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REGULARIZATION FOR TWO INVERSE PROBLEMS FOR CONFORMABLE HEAT EQUATION IN L^s SPACES

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Abstract. The main goal of the paper is to approximating two types of inverse problems for conformable heat equation (or called parabolic equation with conformable operator). The conformable models have many applications in dynamics, chaotic systems, and physics. Up to now, there are very few surveys works on the results of regularization in L^s spaces. Our paper is the first work to investigate the inverse problem for conformable parabolic equation in such spaces. For the inverse source problem, we provide Fourier regularized solution and investigated the error between the approximate solution and the sought solution. For the backward in time problem, we also use Fourier truncation method to approximate problem. The error between the regularized solution and the exact solution is obtained in L^s under some suitable assumptions on the Cauchy data.

1. Introduction

Partial differential equations (PDEs) have applications in many branches of science and engineering; see for example [25, 20, 27, 38, 29, 23, 34, 11, 32]. In this paper, we consider the initial value problem for the conformable heat equation (or called parabolic equation with conformable operator)

$$\begin{cases} \dfrac{{}^C\partial^\alpha}{\partial t^\alpha} u(x,t) - \Delta u(x,t) = F(x,t), & x \in \mathcal{D},\ \ t \in (0,T), \\ u(x,t) = 0, & x \in \partial\mathcal{D}, t \in (0,T), \end{cases} \tag{1.1}$$

Here $\mathcal{D} \subset \mathbb{R}^N$ $(N \geq 1)$ is a bounded domain with the smooth boundary $\partial\mathcal{D}$, and $T > 0$ is a given positive number. The function G represents the external forces or the advection term of a diffusion phenomenon, etc, and the function u_0 is the initial datum which will be specified later. It is an obvious fact that a conformable operator has many practical applications. Let us mention that the applications of conformable derivative models in the harmonic oscillator, the damped oscillator and the forced oscillator (see [15]), electrical circuits (see [24]), chaotic systems in dynamics (see [18]), projectile motion (see [4]).

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Conformable derivative model: Let us take B is a Banach space, and f is a B-valued function on $[0,\infty)$. Let $\frac{{}^C\partial^\beta}{\partial t^\beta}$ be the conformable derivative of order $0<\beta\le 1$ locally defined by

$$\frac{{}^C\partial^\beta f(t)}{\partial t^\beta} := \lim_{h\to 0}\frac{f(t+ht^{1-\beta})-f(t)}{h} \quad \text{in } B,$$

for each $t>0$ and some more knowledge about the above definition, we refer the reader to [21, 1, 2, 19, 6].

There are two interesting points regarding the relationship between conformable and classical derivatives

- Let us assume that B is $\mathbb{R}$ then we get the following statement: If f is a real function and $s>0$, then f has a conformable fractional derivative of order β at s if and only if it is (classically) differentiable at s, and
$$\frac{{}^C\partial^\beta f(s)}{\partial s^\beta} = s^{1-\beta}\frac{\partial f(s)}{\partial s}. \tag{1.2}$$
- If B is not $\mathbb{R}$, for example B are Sobolev spaces then the relation (1.2) does not hold on. This means that for functions with values in Banach spaces, (1.2) seems not to be true. There are not many conformable related results in Banach spaces. What we just said has been proven in [33]. From this statement, we can say, our model in this paper, seems to be more difficult and complex than the ODEs containing the conformable operator.

For the convenience of the reader, we will consider three models related to problem (1.1) that most mathematicians often study. The first type of problem is the initial value problem, which occurs when we consider (1.1) with initial condition $u(x,0)=u_0(x)$. One of the interesting works on this kind of problem is of [33]. The next two types of inverse problems that we are interested in this paper are described as follows.

- The inverse source problem for (1.1) is described as follows. Let the distribution of u at the terminal time
$$u(x,T)=h(x), \tag{1.3}$$
together with the additional condition $u(x,0)=0,\ x\in\mathcal{D}$. Assume that the source function F has a simple form of split type $F(x,t)=\varphi(t)f(x)$. Let us given the function φ. The inverse source problem for (1.1) is understood as finding the function f when the input data h,φ is given.
- The terminal value problem for (1.1) is described as follows. Let us give the terminal value data
$$u(x,T)=\psi(x) \text{ in } \quad \mathcal{D}. \tag{1.4}$$
The problem of restoring the original value function from (1.4) is also called terminal value problem for (1.1). This problem has many applications in physics, image processing, that has been studied in many interesting papers, for example [34, 30, 31].

Main contributions and novelties of this paper are stated as follows.

As we know, two inverse problems are ill-posed in the sense of Hadamard. When the input data is slightly skewed, it will lead to a large error of the main solution. To give a good approximation, we need to regularize these problems. The number of works on the regularized problem with input data in L^2 is quite abundant. However, results for regularized problem in L^s, for $s \neq 2$ are rare. We confirm that our paper is the first result for the inverse problem for the conformable parabolic equation when the observed data is in the L^s space with $s \neq 2$. Under the data not in L^2, it is impossible to apply Parseval equality directly to solving the problem. One way to overcome this weakness is to use the embedding between L^s and Hilbert scales spaces $\mathbb{X}^s(\mathcal{D})$. The main analytical technique in our paper is to use some embeddings and some analysis estimators related to Hölder inequality. To complete our proofs, we learn many interesting techniques from N.H. Tuan [28].

This paper is organized as follows. In Section 2, we state some function spaces and embeddings. In section 3, we deal with the regularized solution for the inverse source problem for (1.1). Section 4 gives a regularized method for the backward in time problem for (1.1). We solve two problems in the case of observed data in L^s space.

2. Preliminary results

In this section, we introduce the notation and the functional setting which shall be used in our paper. Recall that the spectral problem

$$\begin{cases} \Delta\varphi_j(x) = -\lambda_j \varphi_j(x), & x \in \mathcal{D}, \\ \varphi_j(x) = 0, & x \in \partial D, \end{cases}$$

admits the eigenvalues $0 < \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_j \leq \dots$ with $\lambda_j \to \infty$ as $j \to \infty$. The corresponding eigenfunctions are $\psi_j \in H_0^1(\Omega)$.

Definition 2.1. *(Hilbert scale space).* We recall the Hilbert scale space, which is given as follows

$$\mathbb{X}^m(\mathcal{D}) = \left\{ f \in L^2(\mathcal{D}),\ \sum_{j=1}^{\infty} \lambda_j^{2m} \Big(\int_{\mathcal{D}} f(x)\varphi_j(x)dx \Big)^2 < \infty \right\},$$

for any $m \geq 0$. It is well-known that $\mathbb{X}^m(\mathcal{D})$ is a Hilbert space corresponding to the norm

$$\|f\|_{\mathbb{X}^m(\mathcal{D})} = \left(\sum_{n=1}^{\infty} \lambda_j^{2m} \Big(\int_{\mathcal{D}} f(x)\varphi_j(x)dx \Big)^2 \right)^{1/2},\ f \in \mathbb{X}^m(\mathcal{D}).$$

Lemma 2.1. *(See* [33, 30]*) The following statement are true:*

$$\left.\begin{aligned} &L^p(\mathcal{D}) \hookrightarrow \mathbb{X}^\mu(\mathcal{D}), \quad \textit{if} \quad -\frac{N}{4} < \mu \leq 0, \quad p \geq \frac{2N}{N-4\mu}, \\ &\mathbb{X}^s(\mathcal{D}) \hookrightarrow L^p(\mathcal{D}), \quad \textit{if} \quad 0 \leq s < \frac{N}{4}, \quad p \leq \frac{2N}{N-4s}. \end{aligned}\right\} \tag{2.1}$$

3. Regularization of inverse source problem

In order to find a precise formulation for solutions, we consider the spectral decomposition $u(x,t)=\sum_{j=1}^\infty \langle u(.,t),\varphi_j\rangle\varphi_j(x)$. Thanks for the work [33], we get the following equality

$$\langle u(.,t),\varphi_j\rangle = \exp\Big(-\lambda_j\frac{t^\beta}{\beta}\Big)\langle u_0,\varphi_j\rangle + \int_0^t \nu^{\beta-1}\exp\Big(-\lambda_j\frac{t^\beta-\nu^\beta}{\beta}\Big)\langle F(.,\nu),\varphi_j\rangle d\nu. \tag{3.1}$$

Since the fact that $u(x,0)=0$ and $u(x,T)=h(x)$ and $F(x,\nu)=\psi(\nu)f(x)$, we follow from (3.1) that

$$\int_{\mathcal{D}} h(x)\varphi_j(x)dx = \Big[\int_0^t \nu^{\beta-1}\exp\Big(-\lambda_j\frac{t^\beta-\nu^\beta}{\beta}\Big)\psi(\nu)d\nu\Big]\Big[\int_{\mathcal{D}}\theta(x)\varphi_j(x)dx\Big]. \tag{3.2}$$

This equality will give us the following formula

$$\int_{\mathcal{D}} \theta(x)\varphi_j(x)dx = \Big[\int_0^T \nu^{\beta-1}\exp\Big(-\lambda_j\frac{T^\beta-\nu^\beta}{\beta}\Big)\psi(\nu)d\nu\Big]^{-1}\Big[\int_{\mathcal{D}} h(x)\varphi_j(x)dx\Big]. \tag{3.3}$$

In this section, we provide a regularization result concerning on the observed data in L^s spaces. Fourier truncation method is applied to establish approximate solution.

Theorem 3.1. *Let us take $(\varphi_\varepsilon,h_\varepsilon)\in L^s(0,T)\times L^s(\Omega)$ such that $\psi_\varepsilon(t)>M_0>0$ for any $0\le t\le T$ for any $\frac{1}{\beta}<s<2$ and*

$$\Big\|\psi_\varepsilon-\psi\Big\|_{L^s(0,T)}+\Big\|h_\varepsilon-h\Big\|_{L^s(\mathcal{D})}\le\varepsilon. \tag{3.4}$$

Let us assume that $\theta\in\mathbb{X}^{m+r}(\mathcal{D})$ for $r>0$ and $0<m<\frac{N}{4}$. We construct a regularized solution as follows

$$\theta_\varepsilon(x)=\sum_{\lambda_j\le B_\varepsilon}\Big[\int_0^T \nu^{\beta-1}\exp\Big(-\lambda_j\frac{T^\beta-\nu^\beta}{\beta}\Big)\psi_\varepsilon(\nu)d\nu\Big]^{-1}\Big[\int_{\mathcal{D}} h_\varepsilon(x)\varphi_j(x)dx\Big]\varphi_j(x). \tag{3.5}$$

Then the error estimate between the regularized solution and the sought solution is bounded by

$$\Big\|\theta_\varepsilon-\theta\Big\|_{L^{\frac{2N}{N-4m}}(\mathcal{D})}\lesssim |B_\varepsilon|^{-r}\big\|\theta\big\|_{\mathbb{X}^{m+r}(\mathcal{D})}+\big\|\theta\big\|_{\mathbb{X}^m(\mathcal{D})}B_\varepsilon\varepsilon+(B_\varepsilon)^{m+1+\frac{N}{2s}-\frac{N}{4}}\varepsilon. \tag{3.6}$$

Here B_ε satisfies that

$$\lim_{\varepsilon\to0}B_\varepsilon\varepsilon=\lim_{\varepsilon\to0}\Big((B_\varepsilon)^{m+1+\frac{N}{2s}-\frac{N}{4}}\varepsilon\Big)=0,\quad \lim_{\varepsilon\to0}B_\varepsilon=+\infty. \tag{3.7}$$

Remark 3.1. One choice for B_ε could be

$$B_\varepsilon=\varepsilon^{\frac{\delta-1}{m+1+\frac{N}{2s}-\frac{N}{4}}},\ 0<\delta<1.$$

Proof. In view of triangle inequality, we find that

$$\Big\|\theta_\varepsilon-\theta\Big\|_{\mathbb{X}^m(\mathcal{D})}\le\Big\|\Upsilon_\varepsilon(.)-\theta(.)\Big\|_{\mathbb{X}^m(\mathcal{D})}+\Big\|\Upsilon_\varepsilon(x)-\Theta_\varepsilon(x)\Big\|_{\mathbb{X}^m(\mathcal{D})}+\Big\|\Theta_\varepsilon(x)-\theta_\varepsilon\Big\|_{\mathbb{X}^m(\mathcal{D})},\tag{3.8}$$

where we denote some following functions

$$\Theta_\varepsilon(x)=\sum_{\lambda_j\le B_\varepsilon}\Big[\int_0^T\nu^{\beta-1}\exp\Big(-\lambda_j\frac{T^\beta-\nu^\beta}{\beta}\Big)\psi_\varepsilon(\nu)d\nu\Big]^{-1}\times\Big[\int_{\mathcal{D}}h(x)\varphi_j(x)dx\Big]\varphi_j(x).\tag{3.9}$$

and

$$\Upsilon_\varepsilon(x)=\sum_{\lambda_j\le B_\varepsilon}\Big[\int_0^T\nu^{\beta-1}\exp\Big(-\lambda_j\frac{T^\beta-\nu^\beta}{\beta}\Big)\psi(\nu)d\nu\Big]^{-1}\times\Big[\int_{\mathcal{D}}h(x)\varphi_j(x)dx\Big]\varphi_j(x).\tag{3.10}$$

Now, we need to establish the upper bound of the expressions on the right of (3.8). For convenience, we consider the following step.

Step 1. Estimate of $\Big\|\Upsilon_\varepsilon(.)-\theta(.)\Big\|_{\mathbb{X}^m(\mathcal{D})}$.

Let us recall the function θ as follows.

$$\theta(x)=\sum_{j=1}^\infty\Big[\int_0^T\nu^{\beta-1}\exp\Big(-\lambda_j\frac{T^\beta-\nu^\beta}{\beta}\Big)\psi(\nu)d\nu\Big]^{-1}\Big[\int_{\mathcal{D}}h(x)\varphi_j(x)dx\Big]\varphi_j(x).$$

This expression together with the formula (3.10) gives us the claim of the following difference

$$\begin{aligned}\theta(x)-\Upsilon_\varepsilon(x)&=\sum_{\lambda_j>B_\varepsilon}\Big[\int_0^T\nu^{\beta-1}\exp\Big(-\lambda_j\frac{T^\beta-\nu^\beta}{\beta}\Big)\psi(\nu)d\nu\Big]^{-1}\\&\qquad\times\Big[\int_{\mathcal{D}}h(x)\varphi_j(x)dx\Big]\varphi_j(x)\\&=\sum_{\lambda_j>B_\varepsilon}\Big[\int_{\mathcal{D}}\theta(x)\varphi_j(x)dx\Big]\varphi_j(x).\end{aligned}\tag{3.11}$$

The norm on $\mathbb{X}^m(\mathcal{D})$ of the left above is calculated through the Parseval equality as follows

$$\begin{aligned}\Big\|\Upsilon_\varepsilon(.)-\theta(.)\Big\|^2_{\mathbb{X}^m(\mathcal{D})}&=\sum_{\lambda_j>B_\varepsilon}\lambda_j^{2m}\Big[\int_{\mathcal{D}}\theta(x)\varphi_j(x)dx\Big]^2\\&=\sum_{\lambda_j>B_\varepsilon}\lambda_j^{-2r}\lambda_j^{2m+2r}\Big[\int_{\mathcal{D}}\theta(x)\varphi_j(x)dx\Big]^2.\end{aligned}$$

It is obvious to see that $\lambda_j^{-2r} \le |B_\varepsilon|^{-2r}$ if $\lambda_j > B_\varepsilon$ and $r > 0$. Therefore, we get that

$$\Big\|\Upsilon_\varepsilon(.) - \theta(.)\Big\|^2_{\mathbb{X}^m(\mathcal{D})} \le |B_\varepsilon|^{-2r} \sum_{\lambda_j > B_\varepsilon} \lambda_j^{2m+2r} \Big[\int_{\mathcal{D}} \theta(x)\varphi_j(x)dx\Big]^2$$
$$= |B_\varepsilon|^{-2r} \big\|\theta\big\|^2_{\mathbb{X}^{m+r}(\mathcal{D})}, \tag{3.12}$$

which gives that

$$\Big\|\Upsilon_\varepsilon(.) - \theta(.)\Big\|_{\mathbb{X}^m(\mathcal{D})} \le |B_\varepsilon|^{-r} \big\|\theta\big\|_{\mathbb{X}^{m+r}(\mathcal{D})}. \tag{3.13}$$

Step 2. Estimate of $\Big\|\Upsilon_\varepsilon(x) - \Theta_\varepsilon(x)\Big\|_{\mathbb{X}^m(\mathcal{D})}$.
Based on two formulas (3.9) and (3.10), we have the difference of two functions Υ_ε and Θ_ε as follows

$$\Upsilon_\varepsilon(x) - \Theta_\varepsilon(x)$$
$$= \sum_{\lambda_j \le B_\varepsilon} \frac{\Big[\int_0^T \nu^{\beta-1} \exp\Big(-\lambda_j \frac{T^\beta - \nu^\beta}{\beta}\Big)\Big(\psi_\varepsilon(\nu) - \psi(\nu)\Big)d\nu\Big]}{\Big[\int_0^T \nu^{\beta-1} \exp\Big(-\lambda_j \frac{T^\beta - \nu^\beta}{\beta}\Big)\psi_\varepsilon(\nu)d\nu\Big]} \varphi_j(x)$$
$$\times \frac{\int\limits_{\mathcal{D}} h(x)\varphi_j(x)dx}{\int_0^T \nu^{\beta-1} \exp\Big(-\lambda_j \frac{T^\beta - \nu^\beta}{\beta}\Big)\psi(\nu)d\nu} \varphi_j(x). \tag{3.14}$$

In view of the equality (3.3), we follows from (3.14) that

$$\Upsilon_\varepsilon(x) - \Theta_\varepsilon(x)$$
$$= \sum_{\lambda_j \le B_\varepsilon} \frac{\Big[\int_0^T \nu^{\beta-1} \exp\Big(-\lambda_j \frac{T^\beta - \nu^\beta}{\beta}\Big)\Big(\psi_\varepsilon(\nu) - \psi(\nu)\Big)d\nu\Big]}{\int_0^T \nu^{\beta-1} \exp\Big(-\lambda_j \frac{T^\beta - \nu^\beta}{\beta}\Big)\psi_\varepsilon(\nu)d\nu}$$
$$\times \Big[\int_{\mathcal{D}} \theta(x)\varphi_j(x)dx\Big]\varphi_j(x). \tag{3.15}$$

By taking the norm of both sides of the above expression in space $\mathbb{X}^m(\mathcal{D})$ and using Parseval' s equality, we provide that

$$\Big\|\Upsilon_\varepsilon(x) - \Theta_\varepsilon(x)\Big\|^2_{\mathbb{X}^m(\mathcal{D})}$$
$$= \sum_{\lambda_j \le B_\varepsilon} \left[\frac{\int_0^T \nu^{\beta-1} \exp\Big(-\lambda_j \frac{T^\beta - \nu^\beta}{\beta}\Big)\Big(\psi_\varepsilon(\nu) - \psi(\nu)\Big)d\nu}{\int_0^T \nu^{\beta-1} \exp\Big(-\lambda_j \frac{T^\beta - \nu^\beta}{\beta}\Big)\psi_\varepsilon(\nu)d\nu}\right]^2$$
$$\times \lambda_j^{2m}\Big[\int_{\mathcal{D}} \theta(x)\varphi_j(x)dx\Big]^2. \tag{3.16}$$

Let us continue to look at the first expression on the right above. By applying Hölder inequality, we have that the upper bound

$$\Big|\int_0^T \nu^{\beta-1}\exp\Big(-\lambda_j\frac{T^\beta-\nu^\beta}{\beta}\Big)\Big(\psi_\varepsilon(\nu)-\psi(\nu)\Big)d\nu\Big|$$
$$\le \Big[\int_0^T \Big|\psi_\varepsilon(\nu)-\psi(\nu)\Big|^s d\nu\Big]^{\frac{1}{s}}\Big[\int_0^T \nu^{s^*(\beta-1)}\exp\Big(-s^*\lambda_j\frac{T^\beta-\nu^\beta}{\beta}\Big)d\nu\Big]^{\frac{1}{s^*}}, \quad (3.17)$$

where $s^*=\frac{s}{s-1}$. It is obvious to provide the following statement

$$\Big[\int_0^T \Big|\psi_\varepsilon(\nu)-\psi(\nu)\Big|^s d\nu\Big]^{\frac{1}{s}} = \big\|\psi_\varepsilon-\psi\big\|_{L^s(0,T)}, \quad (3.18)$$

and

$$\int_0^T \nu^{s^*(\beta-1)}\exp\Big(-s^*\lambda_j\frac{T^\beta-\nu^\beta}{\beta}\Big)d\nu \le \int_0^T \nu^{s^*(\beta-1)}d\nu$$
$$= \frac{T^{s^*(\beta-1)+1}}{s^*(\beta-1)+1} = \frac{s-1}{\beta s-1}T^{\frac{\beta s-1}{s-1}} \quad (3.19)$$

where we note that $s>\frac{1}{\beta}$ and we also have used the fact that $\exp\Big(-s^*\lambda_j\frac{T^\beta-\nu^\beta}{\beta}\Big)$ less than 1. Combining three evaluations (3.17), (3.18) and (3.19), we derive that the following estimate

$$\Big|\int_0^T \nu^{\beta-1}\exp\Big(-\lambda_j\frac{T^\beta-\nu^\beta}{\beta}\Big)\Big(\psi_\varepsilon(\nu)-\psi(\nu)\Big)d\nu\Big|$$
$$\le \Big(\frac{s-1}{\beta s-1}\Big)^{\frac{s-1}{s}}T^{\beta-\frac{1}{s}}\big\|\psi_\varepsilon-\psi\big\|_{L^s(0,T)}. \quad (3.20)$$

Next, from the lower bound property of the function ψ_ε by a positive constant M_0, and after a simple transformation for an integral, we have immediately

$$\int_0^T \nu^{\beta-1}\exp\Big(-\lambda_j\frac{T^\beta-\nu^\beta}{\beta}\Big)\psi_\varepsilon(\nu)d\nu$$
$$\ge M_0\int_0^T \nu^{\beta-1}\exp\Big(-\lambda_j\frac{T^\beta-\nu^\beta}{\beta}\Big)d\nu$$
$$= M_0\int_0^{\frac{T^\beta}{\beta}} \exp\Big(-\lambda_j(\frac{T^\beta}{\beta}-z)\Big)dz$$
$$= M_0\frac{1-\exp\Big(\frac{-\lambda_j T^\beta}{\beta}\Big)}{\lambda_j} \ge M_0\frac{1-\exp\Big(\frac{-\lambda_1 T^\beta}{\beta}\Big)}{\lambda_j}. \quad (3.21)$$

From the two closest observations, we assert that

$$\frac{\int_0^T \nu^{\beta-1}\exp\Big(-\lambda_j\frac{T^\beta-\nu^\beta}{\beta}\Big)\Big(\psi_\varepsilon(\nu)-\psi(\nu)\Big)d\nu}{\int_0^T \nu^{\beta-1}\exp\Big(-\lambda_j\frac{T^\beta-\nu^\beta}{\beta}\Big)\psi_\varepsilon(\nu)d\nu}$$
$$\le \overline{M}_1(T,s,\beta)\lambda_j\big\|\psi_\varepsilon-\psi\big\|_{L^s(0,T)}, \quad (3.22)$$

where we denote

$$\overline{M}_1(T,s,\beta)=\Big(\frac{s-1}{\beta s-1}\Big)^{\frac{s-1}{s}}T^{\beta-\frac{1}{s}}\Big[M_0\left(1-\exp\Big(\frac{-\lambda_1 T^\beta}{\beta}\Big)\right)\Big]^{-1}.$$

Combining (3.16) and (3.22), we find that

$$\Big\|\Upsilon_\varepsilon(.)-\Theta_\varepsilon(.)\Big\|^2_{\mathbb{X}^m(\mathcal{D})}\le|\overline{M}_1(T,s,\beta)|^2\big\|\psi_\varepsilon-\psi\big\|^2_{L^s(0,T)}\times\sum_{\lambda_j\le B_\varepsilon}\lambda_j^{2m+2}\Big[\int_{\mathcal{D}}\theta(x)\varphi_j(x)dx\Big]^2. \tag{3.23}$$

Noting that the finite sum $\sum_{\lambda_j\le B_\varepsilon}\lambda_j^{2m+2}\Big[\int_{\mathcal{D}}\theta(x)\varphi_j(x)dx\Big]^2$ is bounded by

$$|B_\varepsilon|^2\sum_{\lambda_j\le B_\varepsilon}\lambda_j^{2m}\Big[\int_{\mathcal{D}}\theta(x)\varphi_j(x)dx\Big]^2\le|B_\varepsilon|^2\big\|\theta\big\|^2_{\mathbb{X}^m(\mathcal{D})}$$

. Therefore, we follows from (3.23) that

$$\begin{aligned}\Big\|\Upsilon_\varepsilon(.)-\Theta_\varepsilon(.)\Big\|_{\mathbb{X}^m(\mathcal{D})}&\le\overline{M}_1(T,s,\beta)\big\|\theta\big\|_{\mathbb{X}^m(\mathcal{D})}B_\varepsilon\big\|\psi_\varepsilon-\psi\big\|_{L^s(0,T)}\\&\le\overline{M}_1(T,s,\beta)\big\|\theta\big\|_{\mathbb{X}^m(\mathcal{D})}B_\varepsilon\varepsilon.\end{aligned} \tag{3.24}$$

Here we have used (3.4).

Step 3. Estimate of $\Big\|\Theta_\varepsilon(.)-\theta_\varepsilon(.)\Big\|_{\mathbb{X}^m(\mathcal{D})}$.

Due to the formulas (3.5) and (3.9), we derive that

$$\Theta_\varepsilon(x)-\theta_\varepsilon(x)=\sum_{\lambda_j\le B_\varepsilon}\Big[\int_0^T\nu^{\beta-1}\exp\Big(-\lambda_j\frac{T^\beta-\nu^\beta}{\beta}\Big)\psi_\varepsilon(\nu)d\nu\Big]^{-1}\times\Big[\int_{\mathcal{D}}\big(h_\varepsilon(x)-h(x)\big)\varphi_j(x)dx\Big]\varphi_j(x). \tag{3.25}$$

By taking the norm of both sides of the above expression in space $\mathbb{X}^m(\mathcal{D})$ and using Parseval' s equality, we obtain that

$$\begin{aligned}&\Big\|\Theta_\varepsilon(.)-\theta_\varepsilon(.)\Big\|^2_{\mathbb{X}^m(\mathcal{D})}\\&=\sum_{\lambda_j\le B_\varepsilon}\Big[\int_0^T\nu^{\beta-1}\exp\Big(-\lambda_j\frac{T^\beta-\nu^\beta}{\beta}\Big)\psi_\varepsilon(\nu)d\nu\Big]^{-2}\\&\qquad\times\lambda_j^{2m}\Big[\int_{\mathcal{D}}\big(h_\varepsilon(x)-h(x)\big)\varphi_j(x)dx\Big]^2.\end{aligned} \tag{3.26}$$

By looking back the inequality (3.21), we get

$$\begin{aligned}&\Big\|\Theta_\varepsilon(.)-\theta_\varepsilon(.)\Big\|^2_{\mathbb{X}^m(\mathcal{D})}\\&=\Big[M_0\left(1-\exp\Big(\frac{-\lambda_1T^\beta}{\beta}\Big)\right)\Big]^{-2}\sum_{\lambda_j\le B_\varepsilon}\lambda_j^{2m+2}\Big[\int_{\mathcal{D}}\big(h_\varepsilon(x)-h(x)\big)\varphi_j(x)dx\Big]^2.\end{aligned} \tag{3.27}$$

We continue to deal with the finite series on the right above. Indeed, we have

$$\begin{aligned}
&\sum_{\lambda_j \le B_\varepsilon} \lambda_j^{2m+2} \Big[\int_{\mathcal{D}} \big(h_\varepsilon(x) - h(x)\big)\varphi_j(x)dx \Big]^2 \\
&\quad = \sum_{\lambda_j \le B_\varepsilon} \lambda_j^{2m+2+\frac{N}{s}-\frac{N}{2}} \lambda_j^{\frac{Ns-2N}{2s}} \Big[\int_{\mathcal{D}} \big(h_\varepsilon(x) - h(x)\big)\varphi_j(x)dx \Big]^2 \\
&\quad \le (B_\varepsilon)^{2m+2+\frac{N}{s}-\frac{N}{2}} \sum_{\lambda_j \le B_\varepsilon} \lambda_j^{\frac{Ns-2N}{2s}} \Big[\int_{\mathcal{D}} \big(h_\varepsilon(x) - h(x)\big)\varphi_j(x)dx \Big]^2 \\
&\quad = (B_\varepsilon)^{2m+2+\frac{N}{s}-\frac{N}{2}} \Big\| h_\varepsilon - h \Big\|^2_{\mathbb{X}^{\frac{Ns-2N}{4s}}(\mathcal{D})}. \qquad (3.28)
\end{aligned}$$

Since $1 < s < 2$, we know that $L^s(\mathcal{D}) \hookrightarrow \mathbb{X}^{\frac{Ns-2N}{4s}}(\mathcal{D})$. Therefore, we get that the following bound

$$\Big\| h_\varepsilon - h \Big\|_{\mathbb{X}^{\frac{Ns-2N}{4s}}(\mathcal{D})} \le C(N,p) \Big\| h_\varepsilon - h \Big\|_{L^s(\mathcal{D})} \le C(N,p)\varepsilon. \qquad (3.29)$$

By summarizing all three evaluations (3.27), (3.28) and (3.29), we derive that

$$\Big\| \Theta_\varepsilon(.) - \theta_\varepsilon(.) \Big\|_{\mathbb{X}^m(\mathcal{D})} \le \overline{M}_2 \, (B_\varepsilon)^{m+1+\frac{N}{2s}-\frac{N}{4}} \, \varepsilon, \qquad (3.30)$$

where

$$\overline{M}_2 = \Big[M_0 \Big(1 - \exp\Big(\frac{-\lambda_1 T^\beta}{\beta} \Big) \Big) \Big] C(N,p).$$

From three steps, we can conclude that

$$\begin{aligned}
\Big\| \theta_\varepsilon - \theta \Big\|_{\mathbb{X}^m(\mathcal{D})} &\le \Big\| \Upsilon_\varepsilon(.) - \theta(.) \Big\|_{\mathbb{X}^m(\mathcal{D})} \\
&\qquad + \Big\| \Upsilon_\varepsilon(.) - \Theta_\varepsilon(.) \Big\|_{\mathbb{X}^m(\mathcal{D})} + \Big\| \Theta_\varepsilon(.) - \theta_\varepsilon(.) \Big\|_{\mathbb{X}^m(\mathcal{D})} \\
&\le |B_\varepsilon|^{-r} \big\| \theta \big\|_{\mathbb{X}^{m+r}(\mathcal{D})} \\
&\qquad + \overline{M}_1(T,s,\beta) \big\| \theta \big\|_{\mathbb{X}^m(\mathcal{D})} B_\varepsilon \varepsilon + \overline{M}_2 \, (B_\varepsilon)^{m+1+\frac{N}{2s}-\frac{N}{4}} \, \varepsilon. \qquad (3.31)
\end{aligned}$$

By using Lemma 2.1, since $0 < m < \frac{N}{4}$, we know that $\mathbb{X}^m(\mathcal{D}) \hookrightarrow L^{\frac{2N}{N-4m}}(\mathcal{D})$, which yields to the desired result (3.6).

□

4. Backward problem for linear parabolic equation with conformable operator

In this section, we consider the backward in time problem for parabolic equation with conformable operator as follows

$$\begin{cases} \dfrac{{}^C\partial^\alpha}{\partial t^\alpha} u(x,t) - \Delta u(x,t) = F(x,t), & x \in \mathcal{D}, \ t \in (0,T), \\ u(x,t) = 0, & x \in \partial\mathcal{D}, t \in (0,T), \end{cases} \qquad (4.1)$$

with the following terminal observation

$$u(x,T) = \psi(x) \quad \text{in} \quad \mathcal{D}. \qquad (4.2)$$

Let us assume that problem (4.1) has a solution u. Suppose that $u(x,0) = u_0(x)$, then we get the following equality

$$\langle u(.,t), \varphi_j\rangle = \exp\Big(-\lambda_j \frac{t^\beta}{\beta}\Big)\langle u_0, \varphi_j\rangle + \int_0^t \nu^{\beta-1} \exp\Big(-\lambda_j \frac{t^\beta - \nu^\beta}{\beta}\Big)\langle F(.,\nu), \varphi_j\rangle d\nu. \tag{4.3}$$

By giving $t = T$ into the above expression and noting that $u(x,T) = \psi(x)$, we get

$$\langle \theta, \varphi_j\rangle = \exp\Big(\frac{-\lambda_j T^\beta}{\beta}\Big)\langle u_0, \varphi_j\rangle + \int_0^T \nu^{\beta-1} \exp\Big(-\lambda_j \frac{T^\beta - \nu^\beta}{\beta}\Big)\langle F(.,\nu), \varphi_j\rangle d\nu. \tag{4.4}$$

Hence, we derive that

$$\langle u_0, \varphi_j\rangle = \exp\Big(\frac{\lambda_j T^\beta}{\beta}\Big)\Big[\langle \psi, \varphi_j\rangle - \int_0^T \nu^{\beta-1} \exp\Big(-\lambda_j \frac{T^\beta - \nu^\beta}{\beta}\Big)\langle F(.,\nu), \varphi_j\rangle d\nu\Big]. \tag{4.5}$$

Combining (4.3) and (4.5) and after a reduction, we arrive at

$$\langle u(.,t), \varphi_j\rangle = \exp\Big(\frac{\lambda_j (T^\beta - t^\beta)}{\beta}\Big)\langle \psi, \varphi_j\rangle - \int_t^T \nu^{\beta-1} \exp\Big(\lambda_j \frac{\nu^\beta - t^\beta}{\beta}\Big)\langle F(.,\nu), \varphi_j\rangle d\nu. \tag{4.6}$$

Hence, we get the explicit formula of the mild solution

$$u(x,t) = \sum_{j=1}^\infty \exp\Big(\frac{\lambda_j (T^\beta - t^\beta)}{\beta}\Big)\langle \psi, \varphi_j\rangle \varphi_j - \sum_{j=1}^\infty \Big[\int_t^T \nu^{\beta-1} \exp\Big(\lambda_j \frac{\nu^\beta - t^\beta}{\beta}\Big)\langle F(.,\nu), \varphi_j\rangle d\nu\Big]\varphi_j. \tag{4.7}$$

By applying Fourier truncation method, we provide the regularized solution as follows

$$u^\varepsilon(x,t) = \sum_{\lambda_j \le E_\varepsilon} \exp\Big(\frac{\lambda_j (T^\beta - t^\beta)}{\beta}\Big)\langle \psi_\varepsilon, \varphi_j\rangle \varphi_j - \sum_{\lambda_j \le E_\varepsilon} \Big[\int_t^T \nu^{\beta-1} \exp\Big(\lambda_j \frac{\nu^\beta - t^\beta}{\beta}\Big)\langle F_\varepsilon(.,\nu), \varphi_j\rangle d\nu\Big]\varphi_j. \tag{4.8}$$

Here E_ε called parameter regularization which is defined later.

Theorem 4.1. *Let the input data $\psi \in L^s(\mathcal{D})$ and $F \in L^\infty(0,T; L^s(\mathcal{D}))$ for any $\frac{1}{\beta} < s < 2$. Let us assume the observed data $\psi_\varepsilon \in L^s(\mathcal{D})$ and $F_\varepsilon \in$*

$L^\infty(0,T;L^s(\mathcal{D}))$ *such that*

$$\Big\|\psi_\varepsilon-\psi\Big\|_{L^s(\mathcal{D})}+\Big\|F_\varepsilon-F\Big\|_{L^\infty(0,T;L^s(\mathcal{D}))}\le\varepsilon,\ \varepsilon>0. \tag{4.9}$$

Let us assume that $u\in L^\infty(0,T;\mathbb{X}^{m+\alpha}(\mathcal{D}))$ *for any* $\alpha>0$. *Here* $0<m<\frac{N}{4}$. *Let us choose* E_ε *such that*

$$\lim_{\varepsilon\to0}E_\varepsilon=+\infty,\ \lim_{\varepsilon\to0}\Big|E_\varepsilon\Big|^{m+\frac{N}{2s}-\frac{N}{4}}\exp\Big(T^\beta\beta^{-1}E_\varepsilon\Big)\varepsilon=0. \tag{4.10}$$

Then the error $\Big\|u^\varepsilon(.,t)-u(.,t)\Big\|_{L^{\frac{2N}{N-4m}}(\mathcal{D})}$ *is of order*

$$\max\Big(\Big|E_\varepsilon\Big|^{m+\frac{N}{2s}-\frac{N}{4}}\exp\Big(T^\beta\beta^{-1}E_\varepsilon\Big)\varepsilon,\ \Big|E_\varepsilon\Big|^{-\alpha}\Big) \tag{4.11}$$

Remark 4.1. Let us choose E_ε such that

$$E_\varepsilon=T^{-\beta}\beta(1-\gamma)\log\big(\frac{1}{\varepsilon}\big),\ 0<\gamma<1. \tag{4.12}$$

Then the error $\Big\|u^\varepsilon(.,t)-u(.,t)\Big\|_{L^{\frac{2N}{N-4m}}(\mathcal{D})}$ is of order

$$\max\left(\varepsilon^\gamma\Big(\log\big(\frac{1}{\varepsilon}\big)\Big)^{m+\frac{N}{2s}-\frac{N}{4}},\ \Big[\log\big(\frac{1}{\varepsilon}\big)\Big]^{-\alpha}\right).$$

Proof. Let us give the following function

$$\begin{aligned}Z^\varepsilon(x,t)=&\sum_{\lambda_j\le E_\varepsilon}\exp\Big(\frac{\lambda_j(T^\beta-t^\beta)}{\beta}\Big)\langle\psi,\varphi_j\rangle\varphi_j\\&-\sum_{\lambda_j\le E_\varepsilon}\Big[\int_t^T\nu^{\beta-1}\exp\Big(\lambda_j\frac{\nu^\beta-t^\beta}{\beta}\Big)\langle F(.,\nu),\varphi_j\rangle d\nu\Big]\varphi_j.\end{aligned} \tag{4.13}$$

The triangle inequality implies that

$$\Big\|u^\varepsilon(.,t)-u(.,t)\Big\|_{\mathbb{X}^m(\mathcal{D})}\le\Big\|u^\varepsilon(.,t)-Z^\varepsilon(.,t)\Big\|_{\mathbb{X}^m(\mathcal{D})}+\Big\|u(.,t)-Z^\varepsilon(.,t)\Big\|_{\mathbb{X}^m(\mathcal{D})}. \tag{4.14}$$

We continue to consider the two components of the right-hand side.
Step 1.Estimation of $\Big\|u^\varepsilon(.,t)-Z^\varepsilon(.,t)\Big\|_{\mathbb{X}^m(\mathcal{D})}$.
From two definitions of two functions u^ε and Z^ε, we find that

$$\begin{aligned}&u^\varepsilon(x,t)-Z^\varepsilon(x,t)\\&=\sum_{\lambda_j\le E_\varepsilon}\exp\Big(\frac{\lambda_j(T^\beta-t^\beta)}{\beta}\Big)\langle\psi_\varepsilon-\psi,\varphi_j\rangle\varphi_j\\&-\sum_{\lambda_j\le E_\varepsilon}\Big[\int_t^T\nu^{\beta-1}\exp\Big(\lambda_j\frac{\nu^\beta-t^\beta}{\beta}\Big)\langle F_\varepsilon(.,\nu)-F(.,\nu),\varphi_j\rangle d\nu\Big]\varphi_j\\&=J_1(x,t)+J_2(x,t).\end{aligned} \tag{4.15}$$

The first term J_1 on the space $\mathbb{X}^m(\mathcal{D})$ is bounded by

$$\begin{aligned}\Big\|J_1(.,t)\Big\|^2_{\mathbb{X}^m(\mathcal{D})} &= \sum_{\lambda_j\le E_\varepsilon} \lambda_j^{2m} \exp\Big(2\frac{\lambda_j(T^\beta-t^\beta)}{\beta}\Big)\langle \psi_\varepsilon-\psi,\varphi_j\rangle^2\\ &= \sum_{\lambda_j\le E_\varepsilon} \lambda_j^{2m+\frac{N}{s}-\frac{N}{2}}\lambda_j^{\frac{Ns-2N}{2s}} \exp\Big(\frac{2\lambda_j T^\beta}{\beta}\Big)\langle \psi_\varepsilon-\psi,\varphi_j\rangle^2\\ &\le \Big|E_\varepsilon\Big|^{2m+\frac{N}{s}-\frac{N}{2}} \exp\Big(\frac{2E_\varepsilon T^\beta}{\beta}\Big)\sum_{\lambda_j\le E_\varepsilon}\lambda_j^{\frac{Ns-2N}{2s}}\langle \psi_\varepsilon-\psi,\varphi_j\rangle^2. \quad (4.16)\end{aligned}$$

Noting that the infinite sum on the right above is bounded by

$$\sum_{\lambda_j\le E_\varepsilon}\lambda_j^{\frac{Ns-2N}{2s}}\langle \psi_\varepsilon-\psi,\varphi_j\rangle^2 \le \sum_{j=1}^\infty \lambda_j^{\frac{Ns-2N}{2s}}\langle \psi_\varepsilon-\psi,\varphi_j\rangle^2 = \Big\|\psi_\varepsilon-\psi\Big\|^2_{\mathbb{X}^{\frac{Ns-2N}{4s}}(\mathcal{D})}.$$

Since the Sobolev embedding $L^s(\mathcal{D})\hookrightarrow \mathbb{X}^{\frac{Ns-2N}{4s}}(\mathcal{D})$, we have the following statement

$$\begin{aligned}&\Big\|\psi_\varepsilon-\psi\Big\|_{\mathbb{X}^{\frac{Ns-2N}{4s}}(\mathcal{D})} + \Big\|F_\varepsilon - F\Big\|_{L^\infty(0,T;\mathbb{X}^{\frac{Ns-2N}{4s}}(\mathcal{D}))}\\ &\le C(N,p)\Big\|\psi_\varepsilon-\psi\Big\|_{L^s(\mathcal{D})} + C(N,s)\Big\|F_\varepsilon-F\Big\|_{L^\infty(0,T;L^s(\mathcal{D}))} \le C(N,s)\varepsilon. \quad (4.17)\end{aligned}$$

This follows from (4.16) that

$$\Big\|J_1(.,t)\Big\|_{\mathbb{X}^m(\mathcal{D})} \le \Big|E_\varepsilon\Big|^{m+\frac{N}{2s}-\frac{N}{4}} \exp\Big(\frac{E_\varepsilon T^\beta}{\beta}\Big) C(N,s)\varepsilon. \quad (4.18)$$

Let us continue to treat the second term on the right hand side of (4.15). Indeed, we get that

$$\begin{aligned}&\Big\|J_2(.,t)\Big\|^2_{\mathbb{X}^m(\mathcal{D})}\\ &= \sum_{\lambda_j\le E_\varepsilon}\lambda_j^{2m}\Big[\int_t^T \nu^{\beta-1}\exp\Big(\lambda_j\frac{\nu^\beta-t^\beta}{\beta}\Big)\langle F_\varepsilon(.,\nu)-F(.,\nu),\varphi_j\rangle d\nu\Big]^2\\ &\le \exp\Big(\frac{2E_\varepsilon T^\beta}{\beta}\Big)\sum_{\lambda_j\le E_\varepsilon}\lambda_j^{2m}\Big[\int_t^T \nu^{\beta-1}\langle F_\varepsilon(.,\nu)-F(.,\nu),\varphi_j\rangle d\nu\Big]^2, \quad (4.19)\end{aligned}$$

where we note that $\exp\Big(\lambda_j\frac{\nu^\beta-t^\beta}{\beta}\Big)\le \exp\Big(\frac{E_\varepsilon T^\beta}{\beta}\Big)$ for $\lambda_j\le E_\varepsilon$ and $0\le t\le \nu\le T$. Using Hölder inequality, we get that

$$\begin{aligned}&\sum_{\lambda_j\le E_\varepsilon}\lambda_j^{2m}\Big[\int_t^T \nu^{\beta-1}\langle F_\varepsilon(.,\nu)-F(.,\nu),\varphi_j\rangle d\nu\Big]^2\\ &\le \sum_{\lambda_j\le E_\varepsilon}\lambda_j^{2m+\frac{N}{s}-\frac{N}{2}}\lambda_j^{\frac{Ns-2N}{2s}}\Big(\int_t^T \nu^{\beta-1}d\nu\Big)\Big[\int_t^T \nu^{\beta-1}\langle F_\varepsilon(.,\nu)-F(.,\nu),\varphi_j\rangle^2 d\nu\Big].\end{aligned} \quad (4.20)$$

It is obvious to see that

$$\lambda_j^{2m+\frac{N}{s}-\frac{N}{2}} \leq \Big|E_\varepsilon\Big|^{2m+\frac{N}{s}-\frac{N}{2}}, \text{ since } \lambda_j \leq E_\varepsilon.$$

and $\int_t^T \nu^{\beta-1} d\nu = \frac{T^\beta - t^\beta}{\beta} \leq \frac{T^\beta}{\beta}$. It follows from (4.20) that

$$\begin{aligned}
&\sum_{\lambda_j \leq E_\varepsilon} \lambda_j^{2m} \Big[\int_t^T \nu^{\beta-1} \langle F_\varepsilon(.,\nu) - F(.,\nu), \varphi_j \rangle d\nu \Big]^2 \\
&\leq \frac{T^\beta \Big|E_\varepsilon\Big|^{2m+\frac{N}{s}-\frac{N}{2}}}{\beta} \Big[\int_t^T \nu^{\beta-1} \sum_{\lambda_j \leq E_\varepsilon} \lambda_j^{2m+\frac{N}{s}-\frac{N}{2}} \langle F_\varepsilon(.,\nu) - F(.,\nu), \varphi_j \rangle^2 d\nu \Big] \\
&\leq \frac{T^\beta \Big|E_\varepsilon\Big|^{2m+\frac{N}{s}-\frac{N}{2}}}{\beta} \Big[\int_t^T \nu^{\beta-1} \Big\| F_\varepsilon(.,\nu) - F(.,\nu) \Big\|^2_{\mathbb{X}^{\frac{Ns-2N}{4s}}(\mathcal{D})} d\nu \Big].
\end{aligned} \tag{4.21}$$

The latter inequality together with Sobolev embedding $L^s(\mathcal{D}) \hookrightarrow \mathbb{X}^{\frac{Ns-2N}{4s}}(\mathcal{D})$ gives us

$$\begin{aligned}
&\sum_{\lambda_j \leq E_\varepsilon} \lambda_j^{2m} \Big[\int_t^T \nu^{\beta-1} \langle F_\varepsilon(.,\nu) - F(.,\nu), \varphi_j \rangle d\nu \Big]^2 \\
&\leq \frac{T^\beta |\overline{C}(N,s)|^2 \Big|E_\varepsilon\Big|^{2m+\frac{N}{s}-\frac{N}{2}}}{\beta} \Big[\int_t^T \nu^{\beta-1} \Big\| F_\varepsilon(.,\nu) - F(.,\nu) \Big\|^2_{L^s(\mathcal{D})} d\nu \Big] \\
&\leq \frac{T^\beta |\overline{C}(N,s)|^2 \Big|E_\varepsilon\Big|^{2m+\frac{N}{s}-\frac{N}{2}}}{\beta} \Big\| F_\varepsilon - F \Big\|^2_{L^\infty(0,T;\mathbb{X}^{\frac{Ns-2N}{4s}}(\mathcal{D}))} \Big(\int_t^T \nu^{\beta-1} d\nu \Big) \\
&\leq \frac{T^{2\beta} |\overline{C}(N,s)|^2 \Big|E_\varepsilon\Big|^{2m+\frac{N}{s}-\frac{N}{2}}}{\beta^2} \Big\| F_\varepsilon - F \Big\|^2_{L^\infty(0,T;\mathbb{X}^{\frac{Ns-2N}{4s}}(\mathcal{D}))} \\
&\leq \frac{T^{2\beta} |\overline{C}(N,s)|^2 |C(N,s)|^2 \Big|E_\varepsilon\Big|^{2m+\frac{N}{s}-\frac{N}{2}} \varepsilon^2}{\beta^2}.
\end{aligned} \tag{4.22}$$

Combining (4.19) and (4.22), we have

$$\Big\| J_2(.,t) \Big\|^2_{\mathbb{X}^m(\mathcal{D})} \leq \frac{T^{2\beta} |\overline{C}(N,s)|^2}{\beta^2} \exp\Big(2T^\beta \beta^{-1} E_\varepsilon \Big) \Big|E_\varepsilon\Big|^{2m+\frac{N}{s}-\frac{N}{2}} \varepsilon^2, \tag{4.23}$$

which allows us to obtain that

$$\Big\| J_2(.,t) \Big\|_{\mathbb{X}^m(\mathcal{D})} \leq \frac{T^\beta C(N,s)}{\beta} \exp\Big(T^\beta \beta^{-1} E_\varepsilon \Big) \Big|E_\varepsilon\Big|^{m+\frac{N}{2s}-\frac{N}{4}} \varepsilon. \tag{4.24}$$

Combining (4.15), (4.18), (4.24), we deduce that

$$\begin{aligned}
&\left\|u^\varepsilon(.,t) - Z^\varepsilon(.,t)\right\|_{\mathbb{X}^m(\mathcal{D})} \le \left\|J_1(.,t)\right\|_{\mathbb{X}^m(\mathcal{D})} + \left\|J_2(.,t)\right\|_{\mathbb{X}^m(\mathcal{D})} \\
&\le C(N,s)\Big|E_\varepsilon\Big|^{m+\frac{N}{2s}-\frac{N}{4}} \exp\Big(T^\beta \beta^{-1} E_\varepsilon\Big)\varepsilon \\
&\quad + \frac{T^\beta C(N,s)}{\beta} \exp\Big(T^\beta \beta^{-1} E_\varepsilon\Big)\Big|E_\varepsilon\Big|^{m+\frac{N}{2s}-\frac{N}{4}} \varepsilon \\
&\le \Big(C(N,s) + \frac{T^\beta C(N,s)}{\beta}\Big)\Big|E_\varepsilon\Big|^{m+\frac{N}{2s}-\frac{N}{4}} \exp\Big(T^\beta \beta^{-1} E_\varepsilon\Big)\varepsilon. \qquad (4.25)
\end{aligned}$$

Step 2. Estimation of $\Big\|u(.,t) - Z^\varepsilon(.,t)\Big\|_{\mathbb{X}^m(\mathcal{D})}$.

Due to the definitions of u and Z^ε, we derive that

$$\begin{aligned}
\Big\|u(.,t) - Z^\varepsilon(.,t)\Big\|^2_{\mathbb{X}^m(\mathcal{D})} &= \sum_{\lambda_j > E_\varepsilon} \lambda_j^{2m}\Big(\int_{\mathcal{D}} u(x,t)e_j(x)dx\Big)^2 \\
&= \sum_{\lambda_j > E_\varepsilon} \lambda_j^{-2\alpha}\lambda_j^{2m+2\alpha}\Big(\int_{\mathcal{D}} u(x,t)e_j(x)dx\Big)^2 \\
&\le \Big|E_\varepsilon\Big|^{-2\alpha}\big\|u\big\|^2_{L^\infty(0,T;\mathbb{X}^{m+\alpha}(\mathcal{D}))}. \qquad (4.26)
\end{aligned}$$

Therefore, we find that the following estimate

$$\Big\|u(.,t) - Z^\varepsilon(.,t)\Big\|_{\mathbb{X}^m(\mathcal{D})} \le \Big|E_\varepsilon\Big|^{-\alpha}\big\|u\big\|_{L^\infty(0,T;\mathbb{X}^{m+\alpha}(\mathcal{D}))}. \qquad (4.27)$$

Combining two steps and noting that $\mathbb{X}^m(\mathcal{D}) \hookrightarrow L^{\frac{2N}{N-4m}}(\mathcal{D})$ $(0 < m < \frac{N}{4})$, we deduce that

$$\begin{aligned}
&\Big\|u^\varepsilon(.,t) - u(.,t)\Big\|_{L^{\frac{2N}{N-4m}}(\mathcal{D})} \\
&\le C(m,N)\Big\|u^\varepsilon(.,t) - u(.,t)\Big\|_{\mathbb{X}^m(\mathcal{D})} \\
&\le C(m,N)\Big\|u^\varepsilon(.,t) - Z^\varepsilon(.,t)\Big\|_{\mathbb{X}^m(\mathcal{D})} + C(m,N)\Big\|u(.,t) - Z^\varepsilon(.,t)\Big\|_{\mathbb{X}^m(\mathcal{D})} \\
&\le \Big(C(N,s,m) + \frac{T^\beta C(N,s,m)}{\beta}\Big)\Big|E_\varepsilon\Big|^{m+\frac{N}{2s}-\frac{N}{4}} \exp\Big(T^\beta\beta^{-1}E_\varepsilon\Big)\varepsilon \\
&+ \Big|E_\varepsilon\Big|^{-\alpha}\big\|u\big\|_{L^\infty(0,T;\mathbb{X}^{m+\alpha}(\mathcal{D}))}. \qquad (4.28)
\end{aligned}$$

The proof of Theorem (4.1) is completed.

In the following theorem, we give a regularization result in the case that F has a split form $F(x,t) = \Psi(t)f(x)$.

Theorem 4.2. *Let us assume that the input data* Ψ, ψ, f *and the observation data* $\Psi_\varepsilon, \psi_\varepsilon, f_\varepsilon$ *such that*

$$\Big\|\Psi - \Psi_\varepsilon\Big\|_{L^s(0,T)} + \Big\|\psi - \psi_\varepsilon\Big\|_{L^s(\mathcal{D})} + \Big\|f_\varepsilon - f\Big\|_{L^s(\mathcal{D})} \le \varepsilon. \qquad (4.29)$$

Let us assume that $u \in L^\infty(0,T;\mathbb{X}^{m+\alpha}(\mathcal{D}))$ *for any* $\alpha > 0$. *Then we construct a regularized solution defined by*

$$W_\varepsilon(x,t) = \sum_{\lambda_j \le H_\varepsilon} \exp\Big(\frac{\lambda_j(T^\beta - t^\beta)}{\beta}\Big)\langle \psi_\varepsilon, \varphi_j\rangle\varphi_j - \sum_{\lambda_j \le H_\varepsilon}\Big[\int_t^T \nu^{\beta-1}\exp\Big(\lambda_j\frac{\nu^\beta - t^\beta}{\beta}\Big)\Psi_\varepsilon(\nu)d\nu\Big]\langle f_\varepsilon, \varphi_j\rangle\varphi_j. \tag{4.30}$$

Then the error $\Big\|W_\varepsilon(.,t) - u(.,t)\Big\|_{L^{\frac{2N}{N-4m}}(\mathcal{D})}$ *is of order*

$$\max\Big(\varepsilon\Big|H_\varepsilon\Big|^{-\alpha},\ \Big|H_\varepsilon\Big|^{m+\frac{N}{2s}-\frac{N}{4}}\exp\Big(\frac{T^\beta H_\varepsilon}{\beta}\Big)\varepsilon\Big). \tag{4.31}$$

Remark 4.2. Let us define

$$H_\varepsilon = \beta T^{-\beta}(1-\gamma)\log\big(\frac{1}{\varepsilon}\big). \tag{4.32}$$

Then the error $\Big\|W_\varepsilon(.,t) - u(.,t)\Big\|_{L^{\frac{2N}{N-4m}}(\mathcal{D})}$ is of order

$$\max\Big(\varepsilon\Big|\log\big(\frac{1}{\varepsilon}\big)\Big|^{-\alpha},\ \Big|\log\big(\frac{1}{\varepsilon}\big)\Big|^{m+\frac{N}{2s}-\frac{N}{4}}\varepsilon^\gamma\Big). \tag{4.33}$$

□

Proof. Since $F(x,t) = \Psi(t)f(x)$, we know that

$$u(x,t) = \sum_{j=1}^\infty \exp\Big(\frac{\lambda_j(T^\beta - t^\beta)}{\beta}\Big)\langle \psi, \varphi_j\rangle\varphi_j - \sum_{j=1}^\infty\Big[\int_t^T \nu^{\beta-1}\exp\Big(\lambda_j\frac{\nu^\beta - t^\beta}{\beta}\Big)\psi(\nu)d\nu\Big]\langle f, \varphi_j\rangle\varphi_j. \tag{4.34}$$

The triangle inequality allows us to obtain that

$$\Big\|W_\varepsilon(.,t) - u(.,t)\Big\|_{\mathbb{X}^m(\mathcal{D})} \le \Big\|W_\varepsilon(.,t) - U_\varepsilon(.,t)\Big\|_{\mathbb{X}^m(\mathcal{D})} + \Big\|U_\varepsilon(.,t) - u(.,t)\Big\|_{\mathbb{X}^m(\mathcal{D})}, \tag{4.35}$$

where

$$U_\varepsilon(x,t) = \sum_{\lambda_j \le H_\varepsilon} \exp\Big(\frac{\lambda_j(T^\beta - t^\beta)}{\beta}\Big)\langle \psi, \varphi_j\rangle\varphi_j - \sum_{\lambda_j \le H_\varepsilon}\Big[\int_t^T \nu^{\beta-1}\exp\Big(\lambda_j\frac{\nu^\beta - t^\beta}{\beta}\Big)\Psi(\nu)d\nu\Big]\langle f, \varphi_j\rangle\varphi_j. \tag{4.36}$$

It is worth noting that the second component on the right hand side of (4.35) is exactly the same as step 2 of Theorem 4.1. Therefore, we get that

$$\Big\|u(.,t) - W_\varepsilon(.,t)\Big\|_{\mathbb{X}^m(\mathcal{D})} \le \Big|H_\varepsilon\Big|^{-\alpha}\|u\|_{L^\infty(0,T;\mathbb{X}^{m+\alpha}(\mathcal{D}))}. \tag{4.37}$$

Let us contitnute to treat the first term on the right hand side of (4.35). It is easy to see that

$$W_\varepsilon(x,t) - U_\varepsilon(x,t) = \overline{J}_1(x,t) + \overline{J}_2(x,t) + \overline{J}_3(x,t). \tag{4.38}$$

Here we have that

$$\begin{aligned} \overline{J}_1(x,t) &= \sum_{\lambda_j \le H_\varepsilon} \exp\Big(\frac{\lambda_j(T^\beta - t^\beta)}{\beta}\Big) \langle \psi_\varepsilon - \psi, \varphi_j\rangle \varphi_j, \\ \overline{J}_2(x,t) &= -\sum_{\lambda_j \le H_\varepsilon} \Big[\int_t^T \nu^{\beta-1} \exp\Big(\lambda_j \frac{\nu^\beta - t^\beta}{\beta}\Big)\Psi_\varepsilon(\nu) d\nu\Big] \langle f_\varepsilon - f, \varphi_j\rangle\varphi_j, \end{aligned} \tag{4.39}$$

and

$$\overline{J}_3(x,t) = \sum_{\lambda_j \le H_\varepsilon} \Big[\int_t^T \nu^{\beta-1} \exp\Big(\lambda_j \frac{\nu^\beta - t^\beta}{\beta}\Big)\Big(\Psi(\nu) - \Psi_\varepsilon(\nu)\Big) d\nu\Big] \langle f, \varphi_j\rangle\varphi_j. \tag{4.40}$$

It is obvious to find that the form $\overline{J}_1$ is the same as that of J_1 on the right hand side of (4.15). Hence, we obtain that

$$\|\overline{J}_1(.,t)\|_{\mathbb{X}^m(\mathcal{D})} \le C(N,s)\Big|H_\varepsilon\Big|^{m+\frac{N}{2s}-\frac{N}{4}} \exp\Big(\frac{H_\varepsilon T^\beta}{\beta}\Big)\varepsilon. \tag{4.41}$$

Our next aim is to treating the term $\Big\|\overline{J}_2(.,t)\Big\|_{\mathbb{X}^m(\mathcal{D})}$. In view of Parseval's equality, one has

$$\|\overline{J}_2(.,t)\|^2_{\mathbb{X}^m(\mathcal{D})} = \sum_{\lambda_j \le H_\varepsilon} \lambda_j^{2m} \Big[\int_t^T \nu^{\beta-1} \exp\Big(\lambda_j \frac{\nu^\beta - t^\beta}{\beta}\Big)\Psi_\varepsilon(\nu) d\nu\Big]^2 \langle f_\varepsilon - f, \varphi_j\rangle^2. \tag{4.42}$$

Thanks to Hölder inequality, we derive that for $s > 1$ and $s^* = \frac{s}{s-1}$

$$\begin{aligned} &\Big|\int_t^T \nu^{\beta-1} \exp\Big(\lambda_j \frac{\nu^\beta - t^\beta}{\beta}\Big)\Psi_\varepsilon(\nu) d\nu\Big| \\ &\quad \le \Big(\int_0^T |\Psi_\varepsilon(\nu)|^s d\nu\Big)^{1/s}\Big(\int_t^T \nu^{s^*(\beta-1)} \exp\Big(s^*\lambda_j \frac{\nu^\beta - t^\beta}{\beta}\Big) d\nu\Big)^{1/s^*} \\ &\quad \le \exp\Big(\frac{T^\beta\lambda_j}{\beta}\Big)\Big\|\Psi_\varepsilon\Big\|_{L^s(0,T)}\Big(\int_t^T \nu^{s^*(\beta-1)} d\nu\Big)^{1/s^*} \\ &\qquad \le \Big(\frac{s-1}{s\beta-1}\Big)^{\frac{s-1}{s}} T^{\frac{s\beta-1}{s}} \exp\Big(\frac{T^\beta\lambda_j}{\beta}\Big)\Big\|\Psi_\varepsilon\Big\|_{L^s(0,T)}. \end{aligned} \tag{4.43}$$

The latter inequality leads to

$$
\begin{aligned}
&\left\|\overline{J}_2(.,t)\right\|^2_{\mathbb{X}^m(\mathcal{D})}\\
&= \sum_{\lambda_j \le H_\varepsilon} \lambda_j^{2m} \Big[\int_t^T \nu^{\beta-1} \exp\Big(\lambda_j \frac{\nu^\beta - t^\beta}{\beta}\Big) \Psi_\varepsilon(\nu) d\nu \Big]^2 \langle f_\varepsilon - f, \varphi_j\rangle^2\\
&\le \Big(\frac{s-1}{s\beta-1}\Big)^{\frac{2s-2}{s}} T^{\frac{2s\beta-2}{s}} \Big\|\Phi_\varepsilon(\theta)\Big\|^2_{L^s(0,T)}\\
&\qquad \times \sum_{\lambda_j \le H_\varepsilon} \lambda_j^{2m+\frac{N}{s}-\frac{N}{2}} \lambda_j^{\frac{Ns-2N}{2s}} \exp\Big(\frac{2T^\beta \lambda_j}{\beta}\Big) \langle f_\varepsilon - f, \varphi_j\rangle^2\\
&\le \Big(\frac{s-1}{s\beta-1}\Big)^{\frac{2s-2}{s}} T^{\frac{2s\beta-2}{s}} \Big\|\Phi_\varepsilon(\theta)\Big\|^2_{L^s(0,T)}\\
&\qquad \times \Big|H_\varepsilon\Big|^{2m+\frac{N}{s}-\frac{N}{2}} \exp\Big(\frac{2T^\beta H_\varepsilon}{\beta}\Big) \Big\|f_\varepsilon - f\Big\|^2_{\mathbb{X}^{\frac{Ns-2N}{4s}}(\mathcal{D})}, \qquad (4.44)
\end{aligned}
$$

where $s > \frac{1}{\beta}$. In view of Sobolev embedding $L^s(\mathcal{D}) \hookrightarrow \mathbb{X}^{\frac{Ns-2N}{4s}}(\mathcal{D})$, we derive that the following estimate

$$
\begin{aligned}
&\left\|\overline{J}_2(.,t)\right\|_{\mathbb{X}^m(\mathcal{D})}\\
&\le \overline{A}(s,\beta,T,N) \Big\|\Phi_\varepsilon\Big\|_{L^s(0,T)} \Big|H_\varepsilon\Big|^{m+\frac{N}{2s}-\frac{N}{4}} \exp\Big(\frac{T^\beta H_\varepsilon}{\beta}\Big) \big\|f_\varepsilon - f\big\|_{L^s(\mathcal{D})}\\
&\le \overline{A}(s,\beta,T,N) \Big\|\Phi_\varepsilon\Big\|_{L^s(0,T)} \Big|H_\varepsilon\Big|^{m+\frac{N}{2s}-\frac{N}{4}} \exp\Big(\frac{T^\beta H_\varepsilon}{\beta}\Big) \varepsilon, \qquad (4.45)
\end{aligned}
$$

where

$$
\overline{A}(s,\beta,T,N) = \Big(\frac{s-1}{s\beta-1}\Big)^{\frac{s-1}{s}} T^{\frac{s\beta-1}{s}} C(N,s).
$$

Let us now to consider the term $\left\|\overline{J}_3(.,t)\right\|_{\mathbb{X}^m(\mathcal{D})}$. By applying Hölder inequality, we get that for $s > 1$ and $s^* = \frac{s}{s-1}$

$$
\begin{aligned}
&\Big| \int_t^T \nu^{\beta-1} \exp\Big(\lambda_j \frac{\nu^\beta - t^\beta}{\beta}\Big) \Big(\Psi(\nu) - \Psi_\varepsilon(\nu)\Big) d\nu \Big|\\
&\quad \le \Big(\int_0^T |\Psi(\nu) - \Psi_\varepsilon(\nu)|^s \, d\nu \Big)^{1/s} \Big(\int_t^T \nu^{s^*(\beta-1)} \exp\Big(s^* \lambda_j \frac{\nu^\beta - t^\beta}{\beta}\Big) d\nu \Big)^{1/s^*}\\
&\quad \le \exp\Big(\frac{T^\beta \lambda_j}{\beta}\Big) \Big\|\Psi - \Psi_\varepsilon\Big\|_{L^s(0,T)} \Big(\int_t^T \nu^{s^*(\beta-1)} d\nu \Big)^{1/s^*}\\
&\quad \le \Big(\frac{s-1}{s\beta-1}\Big)^{\frac{s-1}{s}} T^{\frac{s\beta-1}{s}} \exp\Big(\frac{T^\beta \lambda_j}{\beta}\Big) \Big\|\Psi - \Psi_\varepsilon\Big\|_{L^s(0,T)}\\
&\qquad \le \Big(\frac{s-1}{s\beta-1}\Big)^{\frac{s-1}{s}} T^{\frac{s\beta-1}{s}} \exp\Big(\frac{T^\beta \lambda_j}{\beta}\Big) \varepsilon. \qquad (4.46)
\end{aligned}
$$

This inequality together with Parseval's equality allows us to derive that

$$\begin{aligned}
&\big\|\overline{J}_3(.,t)\big\|^2_{\mathbb{X}^m(\mathcal{D})}\\
&= \sum_{\lambda_j \le H_\varepsilon} \lambda_j^{2m} \Big[\int_t^T \nu^{\beta-1} \exp\Big(\lambda_j \frac{\nu^\beta - t^\beta}{\beta}\Big)\Big(\Psi(\nu) - \Psi_\varepsilon(\nu)\Big) d\nu\Big]^2 \langle f, \varphi_j\rangle^2\\
&\le \Big(\frac{s-1}{s\beta-1}\Big)^{\frac{2s-2}{s}} T^{\frac{2s\beta-2}{s}} \varepsilon^2 \sum_{\lambda_j \le H_\varepsilon} \lambda_j^{2m+\frac{N}{s}-\frac{N}{2}} \lambda_j^{\frac{Ns-2N}{2s}} \exp\Big(\frac{2T^\beta \lambda_j}{\beta}\Big) \langle f, \varphi_j\rangle^2\\
&\le \Big(\frac{s-1}{s\beta-1}\Big)^{\frac{2s-2}{s}} T^{\frac{2s\beta-2}{s}} \varepsilon^2 \Big|H_\varepsilon\Big|^{2m+\frac{N}{s}-\frac{N}{2}} \exp\Big(\frac{2T^\beta H_\varepsilon}{\beta}\Big) \big\|f\big\|^2_{\mathbb{X}^{\frac{Ns-2N}{4s}}(\mathcal{D})}. \qquad (4.47)
\end{aligned}$$

By the fact that $L^s(\mathcal{D}) \hookrightarrow \mathbb{X}^{\frac{Ns-2N}{4s}}(\mathcal{D})$, we deduce that

$$\big\|\overline{J}_3(.,t)\big\|_{\mathbb{X}^m(\mathcal{D})} \le \overline{A}(s,\beta,T,N) \Big\|f\Big\|_{L^s(\mathcal{D})} \Big|H_\varepsilon\Big|^{m+\frac{N}{2s}-\frac{N}{4}} \exp\Big(\frac{T^\beta H_\varepsilon}{\beta}\Big)\varepsilon. \qquad (4.48)$$

Combining (4.35), (4.37), (4.38), (4.41), (4.45) and (4.48), we infer that

$$\begin{aligned}
\Big\|W_\varepsilon(.,t) - u(.,t)\Big\|_{\mathbb{X}^m(\mathcal{D})} &\le \big\|W_\varepsilon(.,t) - U_\varepsilon(.,t) + \Big\|u(.,t) - W_\varepsilon(.,t)\Big\|_{\mathbb{X}^m(\mathcal{D})}\\
&\le \Big|H_\varepsilon\Big|^{-\alpha} \big\|u\big\|_{L^\infty(0,T;\mathbb{X}^{m+\alpha}(\mathcal{D}))} + \big\|\overline{J}_1(.,t)\big\|_{\mathbb{X}^m(\mathcal{D})}\\
&\qquad + \big\|\overline{J}_2(.,t)\big\|_{\mathbb{X}^m(\mathcal{D})} + \big\|\overline{J}_3(.,t)\big\|_{\mathbb{X}^m(\mathcal{D})}\\
&\le \varepsilon \Big|H_\varepsilon\Big|^{-\alpha} \big\|u\big\|_{L^\infty(0,T;\mathbb{X}^{m+\alpha}(\mathcal{D}))}\\
&\qquad + \overline{A}(s,\beta,T,N) \Big(\Big\|\Phi_\varepsilon\Big\|_{L^s(0,T)} + \Big\|f\Big\|_{L^s(\mathcal{D})} + C(N,s)\Big)\\
&\qquad \times \Big|H_\varepsilon\Big|^{m+\frac{N}{2s}-\frac{N}{4}} \exp\Big(\frac{T^\beta H_\varepsilon}{\beta}\Big)\varepsilon. \qquad (4.49)
\end{aligned}$$

The proof is completed.

□

5. Conclusion

Our paper is the first work to investigate the inverse problem for conformable parabolic equation, we consider the source function $F(x,t)$ which has two forms as follows

- In case one, the source function $F(x,t) = \varphi(t)f(x)$, using the the Fourier regularized solution, we have the error estimate between the approximate solution and the sought solution with two data $\varphi \in L^s(0,T)$ and $\psi \in L^s(\mathcal{D})$.
- In case two, for the backward in time problem (1.1), we also use Fourier truncation method to approximate problem.

The main techniques of the paper is Sobolev embeddings and Hölder inequality.

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