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ON THE TRANSFORMATION OPERATOR FOR THE SCHRÖDINGER EQUATION WITH AN ADDITIONAL INCREASING POTENTIAL

DAVUD H. ORUJOV

Abstract. This paper considers the Schrödinger equation with an additional increasing potential on the entire axis. A transformation operator with a condition at infinity is constructed. Estimates are found for the kernels of the transformation operator.

1. Introduction and the main result

It is known that for many problems of the spectral theory of one-dimensional linear differential equations of the second order, the apparatus of transformation operators is the most natural and powerful research tool. For various Sturm-Liouville equations, transformation operators were constructed in the works [3],[4], [6]-[8],[11],[12]. In the papers [3],[8],[10], transformation operators with conditions at infinity were constructed for equations of the form $-y'' \pm x^j y + q(x)\, y = \lambda y$, $j = 1, 2$. However, after a thorough analysis of the integral equations used in the papers [3], [8], [10] for the kernels of the transformation operators, it became obvious that additional restrictions should be imposed on the growth of the function $q'(x)$. So, for example, from the integral equation

$$\tilde{K}(\xi_0, \eta_0) = \frac{1}{4}\int\limits_{\xi_0}^{\infty} V(\xi_0, \eta_0, \xi, 0)\, q\left(\frac{\xi}{2}\right) d\xi -$$

$$-\frac{1}{4}\int\limits_{0}^{\eta_0}\int\limits_{\xi_0}^{\infty} V(\xi_0, \eta_0, \xi, \eta)\, q\left(\frac{\xi-\eta}{2}\right) \tilde{K}(\xi, \eta)\, d\xi d\eta$$

used in [10] (see formula (1.8)), it follows that in order to establish the asymptotes of the function $\frac{\partial^2 \tilde{K}(\xi_0,\eta_0)}{\partial \xi_0^2}$, additional restrictions on the function $q'(\xi_0)$ for $\xi_0 \to \infty$ are required. Therefore, the proofs given in the works [3],[8],[10] cannot be considered satisfactory.

Let us consider the differential equation

$$-y'' + \theta(x)\, x^2 y + q(x) y = \lambda y, \qquad -\infty < x < +\infty, \tag{1.1}$$

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where $\theta(x)$ is the Heaviside function, i.e. $\theta(x) = \begin{cases} 1, x \geq 0, \\ 0, x < 0 \end{cases}$, and the real potential $q(x)$ satisfies the condition

$$\int\limits_{-\infty}^{0} (1+|x|)\,|q(x)|\,dx + \int\limits_{0}^{+\infty} \left(1+x^2\right)|q(x)|\,dx < \infty\,. \tag{1.2}$$

It was proved in the paper [9] that for $q(x) = 0$ the equation (1.1) has solutions $\psi_{\pm}(x,\lambda)$ for each complex value of λ, which can be represented as

$$\psi_{+}(x,\lambda) = \begin{cases} D_{\frac{\lambda-1}{2}}\left(\sqrt{2x}\right), x \geq 0, \\ \frac{1}{2}\left[D_{\frac{\lambda-1}{2}}(0) - i\sqrt{\frac{2}{\lambda}}D'_{\frac{\lambda-1}{2}}(0)\right]e^{i\sqrt{\lambda}x} + \\ +\frac{1}{2}\left[D_{\frac{\lambda-1}{2}}(0) + i\sqrt{\frac{2}{\lambda}}D'_{\frac{\lambda-1}{2}}(0)\right]e^{-i\sqrt{\lambda}x}, x < 0, \end{cases} \tag{1.3}$$

$$\psi_{-}(x,\lambda) = \begin{cases} \frac{1}{2}\left[D^{-1}_{\frac{\lambda-1}{2}}(0) - i\sqrt{\frac{\lambda}{2}}\left(D'_{\frac{\lambda-1}{2}}(0)\right)^{-1}\right]D_{\frac{\lambda-1}{2}}\left(\sqrt{2}x\right) + \\ +\frac{1}{2}\left[D^{-1}_{\frac{\lambda-1}{2}}(0) + i\sqrt{\frac{\lambda}{2}}\left(D'_{\frac{\lambda-1}{2}}(0)\right)^{-1}\right]D_{\frac{\lambda-1}{2}}\left(-\sqrt{2}x\right), x \geq 0, \\ e^{-i\sqrt{\lambda}x}, x < 0, \end{cases} \tag{1.4}$$

where $D_{\nu}(x)$ is Weber function (see [1], [8]. In this paper, with the help of transformation operators, triangular representations of special solutions $f_{\pm}(x,\lambda)$ of equation (1.1) with asymptotes $f_{\pm}(x,\lambda) = \psi_{\pm}(x,\lambda) + o(1), x \to \pm\infty$ are found. At the same time, in contrast to the works [3], [8], [10], no restrictions are imposed on the smoothness of the potential $q(x)$. Moreover, the method we proposed justifies the results of the works [3], [8], [10].

We shall use the following notation

$$\sigma_{\pm}(x) = \pm\int\limits_{x}^{\pm\infty}\left|\theta(t)\,t^2 - \frac{1\pm 1}{2}t^2 + q(t)\right|dt.$$

The main result of the present paper is as follows.

Theorem 1.1. *If the potential $q(x)$ satisfies conditions (1.2), then for all values λ, the equation (1.1) has solutions $f_{\pm}(x,\lambda)$ representable as*

$$f_{\pm}(x,\lambda) = \psi_{\pm}(x,\lambda) \pm \int\limits_{x}^{\pm\infty} K_{\pm}(x,t)\ \psi_{\pm}(t,\lambda)\,dt, \tag{1.5}$$

where the kernels $K_{\pm}(x,t)$ are continuous functions and satisfy the following conditions

$$K_{\pm}(x,t) = O\left(\sigma_{\pm}\left(\frac{x+t}{2}\right)\right), x+t \to \pm\infty, \tag{1.6}$$

$$K_{\pm}(x,x)=\pm\frac{1}{2}\int\limits_{x}^{\pm\infty}\left[\theta(t)\,t^{2}-\frac{1\pm 1}{2}t^{2}+q\ (t)\right]\,dt. \tag{1.7}$$

2. Proof of the theorem

First of all, note that the "-" case is obtained from the representation of the Jost solution of the Schrödinger equation with rapidly decreasing potential and the known properties of the transformation operators (see [6], Lemma 1.1.2). Consider now the case "+". Let $f_0(x,\lambda)=D_{\frac{\lambda-1}{2}}\left(\sqrt{2}x\right)$, where $D_\nu(x)$ is the Weber function. It is known from [1], [2] that the function $f_0(x,\lambda)$ and its derivative for all λ satisfy the asymptotic equalities

$$f_0(x,\lambda)=\left(\sqrt{2}x\right)^{\frac{\lambda-1}{2}}e^{-\frac{x^2}{2}}\left(1+O\left(x^{-2}\right)\right),\quad x\to+\infty,$$

$$\frac{\partial}{\partial x}f_0(x,\lambda)=-\frac{1}{\sqrt{2}}\left(\sqrt{2}x\right)^{\frac{\lambda+1}{2}}e^{-\frac{x^2}{2}}\left(1+O\left(x^{-2}\right)\right),\ x\to+\infty. \tag{2.1}$$

Let us first prove that equation (1.1) has a solution $f(x,\lambda)$, which can be represented in the triangular form

$$f(x,\lambda)=f_0(x,\lambda)+\int\limits_{x}^{\infty}K(x,t)\,f_0(t,\lambda)\,dt. \tag{2.2}$$

The following relations hold for the kernel $K(x,t)$:

$$|K(x,t)|\le\frac{1}{2}\sigma\left(\frac{x+t}{2}\right)e^{\sigma_1(x)}, \tag{2.3}$$

$$K(x,x)=\frac{1}{2}\int\limits_{x}^{\infty}\left[\theta(t)\,t^{2}-t^{2}+q\ (t)\right]\,dt. \tag{2.4}$$

Without loss of generality, we will assume that $x\ge 0$. Let us consider the function

$$R(\xi,\eta,\xi_0,\eta_0)=J_0\left(2\sqrt{\left(\eta_0^2-\eta^2\right)\left(\xi^2-\xi_0^2\right)}\right)$$

in the domain $0\le\eta\le\eta_0\le\xi_0\le\xi<\infty$, where $J_0(z)$ is the Bessel function of the first kind. The following properties of the function $R(\xi,\eta,\xi_0,\eta_0)$ were established in the work [8]:

$$\frac{\partial R}{\partial\xi}=-2\xi\left(\eta_0^2-\eta^2\right)J_1(z)\,z^{-1},\ \frac{\partial R}{\partial\eta}=2\eta\left(\xi^2-\xi_0^2\right)J_1(z)\,z^{-1},$$

$$\frac{\partial^2 R}{\partial\xi^2}=2\left(\eta^2-\eta_0^2\right)J_1(z)\,z^{-1}+4\xi^2\left(\eta_0^2-\eta^2\right)^2J_2(z)\,z^{-2},$$

$$\frac{\partial^2 R}{\partial\eta^2}=-2\left(\xi^2-\xi_0^2\right)J_1(z)\,z^{-1}+4\eta^2\left(\xi^2-\xi_0^2\right)^2J_2(z)\,z^{-2},$$

$$\frac{\partial^2 R}{\partial\xi\partial\eta}=4\xi\eta J_0(z)\,.$$

$$|R| \le 1,\ \left|\frac{\partial R}{\partial \xi_0}\right| \le 2\xi_0\left(\eta_0^2-\eta^2\right),\ \left|\frac{\partial R}{\partial \eta_0}\right| \le 2\eta_0\left(\xi^2-\xi_0^2\right),$$

$$\left|\frac{\partial^2 R}{\partial \xi_0^2}\right| \le 2\left(\eta_0^2-\eta^2\right)+C\xi_0^2\left(\eta_0^2-\eta^2\right)^2, \tag{2.5}$$

$$\left|\frac{\partial^2 R}{\partial \eta_0^2}\right| \le 2\eta\left(\xi^2-\xi_0^2\right)+C\eta_0^2\left(\xi^2-\xi_0^2\right)^2, \left|\frac{\partial^2 R}{\partial \xi_0 \partial \eta_0}\right| \le 4\xi_0\eta_0 .$$

Further, as shown in [8], if the function $q(x)$ satisfies condition (1.2), then the integral equation

$$U(\xi_0,\eta_0)=\frac{1}{2}\int_{\xi_0}^{\infty} R(\xi,0,\xi_0,\eta_0)\,q(\xi)\,d\xi-$$

$$-\int_{\xi_0}^{\infty} d\xi \int_0^{\eta_0} U(\xi,\eta)\ R(\xi,\eta,\xi_0,\eta_0)\,q(\xi+\eta)\,d\eta \tag{2.6}$$

has a unique solution in the region $0 \le \eta_0 \le \xi_0$. In addition, $U(\xi_0,\eta_0)$ satisfies the estimate

$$|U(\xi_0,\eta_0)| \le \frac{1}{2}\sigma(\xi_0)\,e^{\sigma_1(\xi_0)}. \tag{2.7}$$

Let us now assume that the function $q(x)$ is continuously differentiable on the whole axis and satisfies the condition

$$\int_0^{+\infty} \left[x^2\,|q(x)|+\left|q'(x)\right|\right]dx<\infty. \tag{2.8}$$

Differentiating equation (2.6) and taking into account (2.5), (2.9), we find that the function $U(\xi_0,\eta_0)$ is twice differentiable in the domain $\xi_0 \ge \eta_0$ the following relations are true

$$\frac{\partial U(\xi_0,\eta_0)}{\partial \xi_0}=-\frac{1}{2}q(\xi_0)+\frac{1}{2}\int_{\xi_0}^{\infty}\frac{\partial R(\xi,0,\xi_0,\eta_0)}{\partial \xi_0}q(\xi)\,d\xi+$$

$$+\int_0^{\eta_0} q(\xi_0+\eta)\,U(\xi_0,\eta)\,d\eta-$$

$$-\int_0^{\eta_0}\int_{\xi_0}^{\infty}\frac{\partial R(\xi_0,\eta_0,\xi,\eta)}{\partial \xi_0}q(\xi+\eta)\,U(\xi,\eta)\,d\xi d\eta. \tag{2.9}$$

$$\frac{\partial U(\xi_0,\eta_0)}{\partial \eta_0}=\frac{1}{2}\int_{\xi_0}^{\infty}\frac{\partial R(\xi,0,\xi_0,\eta_0)}{\partial \eta_0}q(\xi)\,d\xi-\int_{\xi_0}^{\infty} q(\xi+\eta_0)\,U(\xi,\eta_0)\,d\xi+$$

$$+\int_0^{\eta_0}\int_{\xi_0}^{\infty}\frac{\partial R(\xi_0,\eta_0,\xi,\eta)}{\partial \eta_0}q(\xi+\eta)\,U(\xi,\eta)\,d\xi d\eta. \tag{2.10}$$

Using (2.5), (2.7), we find from the last equations that

$$\left|\frac{\partial U(\xi_0,\eta_0)}{\partial \xi_0}+\frac{1}{2}q(\xi_0)\right| \le 2\xi_0\eta_0^2\int\limits_{\xi_0}^{\infty}|q(s)|\,ds+\frac{1}{2}\sigma(\xi_0)\,e^{\sigma_1(\xi_0)}\int\limits_{\xi_0}^{\infty}|q(s)|\,ds+$$

$$+\left[2\left(\eta_0^2-\eta^2\right)+C\xi_0^2\left(\eta_0^2-\eta^2\right)^2\right]\frac{1}{2}\sigma(\xi_0)\,e^{\sigma_1(\xi_0)}\int\limits_{\xi_0}^{\infty}\sigma(\xi)\,d\xi \le$$

$$\le 2\xi_0\eta_0^2\sigma(\xi_0)+\frac{1}{2}\sigma(\xi_0)\,e^{\sigma_1(\xi_0)}\left[\sigma(\xi_0)+\eta_0^2\left[2+C\xi_0^2\eta_0^2\right]\sigma_1(\xi_0)\right], \tag{2.11}$$

$$\left|\frac{\partial U(\xi_0,\eta_0)}{\partial \eta_0}\right| \le 2\eta_0\int\limits_{\xi_0}^{\infty}\left(\xi^2-\xi_0^2\right)|q(\xi)|\,d\xi+\frac{1}{2}\sigma(\xi_0)\,e^{\sigma_1(\xi_0)}\int\limits_{\xi_0+\eta_0}^{\infty}|q(s)|\,ds+$$

$$+2\eta_0 e^{\sigma_1(\xi_0)}\int\limits_{0}^{\eta_0}\int\limits_{\xi_0}^{\infty}\left(\xi^2-\xi_0^2\right)\sigma(\xi)\,|q(\xi+\eta)|\,d\xi d\eta=$$

$$=2\eta_0\sigma_2(\xi_0)+\frac{1}{2}\sigma(\xi_0)\,e^{\sigma_1(\xi_0)}\sigma(\xi_0+\eta_0)+$$

$$+2\eta_0 e^{\sigma_1(\xi_0)}\int\limits_{0}^{\eta_0}\int\limits_{\xi_0}^{\infty}\left(\xi^2-\xi_0^2\right)\int\limits_{\xi}^{\infty}|q(s)|\,ds\,|q(\xi+\eta)|\,d\xi d\eta \le$$

$$\le 2\eta_0\sigma_2(\xi_0)+\frac{1}{2}\sigma(\xi_0)\,e^{\sigma_1(\xi_0)}\sigma(\xi_0+\eta_0)+$$

$$+2\eta_0 e^{\sigma_1(\xi_0)}\int\limits_{0}^{\eta_0}\int\limits_{\xi_0}^{\infty}\int\limits_{\xi}^{\infty}\left(s^2-\xi_0^2\right)|q(s)|\,ds\,|q(\xi+\eta)|\,d\xi d\eta \le$$

$$\le 2\eta_0\sigma_2(\xi_0)+\frac{1}{2}\sigma(\xi_0)\,e^{\sigma_1(\xi_0)}\sigma(\xi_0+\eta_0)+2\eta_0 e^{\sigma_1(\xi_0)}\sigma_2(\xi_0)\,\sigma_1(\xi_0). \tag{2.12}$$

Further, differentiating equations (2.9), (2.10), we obtain

$$\frac{\partial^2 U(\xi_0,\eta_0)}{\partial \xi_0^2}=-\frac{1}{2}q'(\xi_0)-\frac{1}{2}\frac{\partial R(\xi_0,0,\xi_0,\eta_0)}{\partial \xi_0}q(\xi_0)+$$

$$+\frac{1}{2}\int\limits_{\xi_0}^{\infty}\frac{\partial^2 R(\xi,0,\xi_0,\eta_0)}{\partial \xi_0^2}q(\xi)\,d\xi+\int\limits_{0}^{\eta_0}q'(\xi_0+\eta)\,U(\xi_0,\eta)\,d\eta+$$

$$+\int\limits_{0}^{\eta_0}q(\xi_0+\eta)\,\frac{\partial U(\xi_0,\eta)}{\partial \xi_0}d\eta+\int\limits_{0}^{\eta_0}\frac{\partial R(\xi_0,\eta_0,\xi_0,\eta)}{\partial \xi_0}q(\xi_0+\eta)\,U(\xi_0,\eta)\,d\eta$$

$$+\int\limits_{0}^{\eta_0}\int\limits_{\xi_0}^{\infty}\frac{\partial^2 R(\xi_0,\eta_0,\xi,\eta)}{\partial \xi_0^2}q(\xi+\eta)\,U(\xi,\eta)\,d\xi d\eta. \tag{2.13}$$

$$\frac{\partial^2 U(\xi_0,\eta_0)}{\partial \eta_0^2}=\frac{1}{2}\int\limits_{\xi_0}^{\infty}\frac{\partial^2 R(\xi,0,\xi_0,\eta_0)}{\partial \eta_0^2}q(\xi)\,d\xi-\int\limits_{\xi_0}^{\infty}q'(\xi+\eta_0)\,U(\xi,\eta_0)\,d\xi+$$

$$+\int_{\xi_0}^{\infty} q(\xi+\eta_0)\frac{\partial U(\xi,\eta_0)}{\partial\eta_0}d\xi-\int_{\xi_0}^{\infty}\frac{\partial R(\xi_0,\eta_0,\xi,\eta_0)}{\partial\eta_0}q(\xi+\eta_0)U(\xi,\eta_0)d\xi+$$

$$+\int_0^{\eta_0}\int_{\xi_0}^{\infty}\frac{\partial^2 R(\xi_0,\eta_0,\xi,\eta)}{\partial\eta_0^2}q(\xi+\eta)U(\xi,\eta)d\xi d\eta. \tag{2.14}$$

In addition, we have

$$\frac{\partial^2 U(\xi_0,\eta_0)}{\partial\xi_0\partial\eta_0}-2\xi_0\eta_0 U(\xi_0,\eta_0)=$$

$$=\frac{1}{2}\int_{\xi_0}^{+\infty}\left[\frac{\partial^2 R(\xi_0,\eta_0,\xi,0)}{\partial\xi_0\partial\eta_0}-2\xi_0\eta_0 R(\xi_0,\eta_0,\xi,0)\right]q(\xi)d\xi-$$

$$-\int_{\xi_0}^{+\infty}d\xi\int_0^{\eta_0}\left[\left[\frac{\partial^2 R(\xi_0,\eta_0,\xi,\eta)}{\partial\xi_0\partial\eta_0}-2\xi_0\eta_0 R(\xi_0,\eta_0,\xi,\eta)\right]\right]U(\xi,\eta)q(\xi+\eta)d\eta+$$

$$+q(\xi_0+\eta_0)U(\xi_0,\eta_0)=q(\xi_0+\eta_0)U(\xi_0,\eta_0). \tag{2.15}$$

From the last equations we get

$$\left|\frac{\partial^2 U(\xi_0,\eta_0)}{\partial\xi_0^2}+\frac{1}{2}q'(\xi_0)\right|\le\xi_0\eta_0^4|q(\xi_0)|+$$

$$+\frac{\eta_0^2}{2}\left[2+C\xi_0^2\eta_0^2\right]\sigma(\xi_0)+\frac{1}{2}\sigma(\xi_0)e^{\sigma_1(\xi_0)}\int_{\xi_0}^{\infty}\left|q'(s)\right|ds+$$

$$+\left\{\frac{1}{2}|q(\xi_0)|+2\xi_0\eta_0^2\sigma(\xi_0)+\frac{1}{2}\sigma(\xi_0)e^{\sigma_1(\xi_0)}\left[\sigma(\xi_0)+\eta_0^2\left[2+C\xi_0^2\eta_0^2\right]\sigma_1(\xi_0)\right]\right\}\times$$

$$\times\int_{\xi_0}^{\infty}|q(s)|ds+\xi_0\eta_0^4\sigma(\xi_0)e^{\sigma_1(\xi_0)}\int_{\xi_0}^{\infty}|q(s)|ds+\frac{\eta_0^3}{2}\left[2+C\xi_0^2\eta_0^2\right]\sigma(\xi_0)e^{\sigma_1(\xi_0)}\times$$

$$\times\int_{\xi_0}^{\infty}|q(s)|ds=\xi_0\eta_0^4|q(\xi_0)|+\frac{\eta_0^2}{2}\left[2+C\xi_0^2\eta_0^2\right]\sigma(\xi_0)+\frac{1}{2}\sigma(\xi_0)e^{\sigma_1(\xi_0)}\int_{\xi_0}^{\infty}\left|q'(s)\right|ds+$$

$$+\left\{\frac{1}{2}|q(\xi_0)|+2\xi_0\eta_0^2\sigma(\xi_0)+\frac{1}{2}\sigma(\xi_0)e^{\sigma_1(\xi_0)}\left[\sigma(\xi_0)+\eta_0^2\left[2+C\xi_0^2\eta_0^2\right]\sigma_1(\xi_0)\right]\right\}\sigma(\xi_0)+$$

$$+\xi_0\eta_0^4\sigma^2(\xi_0)e^{\sigma_1(\xi_0)}+\frac{\eta_0^3}{2}\left[2+C\xi_0^2\eta_0^2\right]\sigma^2(\xi_0)e^{\sigma_1(\xi_0)}$$

$$\left|\frac{\partial^2 U(\xi_0,\eta_0)}{\partial\eta_0^2}\right|\le\left[\eta_0\int_{\xi_0}^{\infty}\left|\xi^2 q(\xi)\right|d\xi+\frac{C\eta_0^2}{2}\int_{\xi_0}^{\infty}\left|\xi^4 q(\xi)\right|d\xi\right]+$$

$$+\frac{1}{2}\sigma(\xi_0)e^{\sigma_1(\xi_0)}\int\limits_{\xi_0+\eta_0}^{\infty}\left|q'(s)\right|ds+$$

$$+\left[2\eta_0\sigma_2(\xi_0)+\frac{1}{2}\sigma(\xi_0)e^{\sigma_1(\xi_0)}\sigma(\xi_0+\eta_0)+2\eta_0 e^{\sigma_1(\xi_0)}\sigma_2(\xi_0)\sigma_1(\xi_0)\right]\sigma(\xi_0+\eta_0)+$$

$$+\eta_0\sigma(\xi_0+\eta_0)e^{\sigma_1(\xi_0)}\sigma_2(\xi_0+\eta_0)+$$

$$+\frac{1}{2}\int\limits_0^{\eta_0}\int\limits_{\xi_0}^{\infty}\left[2\eta_0\xi^2+C\eta_0^2\xi^4\right]\sigma(\xi+\eta)e^{\sigma_1(\xi)}\left|q(\xi+\eta)\right|d\xi d\eta\leq$$

$$\left[\eta_0\int\limits_{\xi_0}^{\infty}\left|\xi^2 q(\xi)\right|d\xi+\frac{C\eta_0^2}{2}\int\limits_{\xi_0}^{\infty}\left|\xi^4 q(\xi)\right|d\xi\right]+\frac{1}{2}\sigma(\xi_0)e^{\sigma_1(\xi_0)}\int\limits_{\xi_0+\eta_0}^{\infty}\left|q'(s)\right|ds+$$

$$+\left[2\eta_0\sigma_2(\xi_0)+\frac{1}{2}\sigma^2(\xi_0)e^{\sigma_1(\xi_0)}+2\eta_0 e^{\sigma_1(\xi_0)}\sigma_2(\xi_0)\sigma_1(\xi_0)\right]\sigma(\xi_0)+$$

$$+\eta_0\sigma(\xi_0)e^{\sigma_1(\xi_0)}\sigma_2(\xi_0)+\eta_0^2\sigma(\xi_0)e^{\sigma_1(\xi_0)}\sigma_2(\xi_0)+\frac{1}{2}C\eta_0^3\sigma_2(\xi_0)e^{\sigma_1(\xi_0)}\sigma_2(\xi_0),$$

$$\left|\frac{\partial^2 U(\xi_0,\eta_0)}{\partial\xi_0\partial\eta_0}\right|\leq 2\xi_0\eta_0\sigma(\xi_0)e^{\sigma_1(\xi_0)}+\frac{1}{2}\left|q(\xi_0+\eta_0)\right|\sigma(\xi_0)e^{\sigma_1(\xi_0)}.$$

Since the function $q(x)$ is continuously differentiable and satisfies condition (2.8), $\lim\limits_{x\to+\infty}q(x)=0$ holds. In addition, since $\xi_0\geq\eta_0$, the condition $\eta_0\to+\infty$ implies the relation $\xi_0\to+\infty$. From this and the last three inequalities it follows that, the following relations are true under the condition $\eta_0\to+\infty$:

$$\frac{\partial U(\xi_0,\eta_0)}{\partial\xi_0}+\frac{1}{2}q(\xi_0)=o\left(\eta_0^3\right),\ \frac{\partial U(\xi_0,\eta_0)}{\partial\eta_0}=o(\eta_0),\tag{2.16}$$

$$\frac{\partial^2 U(\xi_0,\eta_0)}{\partial\xi_0^2}+\frac{1}{2}q'(\xi_0)=o\left(\xi_0\eta_0^4\right),$$

$$\frac{\partial^2 U(\xi_0,\eta_0)}{\partial\eta_0^2}=o\left(\eta_0^3\right),\ \frac{\partial^2 U(\xi_0,\eta_0)}{\partial\xi_0\partial\eta_0}=o(1).\tag{2.17}$$

Next, assuming $\eta_0=0$ in (2.6), we obtain

$$U(\xi_0,0)=\frac{1}{2}\int\limits_{\xi_0}^{+\infty}q(\xi)d\xi.\tag{2.18}$$

It follows from the last relations and (2.7), (2.14), (2.18) that the function $K(x,t)=U\left(\frac{t+x}{2},\frac{t-x}{2}\right)$ is twice continuously differentiable in the domain $t\geq x$ and satisfies the relations

$$\frac{\partial K(x,t)}{\partial x^2}-\frac{\partial K(x,t)}{\partial t^2}-\left(x^2-t^2+q(x)\right)\ K(x,t)=0,\tag{2.19}$$

$$K(x,x)=\frac{1}{2}\int\limits_x^{+\infty}q\ (t)\ dt,\tag{2.20}$$

$$|K(x,t)| \le \frac{1}{2}\sigma\left(\frac{t+x}{2}\right)e^{\sigma_1(x)}. \tag{2.21}$$

In addition, due to (2.16), we find that if x is fixed, then for $t \to +\infty$ we have the relations

$$\frac{\partial K(x,t)}{\partial x} = o\left(t^3\right), \ \frac{\partial K(x,t)}{\partial t} = o\left(t^3\right),$$
$$\frac{\partial^2 K(x,t)}{\partial x^2} + \frac{1}{2}q'\left(\frac{x+t}{2}\right) = o\left(t^5\right), \ \frac{\partial^2 K(x,t)}{\partial t^2} + \frac{1}{2}q'\left(\frac{x+t}{2}\right) = o\left(t^5\right) \tag{2.22}$$

Since the function $f_0(x,\lambda) = D_{\frac{\lambda-1}{2}}\left(\sqrt{2}x\right)$ and its derivative satisfy the asymptotic equalities (2.1) for all λ, it follows from (2.18), (2.22) that the function

$$f(x,\lambda) = f_0(x,\lambda) + \int\limits_x^{+\infty} K(x,t)\, f_0(t,\lambda)\, dt$$

is a solution of equation (1.1). Moreover, the kernel $K(x,t)$ satisfies conditions (2.3),(2.4).

Let now only condition (1.3) be satisfied, so that the functions $U(\xi,\eta)$ and $K(x,t)$ may not have second order derivatives. In this case the kernel $K(x,t)$ satisfies the estimate (2.13). Moreover, it follows from (2.9)-(2.12) that the function $U(\xi,\eta)$ and thus $K(x,t)$ almost everywhere have first order partial derivatives with respect to both variables, and if x is fixed, then for $t \to +\infty$, we have the relations

$$\frac{\partial K(x,t)}{\partial x} + \frac{1}{2}q\left(\frac{x+t}{2}\right) = o\left((t-x)^3\sigma_2\left(\frac{x+t}{2}\right)\right),$$
$$\frac{\partial K(x,t)}{\partial t} + \frac{1}{2}q\left(\frac{x+t}{2}\right) = o\left((t-x)^3\sigma_2\left(\frac{x+t}{2}\right)\right).$$

Let us construct a sequence of continuously differentiable functions $q_n(x)$ so that the following relations would be satisfied:

$$\lim_{n\to\infty}\int\limits_0^{+\infty}\left(1+x^2\right)|q_n(x) - q(x)|\, dx = 0,$$

$$\int\limits_0^{+\infty} x^2\,|q_n(x)|\, dx < \infty, \quad \int\limits_0^{+\infty}\left|q_n'(x)\right| dx < \infty.$$

Without loss of generality, we can assume that the sequence of functions $q_n(x)$ also converges to the function $q(x)$ almost everywhere (otherwise we would be choosing some subsequence of this sequence). Then, as shown above, the equation

$$U^{(n)}(\xi_0,\eta_0) = \frac{1}{2}\int\limits_{\xi_0}^{+\infty} R(\xi,0,\xi_0,\eta_0)\, q_n(\xi)\, d\xi +$$

$$+ \int\limits_{\xi_0}^{+\infty} d\xi \int\limits_0^{\eta_0} U^{(n)}(\xi,\eta)\, R(\xi,\eta,\xi_0,\eta_0)\, q_n(\xi+\eta)\, d\eta \tag{2.23}$$

has a unique solution satisfying relations similar to (2.7), (2.10). Using equations (2.6), (2.23), we find that

$$V^{(n)}(\xi_0,\eta_0)=\frac{1}{2}\int\limits_{\xi_0}^{+\infty}R(\xi,0,\xi_0,\eta_0)\,p_n(\xi)\,d\xi+$$

$$+\int\limits_{\xi_0}^{+\infty}d\xi\int\limits_{0}^{\eta_0}R(\xi,\eta,\xi_0,\eta_0)\,p_n(\xi+\eta)\,U(\xi,\eta)\,d\eta\times$$

$$\times\int\limits_{\xi_0}^{+\infty}d\xi\int\limits_{0}^{\eta_0}R(\xi,\eta,\xi_0,\eta_0)\,p_n(\xi+\eta)\,V^{(n)}(\xi,\eta)\,d\eta,\qquad(2.24)$$

where $p_n(\xi)=q_n(\xi)-q(\xi)$, $V^{(n)}(\xi,\eta)=U^{(n)}(\xi,\eta)-U(\xi,\eta)$. To analyze equation (2.24), let us use the method of successive approximation. Let

$$\rho_n(\xi)=\int\limits_{\xi}^{+\infty}|p_n(s)|\,ds,\ \rho_{1,n}(\xi)=\int\limits_{\xi}^{+\infty}\rho_n(s)\,ds.$$

$$V_0^{(n)}(\xi_0,\eta_0)=\frac{1}{2}\int\limits_{\xi_0}^{+\infty}R(\xi,0,\xi_0,\eta_0)\,p_n(\xi)\,d\xi+$$

$$+\int\limits_{\xi_0}^{+\infty}d\xi\int\limits_{0}^{\eta_0}R(\xi,\eta,\xi_0,\eta_0)\,p_n(\xi+\eta)\,U(\xi,\eta)\,d\eta$$

$$V_k^{(n)}(\xi_0,\eta_0)=\int\limits_{\xi_0}^{+\infty}d\xi\int\limits_{0}^{\eta_0}R(\xi,\eta,\xi_0,\eta_0)\,p_n(\xi+\eta)\,V_{k-1}^{(n)}(\xi,\eta)\,d\eta,\ k=1,2,...$$

In the same way as it was done above, it is established that

$$\left|V_0^{(n)}(\xi_0,\eta_0)\right|\le\frac{1}{2}\rho_n(\xi_0)+\frac{1}{2}\sigma(\xi_0)\,\rho_{1,n}(\xi_0),$$

$$\left|V_1^{(n)}(\xi_0,\eta_0)\right|\le\frac{1}{2}\rho_n(\xi_0)\,\rho_{1,n}(\xi_0)+\frac{1}{2}\sigma(\xi_0)\,\frac{[\rho_{1,n}(\xi_0)]^2}{2!},$$

$$\left|V_k^{(n)}(\xi_0,\eta_0)\right|\le\frac{1}{2}\rho_n(\xi_0)\,\frac{[\rho_{1,n}(\xi_0)]^k}{k!}+\frac{1}{2}\sigma(\xi_0)\,\frac{[\rho_{1,n}(\xi_0)]^{k+1}}{(k+1)!}.$$

Therefore, for the sum $V^{(n)}(\xi_0,\eta_0)=\sum\limits_{k=0}^{\infty}V_k^{(n)}(\xi_0,\eta_0)$ we get

$$\left|V^{(n)}(\xi_0,\eta_0)\right|\le\frac{1}{2}\rho_n(\xi_0)\,e^{\rho_{1,n}(\xi_0)}+\frac{1}{2}\sigma(\xi_0)\left[e^{\rho_{1,n}(\xi_0)}-1\right].$$

The last estimate shows that the sequence of functions $U^{(n)}(\xi_0,\eta_0)$ uniformly converges in the domain $\xi_0\ge\eta_0\ge0$ to the function $U(\xi_0,\eta_0)$. It is obvious that the last statement remains valid for the sequence of functions $K^{(n)}(x,t)=$

$U^{(n)}\left(\frac{t+x}{2},\frac{t-x}{2}\right)$ in the domain $t \geq x$, whose limit function is $K(x,t)$. From this it follows that the sequence

$$f_n(x,\lambda) = f_0(x,\lambda) + \int\limits_x^{\infty} K^{(n)}(x,t)\, f_0(t,\lambda)\, dt.$$

uniformly converges to the function

$$f(x,\lambda) = f_0(x,\lambda) + \int\limits_x^{\infty} K(x,t)\, f_0(t,\lambda)\, dt,$$

with respect to x taken from the domain $x \geq 0$, and with respect to λ taken from any finite region.

Similarly, by differentiating equation (2.24) we find that the sequences of functions $\frac{\partial U^{(n)}(\xi_0,\eta_0)}{\partial \xi_0}$ and $\frac{\partial U^{(n)}(\xi_0,\eta_0)}{\partial \eta_0}$ converge almost everywhere to the functions $\frac{\partial U(\xi_0,\eta_0)}{\partial \xi_0}$ and $\frac{\partial U(\xi_0,\eta_0)}{\partial \eta_0}$, respectively. This implies that sequences of functions $\frac{\partial K^{(n)}(x,t)}{\partial x}$, $\frac{\partial K^{(n)}(x,t)}{\partial t}$ converge almost everywhere to functions $\frac{\partial K(x,t)}{\partial x}$, $\frac{\partial K(x,t)}{\partial t}$, respectively. Due to (2.1) and Lebesgue's theorem on passage to the limit under the integral [5], , the sequence

$$f_n'(x,\lambda) = f_0'(x,\lambda) - K^{(n)}(x,x)\, f_0(x,\lambda) + \int\limits_x^{\infty} \frac{\partial K^{(n)}(x,t)}{\partial x} f_0(t,\lambda)\, dt$$

for every x and λ converges to the function

$$f'(x,\lambda) = f_0'(x,\lambda) - K(x,x)\, f_0(x,\lambda) + \int\limits_x^{\infty} \frac{\partial K(x,t)}{\partial x} f_0(t,\lambda)\, dt.$$

On the other hand, as shown above, the functions $f_n(x,\lambda)$ will satisfy the equations

$$-y'' + q_n(x)\, y = \lambda y.$$

Passing to the limit as $n \to \infty$ in these formulas, we come to the conclusion that the function $f(x,\lambda)$ must satisfy equation (1.1). Thus, we have proved that representation (2.2) holds true only if the potential $q(x)$ satisfies condition (1.2).

Next, we rewrite the unperturbed equation

$$-y'' + \theta(x)\, x^2 y = \lambda y, \ -\infty < x < \infty, \quad \lambda \in C$$

as

$$-y'' + x^2 y + [\theta(x) - 1]\, x^2 y = \lambda y, \ -\infty < x < \infty, \quad \lambda \in C.$$

Then, from the above reasoning it follows that the solution $\psi_+(x,\lambda)$ of the last equation also admits the representation

$$\psi_+(x,\lambda) = f_0(x,\lambda) + \int\limits_x^{\infty} A(x,t)\ \ f_0(t,\lambda)\, dt\,.$$

On other hand, from the well-known properties of the transformation operators it follows that (see [6], Lemma 1.1.2) the function $f_0(x,\lambda)$ also admits the representation

$$f_0(x,\lambda) = \psi_+(x,\lambda) + \int_x^\infty K_0(x,t) \quad \psi_+(t,\lambda)\,dt\,, \tag{2.25}$$

while kernels $A(x,t)\,, K_0(x,t)$ satisfy relations

$$A(x,t) + K_0(x,t) + \int_x^t K_0(x,u) \quad A(u,t)\,du = 0. \tag{2.26}$$

In addition, the kernels $A(x,t)\,, K_0(x,t)$ satisfy the similar estimate (2.21). Substituting (2.25) in (2.2) we get

$$f_+(x,\lambda) = \psi_+(x,\lambda) + \int_x^\infty K(x,t)\left[\psi_+(t,\lambda) + \int_t^\infty K_0(t,u)\,\psi_+(u,\lambda)\,du\right]dt =$$

$$= \psi_+(x,\lambda) + \int_x^\infty K(x,t)\,\psi_+(t,\lambda)\,dt + \int_x^\infty K(x,t)\int_t^\infty K_0(t,u)\,\psi_+(u,\lambda)\,dudt$$

$$= \psi_+(x,\lambda) + \int_x^\infty K(x,t)\,\psi_+(t,\lambda)\,dt + \int_x^\infty\left(\int_x^t K(x,u)\,K_0(u,t)\,du\right)\psi_+(t,\lambda)\,dt.$$

Now, putting

$$K_+(x,t) = K(x,t) + \int_x^t K(x,u) \quad K_0(u,t)\,du \tag{2.27}$$

we finally get

$$f_+(x,\lambda) = \psi_+(x,\lambda) + \int_x^\infty K_+(x,t) \quad \psi_+(t,\lambda)\,dt.$$

From (2.3), (2.26), (2.27) the validity of (1.6) follows for the "+" case. Moreover, taking $t = x$ from (2.27) we find that $K_+(x,x) = K(x,x)$, which implies (1.7). Thus, the proof of the theorem is completed.

References

[1] M. M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, Natl. Bur. Stand. Appl. Math. Ser. 55, U.S. Government Printing Office, Washington, 1964.

[2] D. Chelkak and E. Korotyaev, The inverse problem for perturbed harmonic oscillator on the half-line with a Dirichlet boundary condition, *Ann. Henri Poincaré* **8** (2007), no.6 1115–1150.

[3] M. G. Gasymov, B. A. Mustafaev, The inverse problem of scattering theory for the anharmonic equation on the half-axis, (Russian) *Dokl. Akad. Nauk SSSR* **228**, (1976), no.1, 11–14.

[4] V. V. Kravchenko, V. A. Vicente-Benitez, *Series representation for the Jost solution of the Sturm-Liouville equation in impedance form*, Mathematical Methods in the Applied Sciences, 2022.

[5] A.N.Kolmogorov, S.V.Fomin, *Elements of the Theory of Functions and Functional Analysis,* Published 1957 – 1961, Graylock Press (translated by Leo F. Boron).

[6] B. M. Levitan, *Inverse Sturm–Liouville Problems,* [in Russian], Nauka, Moscow 1984.

[7] V. A. Marchenko, *Sturm–Liouville Operators and Applications.* Translated from the Russian by A. Jacob, Oper. Theory Adv. Appl. 22, Birkhauser, Basel, 1986.

[8] G. M. Masmaliev and A. K. Khanmamedov, Transformation operators for a perturbed harmonic oscillator, *Mat. Zametki*, **105**, (2019) no.5, 740–746.

[9] D.H. Orucov, Spectral analysis of a one-dimensional Shrodinger operator with a growing potential, *News of Baku University, Series of physico-mathematical sciences*, (2021), no.3, 39-47.

[10] Li. Yishen, One special inverse problem of the second order differential equation on the whole real axis, *Chin. Ann. of Math.,* **2** (1981), no.2, 147-155.

[11] Q. Zhang, Y. Lin, M. Qian, Inverse scattering problem for one-dimensional Schordinger operators related to the general Stark effect, *Acta Mathematicae Applicatae Sinica*, **5** (1989),no.2, 116-136.

[12] L. Zampogni, Some remarks concerning the scattering theory for the Sturm- Liouville operator. *J. Dynam. Differential Equations,* **34** (2022), no. 1, 311–339.

Davud H. Orujov
Baku Engineering University, AZ 0101, Baku, Azerbaijan
E-mail address: dorudjov@gmail.com

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