{d\varphi_{s-1}^*}{dt}(t) \in F^*\left(x^*(t); \left(\tilde{x}(t), \tilde{x}^{(s)}(t) \right), t \right), \tag{3.3}$$

where according to (3.2)

$$\varphi_1^*(t) = A_{s-1}^* x^*(t), \varphi_2^*(t) + \frac{d\varphi_1^*}{dt} = A_{s-2}^* x^*(t), \ldots, \varphi_{s-1}^*(t) + \frac{d\varphi_{s-2}^*}{dt} = A_1^* x^*(t).$$

Here, successively differentiating $\varphi_i^*(t)$ and substituting it into the next relation to obtain $\varphi_{i+1}^*(t)$, we have the useful formula

$$\varphi_{s-1}^*(t) = A_1^* x^*(t) - A_2^* x^{*\prime}(t) + A_3^* x^{*\prime\prime} - \cdots + (-1)^s A_{s-1}^* x^{*(s-2)}(t),$$

whereas, differentiating again, we have

$$\frac{d\varphi_{s-1}^*(t)}{dt} = A_1^* x^{*\prime}(t) - A_2^* x^{*\prime\prime}(t) + A_3^* x^{*\prime\prime\prime} - \cdots + (-1)^s A_{s-1}^* x^{*(s-1)}(t). \quad (3.4)$$

Finally, substituting (3.4) into the adjoint DFI (3.3) and taking into account the specifics of systems for the Mayer problem (HD) with high-order DFI, we can formulate the following EulerLagrange type adjoint inclusion and the so-called transversality conditions:

(a) $(-1)^s x^{*(s)}(t) - \sum_{i=1}^{s-1} (-1)^i A_i^* x^{*(i)}(t) \in F^*\left(x^*(t); \left(\tilde{x}(t), \tilde{x}^{(s)}(t)\right), t\right)$, a.e. $t \in [0,T]$

(b) $\tilde{x}^{(s)}(t) - \sum_{i=1}^{s-1} A_i \tilde{x}^{(i)}(t) \in F_A\left(\tilde{x}(t); x^*(t), t\right)$, a.e. $t \in [0,T]$

(c) $\Big((-1)^{s-1} x^{*(s-1)}(0) - \sum_{i=1}^{s-1} (-1)^{i-1} A_i^* x^{*(i-1)}(0), (-1)^s x^{*(s-1)}(T)$
$- \sum_{i=1}^{s-1} (-1)^i A_i^* x^{*(i-1)}(T)\Big) \in \partial_{(x,y)} f(\tilde{x}(0), \tilde{x}(T)),\ x^{*(i)}(T) = 0, i = 0, \ldots, s-2.$

The definition of the solution to the Euler-Lagrange inclusion is defined appropriately to the definition of the solution to problem (HD); a pair of absolutely continuous functions $x^*(t)$, $x^*(\cdot) \in AC^{s-1}([0,T]) \cap W_{1,s}([0,T])$, is called a feasible solution to problem (a) - (c), if $x^*(\cdot)$ satisfy the associated inclusions (a) and the transversality condition (c).

Theorem 3.1. *Let $F(\cdot,t) : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be a convex mapping, $f : \mathbb{R}^{2n} \to \mathbb{R}^1 \cup \{+\infty\}$ be continuous proper convex function and $A_i, i = 1, \ldots, s-1$ be $n \times n$ real matrices. Then, for optimality of the trajectory $\tilde{x}(\cdot)$ in problem (HD), it is sufficient that there exists a pair of functions $x^*(\cdot)$ satisfying a.e. the adjoint Euler-Lagrange type inclusion (a), (b) and the transversality condition (c).*

Proof. Recall that, according to Theorem 2.1 [21, p.62], the Euler-Lagrange inclusion (a) is equivalent to the subdifferential inclusion

$$(-1)^s x^{*(s)}(t) - \sum_{i=1}^{s-1} (-1)^i A_i^* x^{*(i)}(t) \in \partial_x H_F\left(\tilde{x}(t), x^*(t)\right), \quad t \in [0,T],$$

whereas, by definition of Hamiltonian function H_F implies that

$$\begin{aligned} &H_F\left(x(t), x^*(t)\right) - H_F\left(\tilde{x}(t), x^*(t)\right) \\ &\le \left\langle (-1)^s x^{*(s)}(t) - \sum_{i=1}^{s-1} (-1)^i A_i^* x^{*(i)}(t), x(t) - \tilde{x}(t) \right\rangle. \end{aligned} \quad (3.5)$$

Further, by condition (b) of theorem and Proposition 3.1 and definition of Hamiltonian, we have

$$\begin{aligned} H_F\left(\tilde{x}(t), x^*(t)\right) &= \left\langle \tilde{x}^{(s)}(t), x^*(t) \right\rangle - \sum_{i=1}^{s-1} \left\langle A_i \tilde{x}^{(i)}(t), x^*(t) \right\rangle \\ &= \left\langle \tilde{x}^{(s)}(t) - \sum_{i=1}^{s-1} A_i \tilde{x}^{(i)}(t), x^*(t) \right\rangle, \end{aligned} \quad (3.6)$$

$$\begin{aligned} &H_F\left(x(t), x^*(t)\right) = H_G\left(x(t), x^*(t)\right) \\ &- \sum_{i=1}^{s-1} \left\langle A_i x^{(i)}(t), x^*(t) \right\rangle \ge \left\langle x^{(s)}(t) - \sum_{i=1}^{s-1} A_i x^{(i)}(t), x^*(t) \right\rangle. \end{aligned} \quad (3.7)$$

Hence, considering the two relations (3.6),(3.7) from inequality (3.5) we obtain

$$\left\langle x^{(s)}(t) - \sum_{i=1}^{s-1} A_i x^{(i)}(t), x^*(t) \right\rangle - \left\langle \tilde{x}^{(s)}(t) - \sum_{i=1}^{s-1} A_i \tilde{x}^{(i)}(t), x^*(t) \right\rangle$$
$$\leq \left\langle (-1)^s x^{*(s)}(t), x(t) - \tilde{x}(t) \right\rangle - \sum_{i=1}^{s-1} \left\langle (-1)^i x^{*(i)}(t), A_i(x(t) - \tilde{x}(t) \right\rangle \tag{3.8}$$

Hence, from the inequality (3.8) we derive

$$\int_0^T \left[\left\langle x^{(s)}(t) - \tilde{x}^{(s)}(t), x^*(t) \right\rangle - \left\langle (-1)^s x^{*(s)}(t), x(t) - \tilde{x}(t) \right\rangle \right] dt \tag{3.9}$$
$$\leq \sum_{i=1}^{s-1} \int_0^T \left[\left\langle A_i \left(x^{(i)}(t) - \tilde{x}^{(i)}(t) \right), x^*(t) \right\rangle - \left\langle (-1)^i x^{*(i)}(t), A_i(x(t) - \tilde{x}(t) \rangle \right] dt.$$

and finally, from (3.9) we immediately derive that

$$\int_0^T \left[\left\langle x^{(s)}(t) - \tilde{x}^{(s)}(t), x^*(t) \right\rangle - \left\langle (-1)^s x^{*(s)}(t), x(t) - \tilde{x}(t) \right\rangle \right] dt$$
$$- \sum_{i=1}^{s-1} \int_0^T \left[\left\langle A_i x^{(i)}(t) - A_i \tilde{x}^{(i)}(t), x^*(t) \right\rangle \right.$$
$$\left. - \left\langle (-1)^i x^{*(i)}(t), A_i x(t) - A_i \tilde{x}(t) \right\rangle \right] dt \leq 0. \tag{3.10}$$

Hence, denoting $M = \left\langle x^{(s)}(t) - \tilde{x}^{(s)}(t), x^*(t) \right\rangle - \left\langle (-1)^s x^{*(s)}(t), x(t) - \tilde{x}(t) \right\rangle$ in square brackets on the left hand-side of inequality (3.10), we can reduce M to the following useful relation

$$M = -\frac{d}{dt} \left\langle (-1)^s x^{*(s-1)}(t), x(t) - \tilde{x}(t) \right\rangle$$
$$- \frac{d}{dt} \left\langle (-1)^{s-1} x^{*(s-2)}(t), x'(t) - \tilde{x}'(t) \right\rangle - \frac{d}{dt} \left\langle (-1)^{s-2} x^{*(s-3)}(t), x''(t) - \tilde{x}''(t) \right\rangle$$
$$- \frac{d}{dt} \left\langle (-1)^{s-3} x^{*(s-4)}(t), x'''(t) - \tilde{x}'''(t) \right\rangle - \cdots + \frac{d}{dt} \left\langle x^*(t), x^{(s-1)}(t) - \tilde{x}^{(s-1)}(t) \right\rangle. \tag{3.11}$$

Now, if we integrate (3.11) over $[0, T]$ according to higher-order differential calculus [24], we obtain

$$\int_0^T M dt = \sum_{i=0}^{s-1} \left\langle (-1)^{i+1} x^{*(i)}(0), x^{(s-i-1)}(0) - \tilde{x}^{(s-i-1)}(0) \right\rangle$$
$$- \sum_{i=0}^{s-1} \left\langle (-1)^{i+1} x^{*(i)}(T), x^{(s-i-1)}(T) - \tilde{x}^{(s-i-1)}(T) \right\rangle. \tag{3.12}$$

Recall that $x(\cdot)$ and $\tilde{x}(\cdot)$ are feasible $\left(x^{(i)}(0) = \tilde{x}^{(i)}(0) = x_i^0, i = 1, \ldots, s-1 \right)$ and by the second transversality condition $x^{*(i)}(T) = 0, i = 0, \ldots, s-2$. Considering these from the last relation (3.12) we have more compactly

$$\int_0^T M dt = \left\langle (-1)^s x^{*(s-1)}(0), x(0) - \tilde{x}(0) \right\rangle - \left\langle (-1)^s x^{*(s-1)}(T), x(T) - \tilde{x}(T) \right\rangle . \tag{3.13}$$

Let us now calculate the second integral in (3.11); by analogy denoting $A_i x(t) = \eta_i(t), i = 1, \ldots, s-1$ and

$$Q_i = \left\langle A_i x^{(i)}(t) - A_i \tilde{x}^{(i)}(t), x^*(t) \right\rangle - \left\langle (-1)^i x^{*(i)}(t), A_i x(t) - A_i \tilde{x}(t) \right\rangle$$

we should calculate the integral

$$\int_0^T Q_i dt = \int_0^T \left[\left\langle \eta_i^{(i)}(t) - \tilde{\eta}_i^{(i)}(t), x^*(t) \right\rangle - \left\langle (-1)^i x^{*(i)}(t), \eta_i(t) - \tilde{\eta}_i(t) \right\rangle \right] dt,$$

which has the same form as the integral of M with respect to new functions $\eta_i(t), i = 1, \ldots, s-1$. Then taking into account $A_j x^{(j)}(0) = A_j \tilde{x}^{(j)}(0) = A_j x_j^0; j = 1, \ldots, i$ and using again the second transversality condition (c) $\left(x^{*(j)}(T) = 0\ , j = 0, \ldots, i-2\right)$ similarly to (3.13) immediately we have

$$\begin{aligned} \textstyle\int_0^T Q_i dt &= \left\langle (-1)^i x^{*(i-1)}(0), A_i x(0) - A_i \tilde{x}(0) \right\rangle \\ &- \left\langle (-1)^i x^{*(i-1)}(T), A_i x(T) - A_i \tilde{x}(T) \right\rangle \end{aligned}$$

and

$$\begin{aligned} \sum_{i=1}^{s-1} \int_0^T Q_i dt = &\left\langle \sum_{i=1}^{s-1} (-1)^i A_i^* x^{*(i-1)}(0), x(0) - \tilde{x}(0) \right\rangle \\ &- \left\langle \sum_{i=1}^{s-1} (-1)^i A_i^* x^{*(i-1)}(T), x(T) - \tilde{x}(T) \right\rangle . \end{aligned} \tag{3.14}$$

Finally, considering (3.13), (3.14) in inequality (3.10), we obtain

$$\begin{aligned} &\left\langle (-1)^{s-1} x^{*(s-1)}(0) - \sum_{i=1}^{s-1} (-1)^{i-1} A_i^* x^{*(i-1)}(0), x(0) - \tilde{x}(0) \right\rangle \\ &+ \left\langle (-1)^s x^{*(s-1)}(T) - \sum_{i=1}^{s-1} (-1)^{i-1} A_i^* x^{*(i-1)}(T), x(T) - \tilde{x}(T) \right\rangle \geq 0. \end{aligned} \tag{3.15}$$

On the other hand, by the first transversality condition (c) for all feasible arcs $x(\cdot)$ we have

$$\begin{aligned} &\Bigg((-1)^{s-1} x^{*(s-1)}(0) - \sum_{i=1}^{s-1} (-1)^{i-1} A_i^* x^{*(i-1)}(0) \\ &(-1)^s x^{*(s-1)}(T) - \sum_{i=1}^{s-1} (-1)^{i-1} A_i^* x^{*(i-1)}(T) \Bigg) \in \partial_{(x,y)} f(\tilde{x}(0), \tilde{x}(T)) \end{aligned}$$

or by definition of subdifferential

$$f(x(0), x(T)) - f(\tilde{x}(0), \tilde{x}(T))$$

$$\geq \left\langle (-1)^s x^{*(s-1)}(0) - \sum_{i=1}^{s-1}(-1)^{i-1} A_i^* x^{*(i-1)}(0), x(0) - \tilde{x}(0) \right\rangle$$
$$+ \left\langle (-1)^s x^{*(s-1)}(T) - \sum_{i=1}^{s-1}(-1)^{i-1} A_i^* x^{*(i-1)}(T), x(T) - \tilde{x}(T) \right\rangle.$$

Then this inequality and (3.15) imply that for all feasible trajectories $f(x(0), x(T)) - f(\tilde{x}(0), \tilde{x}(T)) \geq 0$ or $f(x(0), x(T)) \geq f(\tilde{x}(0), \tilde{x}(T))$ i.e., $\tilde{x}(\cdot)$ is the optimal trajectory. □

Remark 3.1. We remind that the next approach to constructing the adjoint Euler-Lagrange type DFIs $(a), (b)$ of Theorem 3.1 for such problems is to use the concept of an adjoint differential operator; assume that D^i is an i-th order operator of derivatives of function $x(\cdot)$. Let us rewrite the semilinear DFI of problem (HD) in the following operator form $Lx(t) \in F(x(t), t)$, where $Lx = D^s x - \sum_{i=1}^{s-1} A_i D^i x$ is a s -th order polynomial linear differential operator [28] with matrix coefficients $A_i, i = 1, \ldots, s$, and A_s, is unique matrix. Hence, for the case in the conditions (a), (b) of Theorem 3.1 should be an adjoint inclusion with adjoint operator $L^* x^*(t) = (-1)^s x^{*(s)}(t) + \sum_{i=1}^{s-1}(-1)^{i+1} A_i^* x^{*(i)}(t)$.

Corollary 3.1. *Let $F(\cdot, t)$ be a convex closed set-valued mapping. Then, the conditions $(a), (b)$ of Theorem 3.1 can be rewritten in term of Hamiltonian function as follows*

(i) $(-1)^s x^{*(s)}(t) + \sum_{i=1}^{s-1}(-1)^{i+1} A_i^* x^{*(i)}(t) \in \partial_x H_F(\tilde{x}(t), x^*(t))$, *a.e.* $t \in [0, T]$,

(ii) $\tilde{x}^{(s)}(t) - \sum_{i=1}^{s-1}(-1)^{i+1} A_i x^{(i)}(t) \in \partial_{v^*} H_F(\tilde{x}(t), x^*(t))$, *a.e.* $t \in [0, T]$.

Proof. By Theorem 2.1 [21, p.62] and Theorem 3.1 above the LAM and argmaximum set are the subdifferentials of the Hamiltonian function on x and v^*, respectively:

$$F^*(v^*; (x, v), t) = \partial_x H_F(x, v^*), F_A(x; v^*, t) = \partial_{v^*} H_F(x, v^*).$$

Then the indicated inclusions (i), (ii) of corollary are equivalent with the conditions $(a), (b)$ of Theorem 3.1. □

Remark 3.2. Note that if we consider the problem (HD) with state constraints, $x(t) \in X(t)$, $t \in [0, T]$ without liner part of DFIs (1.2) it can be shown that the Euler-Lagrange type inclusions (a), (c) of Theorem 3.1 should be replaced by

(a_1) $(-1)^s x^{*(s)}(t) \in F^*\left(x^*(t); \left(\tilde{x}(t), \tilde{x}^{(s)}(t)\right), t\right) + K^*_{X(t)}(\tilde{x}(t))$, a.e. $t \in [0, T]$,

(b_1) $\left((-1)^{s-1} x^{*(s-1)}(0), (-1)^s x^{*(s-1)}(T)\right) \in \partial_{(x,y)} f(\tilde{x}(0), \tilde{x}(T))$,,

$x^{*(i)}(T) = 0, i = 0, \ldots, s-2$

where $K_{X(t)}(\tilde{x}(t))$ is a cone of tangent directions at a point $\tilde{x}(t) \in X(t), t \in [0, T]$. To obtain these conditions formally, it suffices to substitute zero matrices instead of $A_i, i = 1, \ldots, s-1$ in the conditions of Theorem 3.1. It is easy to verify that the whole proof of Theorem 3.1 remains valid in this last case, if we remain within the class of absolutely continuous functions. To this end as a solution of the adjoint DFIs $(a_1), (b_1)$ we did not use a function of bounded variation in order to consider the jumps caused by the presence of state constraints in the primal problem. Recall only that every function with bounded variation has a finite derivative almost everywhere, and if it has bounded variation on $[0, T]$, then its set of discontinuities can be at most countable. Note also that, each point of discontinuity is of the first kind and that an absolutely continuous function has a bounded variation.

Remark 3.3. In the work of Loewen and Rockafellar [20], in terms of the Hamiltonian and the normal cone, it was proved that the conditions of Theorem 3.1 are necessary optimality conditions for a first-order DFI. Therefore, it is interesting to note that taking into account $N_{X(t)}(\tilde{x}(t)) = -K^*_{X(t)}(\tilde{x}(t))$, the results of Theorem 4.3 [20] and Theorem 3.1 for the problem (HD) with $s=1$ coincide. Moreover, the transversality conditions of Theorem 4.3 [20] and Theorem 3.1 coincide in the present statement of the problem (HD). Indeed, according to condition (b_1), since $s=1$ is an odd number, we immediately have $(x^*(0), -x^*(T)) \in \partial_{(x,y)} f(\tilde{x}(0), \tilde{x}(T))$. Therefore, in this sense, the adjoint inclusion of the problem (HD) is a natural generalization of the classical Euler-Lagrange inclusion for first-order DFI. On the other hand, we can conclude that, in fact, in the convex case, there is no "gap" between the necessary and sufficient conditions.

4. Some Applications of Theorem 3.1

In this section are given some applications of problem (HD). First of all, we formulate sufficient optimality conditions for a higher-order linear optimal control problem, then for a higher-order polyhedral optimization also. At the end of polyhedral optimization, some of its interpretations related to abstract economics are given.
Let us consider the problem:

$$\begin{aligned} &\text{minimize } f(x(0), x(T)),\\ \text{(LH)}\qquad &\frac{d^s x(t)}{dt^s} \in F(x(t)), \text{ a.e. } t \in [0,T], \quad F(x,y) \equiv \bar{A}_1 x + BU\\ &x^{(i)}(0) = x_i^0, i = 1, \dots, s-1, \end{aligned}$$

where f is continuously differentiable function, $\bar{A}_1$ and B are $n \times n$ and $n \times r$ matrices, respectively, U-convex compact in $\mathbb{R}^r$. The problem is to find a control function $\tilde{u}(t) \in U$ so that the corresponding solution $\tilde{x}(t)$ minimizes $f(x(0), x(T))$.

Theorem 4.1. *The arc $\tilde{x}(t)$ according to the control function $\tilde{u}(t)$ is a solution of the problem (LH), if there exists an absolutely continuous function $x^*(t)$, satisfying the transversality condition (c) of Theorem 3.1, the higher-order adjoint equations and the Pontryagin maximum principle:*

$$\begin{gathered} (-1)^s x^{*(s)}(t) = \bar{A}_1^* x^*(t), \ a.e. \ t \in [0,T],\\ \langle B\tilde{u}(t), x^*(t) \rangle = \max_{u \in U} \langle Bu, x^*(t) \rangle, t \in [0,T]. \end{gathered}$$

Proof. By elementary computations, we find that if $\tilde{v} = \bar{A}_1 \tilde{x} + B\tilde{u}$, then

$$F^*(v^*; (\tilde{x}, \tilde{v})) = \begin{cases} \bar{A}_1^* v^*, & \text{if } -B^* v^* \in K_U^*(\tilde{u}),\\ \varnothing, & \text{if } -B^* v^* \notin K_U^*(\tilde{u}), \end{cases}$$

whereas $\langle u - \tilde{u}, -B^* v^* \rangle \geq 0, u \in U$ or $\langle B\tilde{u}, v^* \rangle = \max_{u \in U} \langle Bu, v^* \rangle$. Thus, applying the conditions of Theorem 3.1 we deduce the adjoint linear differential equation of the higher-order, and the Pontryagin's maximum principle [33]. Indeed, considering that in problem (LH) all matrices $A_i, i = 1, \ldots, s-1$ are zero matrices the LAM in the conditions of Theorem 3.1 is

$$F^*\left(v^*;(\tilde{x},\tilde{v})\right) = \bar{A}_1^* v^*, -B^* v^* \in K_U^*(\tilde{u})$$

we have

$$(-1)^{(s)} x^{*(s)}(t) = \bar{A}_1^* x^*(t), \text{ a.e. } t \in [0,T],$$

At the same time, we note that the inclusion $-B^*v^* \in K_U^*(\tilde{u})$ expresses the fact that the maximum principle is satisfied:

$$\langle B\tilde{u}(t), x^*(t)\rangle = \max_{u\in U} \langle Bu, x^*(t)\rangle, t \in [0,T]. \text{ The proof is completed.}$$

□

The second problem is the following "polyhedral" problem one:

$$\begin{aligned} &\text{minimize } f(x(0), x(T)),\\ \text{(PHR)}\quad &\frac{d^s x(t)}{dt^s} \in F(x(t)), \text{ a.e. } t \in [0,T], F(x) = \{v : Ax - Cv \le d\}\\ &x^{(i)}(0) = x_i^0, i = 1, \ldots, s-1, \end{aligned}$$

where A, C are $m \times n$ dimensional matrices, d is a m-dimensional vector-column, $f(\cdot,\cdot)$ is a proper convex polyhedral function [23](epi f is a polyhedral set in $\mathbb{R}^{2n+1}$). The problem is to find the trajectory $\tilde{x}(\cdot)$ of the problem (PHR) that minimizes the Mayer functional $f(\cdot,\cdot)$. Thus, based on Theorem 3.1 for the problem (PHR), we prove the following theorem.

Theorem 4.2. *For the optimality of the trajectory $\tilde{x}(\cdot)$ in problem* (PH) *with a higher-order polyhedral DFI, it is sufficient that there exists a nonnegative function $\lambda(t) \ge 0, t \in [0,T]$ satisfying (i)- (iii):*

(i) $(-1)^{s-1} C^* \lambda^{(s)}(t) + A^*\lambda(t) = 0$, *a.e.* $t \in [0,T]$,

(ii) $\left\langle A\tilde{x}(t) - C\tilde{x}^{(s)}(t) - d, \lambda(t)\right\rangle = 0$, *a.e.* $t \in [0,T]$,

(iii) $\left((-1)^s C^*\lambda^{(s-1)}(0), (-1)^{s-1} C^*\lambda^{(s-1)}(T)\right) \in \partial_{(x,y)} f(\tilde{x}(0), \tilde{x}(T))$,

$C^*\lambda^{(i)}(T) = 0, i = 0, \ldots, s-2.$

Proof. According to Farkas theorem [21, p. 22], and the LAM calculation technique for polyhedral set-valued mappings [21] it is not hard to calculate that

$$F^*\left(v^*;(\tilde{x},\tilde{v})\right) = \{-A^*\lambda : v^* = -C^*\lambda, \lambda \ge 0, \langle Ax - Cv - d, \lambda\rangle = 0\}. \qquad (4.1)$$

Thus, from Theorem 3.1 and (4.1) we derive that

$$\begin{aligned} (-1)^s x^{*(s)}(t) = -A^*\lambda(t), \text{ a.e. } t \in [0,T], \quad x^*(t) = -C^*\lambda(t), t \in [0,T],\\ \left\langle A\tilde{x}(t) - C\tilde{x}^{(s)}(t) - d, \lambda(t)\right\rangle = 0, \quad \text{a.e. } t \in [0,T]. \end{aligned} \qquad (4.2)$$

Finally, substituting $x^*(t) = -C^*\lambda(t), t \in [0,T]$ into the adjoint inclusion (equation) (4.2) and the transversality condition (c) of Theorem 3.1, we obtain the required result. □

Remark 4.1. From the point of view of abstract economics the "dual" variable $\lambda(\cdot) = \big(\lambda^1(t), \ldots, \lambda^n(t)\big) \geq 0$ can be interpreted as a price of a resource. Wherein, if $\lambda^j(t) = 0 (j = 1, \ldots, n)$ for some i whenever the supply of this resource is not exhausted by the activities. In economic terminology, such a resource is a "free good"; the price of goods that are oversupplied must drop to zero by the law of supply and demand. Generally speaking, from the point of view of the duality of optimal control theory this fact is what justifies interpreting the objective for the dual problem as maximizing the total implicit value of the resources consumed, rather than the resources allocated, where the strong duality means the solution to these matches centralized if $\lambda(t)$ is optimal multiplier.

5. Conclusion

The paper deals with the development of Mayer problem for higher- order semilinear evolution DFIs with endpoint constraints. First are derived sufficient optimality conditions in the form of Euler-Lagrange type inclusions and transversality conditions. It is shown that in the case $s = 1$ the adjoint inclusion for the higher-order DFIs, defined in terms of LAM coincides with the classical Euler-Lagrange inclusion. Hence, the problem posed does not lose its novelty even in the case of first-order DFIs. Comparing with the problem posed by Loewen, and Rockafellar [20] in the case of $s = 1$, it is easy to see that there is almost no "gap" between the necessary and sufficient conditions. Consequently, there can be no doubt that investigations of optimality results to semilinear problems with higher-order DFIs can have great contribution to the modern development of the optimal control theory and the proposed method is reliable for solving the various problems with higher -order DFIs.

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References

[1] G. Arthi, K.Balachandran, Controllability of Damped Second-Order Impulsive Neutral Functional Differential Systems with Infinite Delay. *J. Optim. Theory Appl.* **152** (2012), 799-813.

[2] D. Aussel, Y. Garcia, N. Hadjisavvas, Single-Directional Property of Multivalued Maps and Variational Systems, *SIAM J. Optim.* **20** (2009), 1274-1285.

[3] D.L. Azzam, C. Castaing, L.Thibault, Three Boundary Value Problems for Second Order Differential Inclusions in Banach Spaces. *Contr. Cyber.* **31** (2002), 659-693.

[4] D.L. Azzam, A. Makhlouf, L. Thibault, Existence and Relaxation Theorem for a Second Order Differential Inclusion. *Numer. Funct. Anal. Optim.* **31** (2010), 1103-1119.

[5] H.H. Bauschke, P.L. Combettes, *Convex Analysis and Monotone Operator Theory in Hilbert Spaces.* Springer, 2011.

[6] M. Benchohra, S.K. Ntouyas, Controllability of Second-Order Differential Inclusions in Banach Spaces with Nonlocal Conditions, *J. Optim. Theory Appl.* **107** (2000), 559-571.

[7] R.I. Bot, E.R. Csetnek, Second Order Forward-Backward Dynamical Systems for Monotone Inclusion Problems, *SIAM J. Control Optim.* **54** (2016), 1423-1443.

[8] R.S. Burachik, A.N Iusem, *Set-Valued Mappings and Enlargements of Monotone Operators*, Series Springer Optimization, 8, (2008).

[9] A. Cernea, On the Existence of Solutions for a Higher Order Differential Inclusion Without Convexity, *Electr. J. Qual. Theory Differ. Equ.* **8** (2007), 1-8.

[10] Y.K. Chang, W.T Li, Controllability of Second-Order Differential and IntegroDifferential Inclusions in Banach Spaces. *J. Optim. Theory Appl.* **129** (2006), 77-87.

[11] F.H. Clarke, P.R. Wolenski, Necessary Conditions for Functional Differential Inclusions. *Appl. Math. Optim.* **34** (1996), 51-78.

[12] B.A. Conway, K.M. Larson, Collocation Versus Differential Inclusion in Direct Optimization. *J. Guidance Contr. Dynam.* **21**, (1998),780-785.

[13] P. Cubiotti, J-C. Yao, On the Cauchy Problem for Lower Semicontinuous Differential Inclusions. *Adv. Differ. Equ.* **2016**, 214 (2016).

[14] T. V. Duong, N. D. Yen, Differential Stability of Convex Optimization Problems Under Inclusion Constraints. *Appl. Anal.* **94** (2015), 108-128.

[15] F. Giannessi, A. Maugeri, *Variational Analysis and Applications.* Springer, 2005

[16] A. Grudzka, K. Rykaczewski, On Approximate Controllability of Functional Impulsive Evolution Inclusions in a Hilbert Space , *J. Optim. Theory Appl.* **166** (2015), 414-439.

[17] A.A. Khan, C. Tammer, C. Zalinescu, *Set-Valued Optimization, an Introduction with Applications.* Springer-Verlag Berlin, 2016.

[18] V. Jeyakumar, A Note on Strong Duality in Convex Semidefinite Optimization: Necessary and Sufficient Conditions. *Optim. Letters.* **2** (2008),15-25.

[19] A.B. Kurzhanski, V.M. Veliov, *Set-Valued Analysis and Differential Inclusions*, Progr. Syst. Contr. Theory. 16, Birghauser, Boston, 1993.

[20] P.D. Loewen, R.T. Rockafellar, *Optimal Control of Unbounded Differential Inclusions.* SIAM J. Control Optim. **32** (1994), 442-470.

[21] E.N. Mahmudov, *Approximation and Optimization of Discrete and Differential Inclusions.* Elsevier, Boston, USA 2011.

[22] E.N. Mahmudov, *Single Variable Differential and Integral Calculus*, Springer, Paris, France, 2013.

[23] E.N. Mahmudov, D. Mastaliyeva, Optimization of neutral functional-differential inclusions, *J. Dynam. Contr. Syst.* **21** (2015), 25-46.

[24] E.N. Mahmudov, Optimal Control of Evolution Differential Inclusions with Polynomial Linear Differential Operators, *Evolut. Equ. Contr. Theory.* **8** (2019), 603-619.

[25] E.N. Mahmudov, Optimal Control of Higher Order Viable Differential Inclusions and Duality. *Applicable Anal.* **101** (2022), 2616-2635.

[26] E.N. Mahmudov, M.J. Mardanov, On duality in optimal control problems with second-order differential inclusions and initial-point constraints, *Proceed. Institute Math. Mech.* **46** (2020), 115-128.

[27] E.N. Mahmudov, M.J. Mardanov, Infimal Convolution and Duality in Convex Mathematical Programming, *Proceedings of the Institute of Mathematics And Mechanics*, **48** (2022), 50-62.

[28] E.N. Mahmudov, Optimal Control of Second Order Sweeping Processes with Discrete and Differential Inclusions, *J. Convex Anal.* **29** (2022), 269-290.

[29] L. Marco, J.A. Murillo, Viability Theorems for Higher-Order Differential Inclusions. *Set-Valued Anal.* **6** (1998), 21-37.

[30] M.J. Mardanov, T.K. Melikov, N.I.Mahmudov, On necessary optimality conditions in discrete control systems, *Int. J.Control* **88** (2015), 2097-2106.

[31] M.J. Mardanov, T.K. Melikov, A new discrete analogue of Pontryagin‘s maximum principle, *Doklady math.*, **98** (2018), 549-551.

[32] M.J. Mardanov, T.K. Melikov, S.T. Malik, On strengthening of optimality conditions in discrete control systems, *Proc. Inst. Math. Mech.*, **44** (2018), 135-154.

[33] M.J. Mardanov, T.K. Melikov, S.T. Malik, On the Theory of Optimal Processes in Discrete Systems, *Math. Notes* **106** (2019) 390-401.

[34] M.J. Mardanov, T.K. Melikov, S.T. Malik, K. Malikov, First and second – order necessary conditions with respect to components for discrete optimal control problems, *J. Comput. Appl. Math.*, **364** (2020), 112342.

[35] B.S. Mordukhovich, L. Wang, Optimal Control of Neutral Functional-Differential Inclusions. *SIAM J. Control Optim.* **43** (2004),111-136.

[36] B.S. Mordukhovich, Discrete Approximations and Refined Euler-Lagrange Conditions for Nonconvex Differential Inclusions, *SIAM J. Contr. Optim.* **33** (1995), 882-915.

[37] L.S. Pontryagin, V. G. Boltyanskii, R.V. Gamkrelidze, E.F. Mishchenko, *The Mathematical Theory of Optimal Processes.* New York, London, Sydney: John Wiley & Sons, Inc.; 1965

[38] V.M. Veliov, Second Order Discrete Approximations to Linear Differential Inclusions. *SIAM J. Numer. Anal.* **29** (1992), 439-451.

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