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MILD SOLUTIONS TO A TIME-FRACTIONAL DIFFUSION EQUATION WITH A HYPER-BESSEL OPERATOR HAVE A CONTINUOUS DEPENDENCE WITH REGARD TO FRACTIONAL DERIVATIVE ORDERS

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Abstract. In the current work, this article studies a time-fractional diffusion equation with a Hyper-Bessel operator. The time-fractional derivative is understood in the sense of a regularized hyper-Bessel operator. First, we represent some stability results on the parameters of the Mittag-Leffler functions. Then, in our primary results, we concentrate on analyzing the continuity of the solution of the initial problem that corresponds to the fractional-order. One of the issues encountered when we do this work is estimating all constants independently of fractional orders. Our main idea is to merge Mittag-Leffler function theories, the Banach fixed point theorem, and Sobolev embeddings to achieve a good result.

1. Introduction

Many problems nowadays are related to the time-fractional diffusion equation, which depends on fractional orders. Fractional calculus was used to model processes in physics and engineering, such as porous media [1], predator-prey dynamics [16], and references to these topics. Fractional derivatives come in a variety of forms, including Riemann-Liouville, Caputo, Grunwald, Letnikov, Weyl, and Caputo-Fabrizio. For the physical device under consideration, we frequently need to apply a specific fractional operator. The fractional operators the derivative of Caputo and the derivative of Riemann-Liouville. are two that we usually explore. Many scientists have believed in recent decades that differential equations with fractional derivatives provide a logical foundation for modeling discussion of these types of real-world situations. Fractional derivatives are also used in chemistry in problems like viscous structures, signal processing, diffusion processes, and control processing, [8],[13],[18]. Fractional PDEs have a wide range of uses in a variety of fields, we can see [15, 10, 23, 2, 19, 21, 11, 14, 20, 12, 25, 4]. In this

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paper, we look at the following time fractional diffusion equation.

$$\begin{cases} {}^{\mathscr{C}}\left(t^b\partial_t\right)^a \mathrm{u}(x,t) + \mathscr{A}\mathrm{u}(x,t) = \mathscr{F}\big(\mathrm{u}(x,t)\big), & (x,t) \in \Omega \times (0,T), \\ \mathrm{u}(x,t) = 0, & (x,t) \in \partial\Omega \times (0,T), \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with sufficiently smooth boundary $\partial\Omega$ and T is a positive number, ${}^{\mathscr{C}}\left(t^b\partial_t\right)^a$ is called a regularized Caputo-like counterpart hyper-Bessel operator of order $a \in (0,1), b \in (0,1)$ defined by(see [9])

$$ {}^{\mathscr{C}}\left(t^b\partial_t\right)^a \mathrm{u}(x,t) = \left(t^b\partial_t\right)^a \mathrm{u}(x,t) - \frac{\mathrm{u}(x,0)t^{-a(1-b)}}{(1-b)^{-a}\Gamma(1-a)} \tag{1.2}$$

where $\Gamma(\mu) := \int_0^\infty t^{\mu-1}e^{-t}\mathrm{d}t$ is the Gamma function, $\mathscr{F}$ is a nonlinear function which satisfies some specific assumptions, $\mathscr{A}$ is a symmetric uniformly elliptic operator on Ω and f is the initial data of the problem. The hyper-Bessel operator, or $\left(t^b\partial_t\right)^a$, was introduced by Dimovski [7]. We can see from (1.2) that the study of (1.1) is based on the definition of the hyper-Bessel operator, which is inspired by the fractional extension of the diffusion equation governing the law of fractional Brownian motion. The hyper-Bessel operator is used in several articles([9]) to describe heat diffusion in fractional Brownian motion. Al-Musalhi et al.[3] also provide more information on them.

The following two problems for time fractional diffusion equations (1.1) are of interest to us. The first problem is called an initial value problem, in which we represent a solution $u(x,t)$ satisfying equations (1.1) and the initial condition $u(x,0) = f(x)$. The second problem is referred to as a final value problem because the solution u(x, t) must satisfy equation (1.1) and the final condition u(x, T) = g(x). The results for equation (1.1) were research by some recent work (see [26], [3]). Two direct and inverse source problems of a fractional diffusion equation with a regularized Caputo-like counterpart hyper-Bessel operator were studied by the authors[3]. They proved the existence and uniqueness of the problem's solutions and provided detailed eigenfunction expansions. In work [26, 22] the author used fixed point theorems to analyze the explicit solution of the inhomogeneous linear equation and the semilinear equation. To the best of our knowledge, there are only a few papers on both cases above. In practice, many problems involving time-space fractional equations depend on fractional parameters, i.e., fractional derivative orders. Nevertheless,these fractional parameters are unknown a priori in the modeling process. As a result, for modeling purposes, the stability of solutions on these parameters is critical. Besides that, if this continuity is not maintained, numerical computations are not permitted.

Our first purpose in this paper is to investigate the global existence solution in the case of the globally Lipschitz $\mathscr{F}$and estimate the dependence of fractional order of the time. The second purpose of this article is that establish the continuity of fractional-time derivative of solution. Let us now summarize the key points of the article. Section 2 provides some basic definitions and preliminaries. Section 3 gives a formula of a mild solution, and several lemmas can be related to the main objective of the article. Finally, we apply the results of the estimates above to establish the existence and uniqueness of solution u to (1.1) as well as evaluate the evenness and parameter dependence of the solution under the semi-linear term source for the inital value problem.

2. Some fundamental fractional calculus outcomes

2.1. The Mittag-Leffler function.

Definition 2.1. For $\mu \in \mathbb{C}$ and $a > 0, b > 0$, the Mittag-Leffler function is a function which is signed and determined as follows:

$$E_{a,b}(\mu) = \sum_{n=1}^{\infty} \frac{\mu^n}{\Gamma(na+b)}. \tag{2.1}$$

If b is equal to one, we can denote $E_a(\mu) := E_{a,1}(\mu)$.

There are some lemmas related to the Mittag-Leffler function that are useful for applying the main results of Section 3 and Section 4 below.

Lemma 2.1. *Let $0 < c < d < 1$ and $a \in (c,d)$. There are positive numbers $\mathcal{M}_1$, $\mathcal{M}_2$, and $\mathcal{M}_3$ only depend on c, d such that*

$$\frac{\mathcal{M}_1(c,d)}{1+t} \le |E_a(-t)| \le \frac{\mathcal{M}_2(c,d)}{1+t} \tag{2.2}$$

$$|E_{a,a}(-t)| \le \frac{\mathcal{M}_3(c,d)}{1+t}, \qquad \textit{for all}\, t > 0. \tag{2.3}$$

Proof. Using Lemma 2.3 of [6]. □

Lemma 2.2. *Let $\lambda > 0$ and $0 < a < 1$. For all $t > 0$. Then*

$$\frac{d}{dt} E_a(-\lambda t^a) = -\lambda t^{a-1} E_{a,a}(-\lambda t^a), \tag{2.4}$$

$$\frac{d}{dt}\left(t^{a-1}E_{a,a}(-\lambda t^a)\right) = t^{a-2}E_{a,a-1}(-\lambda t^a), \tag{2.5}$$

$$\partial_t^a E_a(-\lambda t^a) = -\lambda E_{a,a}(-\lambda t^a), \tag{2.6}$$

$$\partial_t^a \left(t^{a-1}E_{a,a}(-\lambda t^a)\right) = -\lambda t^{a-1} E_{a,a}(-\lambda t^a). \tag{2.7}$$

Proof. Applying Lemma 2.2 of [24]. □

Lemma 2.3. *Let $0 < c < a_1 < a_2 < d$ and $0 < t \le T$. For any $\epsilon > 0$ which independent on a_1, a_2, there always exists $\mathcal{M}_\epsilon$ such that*

$$|t^{a_1} - t^{a_2}| \le \max(T^{d+2\epsilon}, 1)\mathcal{M}_\epsilon (a_2 - a_1)^\epsilon t^{a_1-\epsilon}. \tag{2.8}$$

Proof. See [24, Lemma 3.2] □

Lemma 2.4. [24, Section 3] *Assume that $0 < c < a_1 < a_2 < d < 1$ and $\epsilon > 0$. Then there exists a positive constant $\mathcal{M}_1(c,d,\epsilon,\theta,T)$*

$$\begin{aligned} &|E_{a_2}(-\lambda_j t^{a_2}) - E_{a_1}(-\lambda_j t^{a_1})| \\ &\qquad \le \mathcal{M}_1(c,d,\epsilon,\theta,T)\lambda_j^{\theta-1} t^{-d(1-\theta)-\epsilon}\Big[(a_2-a_1)^\epsilon + (a_2-a_1)\Big] \end{aligned} \tag{2.9}$$

for any $0 \le \theta \le 1$ and $0 < t \le T$.

Lemma 2.5. [24, Section 3] *Assume that* $0 < c < a_1 < a_2 < d < 1$. *For any* $0 \le \theta \le 1$ *and* $\epsilon > 0$ *there exists a positive constant* $\mathcal{M}_2(c, d, \epsilon, \theta, T)$

$$\Big|t^{a_2-1}E_{a_2,a_2}(- \lambda_j t^{a_2}) - t^{a_1-1}E_{a_1,a_1}(-\lambda_j t^{a_1})\Big| \\ \le \mathcal{M}_2(c, d, \epsilon, \theta, T)\lambda_j^{\theta-1}t^{c\theta-\epsilon-1}\Big[(a_2 - a_1)^{\epsilon} + (a_2 - a_1)\Big]. \quad (2.10)$$

Lemma 2.6. [5, Lemma 8] *Assume exist two number* $\mu_1 > -1, \mu_2 > -1$ *such that* $\mu_1 + \mu_2 > -1$ *and* $\mu_3 > -1$*then the following estimate holds*

$$\mathscr{C}^{\mu_3}_{\mu_1,\mu_2}(\gamma) := \sup_{t\in[0,T]} t^{\mu_3} \int_0^1 s^{\mu_1}(1 - s)^{\mu_2}e^{-\gamma t(1-s)}\mathrm{d}s \xrightarrow{r\to\infty} 0. \quad (2.11)$$

2.2. Function spaces. In this subsection, we introduce some important function spaces and useful notations. Noting that $L^2(\Omega)$, $H^1_0(\Omega)$, $H^2(\Omega)$ are understood in the usual sense. The symmetric uniform elliptic operator $\mathscr{L} : L^2(\Omega) \to L^2(\Omega)$ is defined by

$$\mathscr{L}\mathrm{u}(\mathrm{x}) = -\sum_{i=1}^{N}\frac{\partial}{\partial x_i}\left(\mathscr{L}_{ij}(\mathrm{x})\frac{\partial}{\partial x_j}\mathrm{u}(\mathrm{x})\right) + \mathscr{L}(\mathrm{x}), \mathrm{x} \in \overline{\Omega}. \quad (2.12)$$

where $D(\mathscr{L}) = H^1_0(\Omega)\cap H^2(\Omega)$, with assumptions that $\mathscr{L}(\mathrm{x}) \in C(\overline{\Omega}, [0, \infty), \mathscr{L}_{ij} \in C^1(\overline{\Omega}), \mathscr{L}_{ij} = \mathscr{L}_{ji}$, $1 \le i, j \le N$, and there exist a positive constant $\widetilde{\mathscr{L}} > 0$, for $\mathrm{x} \in \overline{\Omega}, \mu = (\mu_1, \mu_2, \cdots, \mu_N) \subset R^N$, such that

$$\widetilde{\mathscr{L}}\sum_{i=1}^{N}\mu_i^2 \le \sum_{1\le i,j\le N}\mu_i\mathscr{L}_{ij}(\mathrm{x})\mu_j, \quad (2.13)$$

see e.g. [17, Section 2]. Let us recall that the following spectral problem

$$\mathscr{L}\varphi_j(\mathrm{x}) = \lambda_j\varphi_j(\mathrm{x}) \text{ in } \Omega \quad \text{and} \quad \varphi_j(\mathrm{x}) = 0 \text{ on } \partial\Omega, \quad (2.14)$$

where $\{\varphi_j\}_{j\in\mathbb{Z}^+}$ is a orthonormal basis of $L^2(\Omega)$, admits a family of eigenvalues $\{\lambda_j\}_{j\in\mathbb{Z}^+}$ satisfying

$$0 < \lambda_1 \le \lambda_2 \le \lambda_3 \le ... \le \lambda_j \le ... \nearrow \infty.$$

Definition 2.2. For $q > 0$, we define the Hilbert scale space $\mathbb{H}^q(\Omega)$ in the following way

$$\mathbb{H}^q(\Omega) = \left\{ \mathrm{u} \in \mathbb{L}^2(\Omega) : \|\mathrm{u}\|^2_{\mathbb{H}^q(\Omega)} := \sum_{j=1}^{\infty}|\langle \mathrm{u}, \varphi_j\rangle|^2\lambda_j^{2q} < \infty \right\}, \quad (2.15)$$

where $\langle \mathrm{u}, \varphi_j\rangle := \int_\Omega \mathrm{u}(x)\varphi_j(x)\mathrm{d}x$ is the inner product of $\mathbb{L}^2(\Omega)$.

We can see that $\mathbb{H}^0(\Omega) = L^2(\Omega)$. Let us denote that $\mathbb{H}^q(\Omega)$ has dual space $\mathbb{H}^{-q}(\Omega)$ which is a Hilbert space with respect to the norm

$$\|\mathrm{u}\|^2_{\mathbb{H}^{-q}(\Omega)} := \sum_{j=1}^{\infty}\lambda_j^{-2q}\big|\mathrm{u}, \varphi_j\big|^2_{-q,q}, \quad (2.16)$$

where $\big|\mathrm{u}, \varphi_j\big|$ is dual product between $\mathbb{H}^{-q}(\Omega)$ and $\mathbb{H}^q(\Omega)$.

Definition 2.3. Let $(\mathbb{X}, \|\cdot\|)$ be a Banach space. For $1 \le p \le \infty$, we denote by $\mathbb{L}^p(0,T,\mathbb{X})$ the function space includes all measurable functions $\mathrm{u} : (0,T) \to \mathbb{X}$ with the norm

$$\|\mathrm{u}\|_{\mathbb{L}^p(0,T,\mathbb{X})} := \left(\int_0^T \|\mathrm{u}(t)\|_{\mathbb{X}}^p \mathrm{d}t\right)^{\frac{1}{p}}, \quad 1 \le p < \infty, \tag{2.17}$$

$$\big\|\mathrm{u}\big\|_{\mathbb{L}^\infty(0,T,\mathbb{X})} := \operatorname*{ess\,sup}_{t\in(0,T)} \big\|\mathrm{u}(t)\big\|_{\mathbb{X}}. \tag{2.18}$$

Also, for the real number $\varrho > 0$, we define the Banach space $\mathbb{L}^\infty_\varrho(0,T,\mathbb{X})$ of all functions $\mathrm{u} \in \mathbb{L}^\infty(0,T,\mathbb{X})$ such that

$$\big\|\mathrm{u}\big\|_{\mathbb{L}^\infty_\varrho((0,T,\mathbb{X})} := \operatorname*{ess\,sup}_{t\in(0,T)} e^{-\varrho t^r} \big\|\mathrm{u}(t)\big\|_{\mathbb{X}} < \infty. \tag{2.19}$$

Definition 2.4. We denote by $\mathbb{C}((0,T];\mathbb{X})$ the space of all continuous functions which map $(0,T]$ into $\mathbb{X}$. For $\vartheta \ge 0$ and $\varrho > 0$, we use the notation $\mathbb{C}^\vartheta_\varrho((0,T];\mathbb{X})$ to indicate the subspace of $\mathbb{C}((0,T];\mathbb{X}))$ such that

$$\|\mathrm{u}\|_{\mathbb{C}^\vartheta_\varrho((0,T];\mathbb{X})} := \sup_{0<t\le T} t^\vartheta e^{-\varrho t^r} \|\mathrm{u}(t)\|_{\mathbb{X}} < \infty, \qquad \mathrm{u} \in \mathbb{C}((0,T];\mathbb{X}), \tag{2.20}$$

2.3. Representation of the mild solution. Assume that problem (1.1) has a unique solution, we can transform the form of u in $L^2(\Omega)$ become $\mathrm{u}(x,t) = \sum_{j=1}^\infty \langle \mathrm{u}(\cdot,t), \varphi_j(\cdot)\rangle \varphi_j(x)$. Therefore, we get

$$^{\mathscr{C}}\left(t^b\partial_t\right)^a \langle \mathrm{u}(\cdot,t), \varphi_j\rangle + \langle \mathscr{A}\mathrm{u}(\cdot,t), \varphi_j\rangle = \langle \mathscr{F}((\cdot,t,\mathrm{u}(\cdot,t)), \varphi_j)\rangle. \tag{2.21}$$

Using the Laplace transform method with $\mathrm{u}(x,0) = f(x) = \sum_{j=1}^\infty \langle f_j, \varphi_j(\cdot)\rangle \varphi_j(x)$, we obtain

$$\begin{aligned}\mathrm{u}_j(t) &= E_a\left(-\frac{\lambda_j}{(1-b)^a} t^{a(1-b)}\right) f_j \\ &+ \frac{1}{(1-b)^a}\int_0^t (t^{1-b} - s^{1-b})^{a-1} E_{a,a}\left(-\frac{\lambda_j}{(1-b)^a}(t^{1-b} - s^{1-b})^a\right) \mathscr{F}_j(s,\mathrm{u}(s))\mathrm{d}s^{1-b}.\end{aligned} \tag{2.22}$$

As a result, the mild solution formula can be written as follows.

$$\begin{aligned}\mathrm{u}(x,t) &= \sum_{j=1}^\infty E_a\left(-\lambda_j(\frac{t^r}{r})^a\right) f_j\varphi_j(\mathrm{x}) \\ &+ \sum_{j=1}^\infty \left[\frac{1}{r}\int_0^t (\frac{t^r - s^r}{r})^{a-1} E_{a,a}\left(-\lambda_j(\frac{t^r - s^r}{r})^a\right) \mathscr{F}_j(s,\mathrm{u}(s))\mathrm{d}s^r\right] \varphi_j(x).\end{aligned} \tag{2.23}$$

where $r := 1 - b$ The proof will be based on the results presented in Section 2.1, as well as the estimate some differences of Mittag–Leffler functions. From (2.23), we obtain the solution of Problem (1.1) is given by

$$\mathrm{u}_a(x,t) = \mathcal{M}_a(\frac{t^r}{r})f + \int_0^t \mathcal{N}_a(\frac{t^r - s^r}{r})\mathscr{F}(\mathrm{u}(s))\mathrm{d}s^r. \tag{2.24}$$

where

$$\mathcal{M}_a(\frac{t^r}{r})f = \sum_{j=1}^{\infty} E_a\left(-\lambda_j(\frac{t^r}{r})^a\right) f_j\varphi_j(x), \tag{2.25}$$

$$\begin{aligned}&\mathcal{N}_a(\frac{t^r-s^r}{r})\mathscr{F}(\mathrm{u}(s))\\ &= \sum_{j=1}^{\infty}\left[\frac{1}{r}\int_0^t(\frac{t^r-s^r}{r})^{a-1}E_{a,a}\left(-\lambda_j(\frac{t^r-s^r}{r})^a\right)\mathscr{F}_j(s,\mathrm{u}(s))\mathrm{d}s^r\right]\varphi_j(x).\end{aligned} \tag{2.26}$$

3. The continuous dependence of mild solutions on the fractional-order.

Assume that the source functions $\mathscr{F}$ satisfy the globally Lipschitz condition, as shown below.

$$\begin{cases}\|\mathscr{F}(\mathrm{u})-\mathscr{F}(\mathrm{v})\|_{\mathbb{H}^p(\Omega)} < C_{\mathscr{F}}\|\mathrm{u}-\mathrm{v}\|_{\mathbb{H}^q(\Omega)},\\ \mathscr{F}(0)=0,\end{cases} \tag{3.1}$$

where $C_{\mathscr{F}}$ is a positive constants and p, q are two constants such that $0 < q-p < \frac{1}{2}$.

Lemma 3.1. *Let $0 < c < a_1 \le a \le a_2 < d < 1$ and $\Theta \in \mathbb{H}^p(\Omega)$. The following inequalities hold:*

$$\|\mathcal{M}_a(\frac{t^r}{r})\Theta\|_{\mathbb{H}^q(\Omega)} \le \overline{\mathcal{M}}_2(c,d)(\frac{t^r}{r})^{-d(q-p)}\|\Theta\|_{\mathbb{H}^p(\Omega)}, \tag{3.2}$$

$$\|\mathcal{N}_a(\frac{t^r-s^r}{r})\Theta\|_{\mathbb{H}^q(\Omega)} \le \overline{\mathcal{N}}_3(c,d)(\frac{t^r-s^r}{r})^{c(1+p-q)-1}\|\Theta\|_{\mathbb{H}^p(\Omega)}, \tag{3.3}$$

$$\begin{aligned}&\left\|\mathcal{M}_{a_2}(\frac{t^r}{r})\Theta-\mathcal{M}_{a_1}(\frac{t^r}{r})\Theta\right\|_{\mathbb{H}^q(\Omega)}\\ &\qquad\le \frac{\mathcal{M}_1(c,d,\epsilon,\theta,T)}{(\frac{t^r}{r})^{d(q-p)+\epsilon}}\Big[(a_2-a_1)^\epsilon+(a_2-a_1)\Big]\|\Theta\|_{\mathbb{H}^p(\Omega)},\end{aligned} \tag{3.4}$$

$$\begin{aligned}&\left\|\mathcal{N}_{a_2}(\frac{t^r-s^r}{r})\Theta-\mathcal{N}_{a_1}(\frac{t^r-s^r}{r})\Theta\right\|_{\mathbb{H}^q(\Omega)}\\ &\qquad\le \frac{\mathcal{M}_2(c,d,\epsilon,\theta,T)}{r(\frac{t^r-s^r}{r})^{-c(1+p-q)+\epsilon+1}}\Big[(a_2-a_1)^\epsilon+(a_2-a_1)\Big]\|\Theta\|_{\mathbb{H}^p(\Omega)}.\end{aligned} \tag{3.5}$$

Proof. From the formula (2.24), applying Lemma 2.1, we obtain

$$\begin{aligned}\|\mathcal{M}_a(\frac{t^r}{r})\Theta\|^2_{\mathbb{H}^q(\Omega)} &= \sum_{j=1}^{\infty}\lambda_j^{2q}E_a^2(-\frac{\lambda_j}{r^a}t^{ar})|\langle\Theta,\varphi_j\rangle|^2\\ &\le \sum_{j=1}^{\infty}\lambda_j^{2q}\frac{\mathcal{M}_2^2(c,d)}{(1+\frac{\lambda_j}{r^a}t^{ar})^2}|\langle\Theta,\varphi_j\rangle|^2.\end{aligned} \tag{3.6}$$

Using the assumptions of p, q in the globally Lipschitz condition, we deduce $\theta = 1 + p - q \in (\frac{1}{2}, 1)$ and we get the following estimate

$$\begin{aligned}\|\mathcal{M}_a(t)\Theta\|^2_{\mathbb{H}^q(\Omega)} &\leq \sum_{j=1}^{\infty} \frac{\lambda_j^{2q}\mathcal{M}_2^2(c,d)}{[(1+\frac{\lambda_j}{r^a}t^{ar})^2]^{1-\theta}}|\langle\Theta,\varphi_j\rangle|^2 \\ &\leq \mathcal{M}_2^2(c,d)(\frac{t^r}{r})^{-2a(1-\theta)}\sum_{j=1}^{\infty}\lambda_j^{2q+2\theta-2}|\langle\Theta,\varphi_j\rangle|^2 \\ &\leq \mathcal{M}_2^2(c,d)(\frac{t^r}{r})^{(2d-2a)(1-\theta)}(\frac{t^r}{r})^{-2d(1-\theta)}\sum_{j=1}^{\infty}\lambda_j^{2q+2\theta-2}|\langle\Theta,\varphi_j\rangle|^2.\end{aligned} \tag{3.7}$$

Therefore, we obtain

$$\|\mathcal{M}_a(t)\Theta\|_{\mathbb{H}^q(\Omega)} \leq \overline{\mathcal{M}}_2(c,d)(\frac{t^r}{r})^{-d(q-p)}\|\Theta\|_{\mathbb{H}^p(\Omega)}. \tag{3.8}$$

where $\overline{\mathcal{M}}_2(c,d) := \mathcal{M}_2(c,d)(\frac{T^r}{r})^{(d-a)(q-p)}$.
From Lemma 2.1, we have

$$\begin{aligned}&\|\mathcal{N}_a(\frac{t^r-s^r}{r})\Theta\|^2_{\mathbb{H}^q(\Omega)} \\ &= \sum_{j=1}^{\infty}\frac{1}{r^2}\lambda_j^{2q}\Big(\frac{t^r-s^r}{r}\Big)^{2(a-1)}E^2_{a,a}\left(-\lambda_j\Big(\frac{t^r-s^r}{r}\Big)^a\right)|\langle\Theta,\varphi_j\rangle|^2 \\ &\leq \sum_{j=1}^{\infty}\frac{1}{r^2}\lambda_j^{2q}\Big(\frac{t^r-s^r}{r}\Big)^{2(a-1)}\frac{\mathcal{M}_3^2(c,d)}{\left[1+\lambda_j\Big(\frac{t^r-s^r}{r}\Big)^a\right]^2}|\langle\Theta,\varphi_j\rangle|^2 \\ &\leq \frac{1}{r^2}\Big(\frac{t^r-s^r}{r}\Big)^{2(a-1)}\sum_{j=1}^{\infty}\frac{\lambda_j^{2q}\mathcal{M}_3^2(c,d)}{\left[1+\lambda_j\Big(\frac{t^r-s^r}{r}\Big)^a\right]^{2(1-\theta)}}|\langle\Theta,\varphi_j\rangle|^2 \\ &\leq \frac{1}{r^2}\Big(\frac{t^r-s^r}{r}\Big)^{2(a-1)-2a(1-\theta)}\mathcal{M}_3^2(c,d)\sum_{j=1}^{\infty}\lambda_j^{2q+2\theta-2}|\langle\Theta\varphi_j\rangle|^2. \\ &\leq \frac{1}{r^2}\Big(\frac{t^r-s^r}{r}\Big)^{2a\theta-2}\mathcal{M}_3^2(c,d)\sum_{j=1}^{\infty}\lambda_j^{2q+2\theta-2}\|\Theta\|^2_{\mathbb{H}^p(\Omega)}.\end{aligned} \tag{3.9}$$

Therefore, we obtain

$$\|\mathcal{N}_a(\frac{t^r-s^r}{r})\Theta\|_{\mathbb{H}^q(\Omega)} \leq \overline{\mathcal{N}}_3(c,d)\Big(\frac{t^r-s^r}{r}\Big)^{c(1+p-q)-1}\|\Theta\|_{\mathbb{H}^p(\Omega)}, \tag{3.10}$$

where $\overline{\mathcal{N}}_3(c,d) := \mathcal{N}_3(c,d)\frac{1}{r}\big(\frac{T^r}{r}\big)^{(a-c)(1+p-q)}$.
Thanks to Lemma 2.4, we get the estimates as follow.

$$\begin{aligned}
&\Big\|\mathcal{M}_{a_2}(\frac{t^r}{r})\Theta - \mathcal{M}_{a_1}(\frac{t^r}{r})\Theta\Big\|^2_{\mathbb{H}^q(\Omega)}\\
&= \sum_{j=1}^{\infty}\lambda_j^{2q}\Big[E_{a_2}(-\frac{\lambda_j}{r^{a_2}}t^{a_2 r}) - E_{a_1}(-\frac{\lambda_j}{r^{a_1}}t^{a_1 r})\Big]^2|\langle\Theta,\varphi_j\rangle|^2\\
&\le \sum_{j=1}^{\infty}\lambda_j^{2q}\left[\mathcal{M}_1(c,d,\epsilon,\theta,T)\lambda_j^{\theta-1}(\frac{t^r}{r})^{-d(1-\theta)-\epsilon}\Big[(a_2-a_1)^\epsilon + (a_2-a_1)\Big]\right]^2|\langle\Theta,\varphi_j\rangle|^2\\
&\le \left[\mathcal{M}_1(\sigma,\nu,\epsilon,\theta,T)(\frac{t^r}{r})^{-d(q-p)-\epsilon}\Big[(a_2-a_1)^\epsilon + (a_2-a_1)\Big]\right]^2\|\Theta\|^2_{\mathbb{H}^p(\Omega)}. \qquad (3.11)
\end{aligned}$$

we deduce that

$$\begin{aligned}
&\Big\|\mathcal{M}_{a_2}(\frac{t}{r})\Theta - \mathcal{M}_{a_1}(\frac{t}{r})\Theta\Big\|_{\mathbb{H}^q(\Omega)}\\
&\le \mathcal{M}_1(c,d,\epsilon,\theta,T)(\frac{t^r}{r})^{-d(q-p)-\epsilon}\Big[(a_2-a_1)^\epsilon + (a_2-a_1)\Big]\|\Theta\|_{\mathbb{H}^p(\Omega)}. \qquad (3.12)
\end{aligned}$$

Finally, we estimate $\Big\|\mathcal{N}_{a_2}(\frac{t^r-s^r}{r})\Theta - \mathcal{N}_{a_1}(\frac{t^r-s^r}{r})\Theta\Big\|_{\mathbb{H}^q(\Omega)}$, using Lemma 2.5 we obtain

$$\begin{aligned}
&\|\mathcal{N}_{a_2}(\frac{t^r-s^r}{r})\Theta - \mathcal{N}_{a_1}(\frac{t^r-s^r}{r})\Theta\|^2_{\mathbb{H}^q(\Omega)}\\
&= \sum_{j=1}^{\infty}\frac{\lambda_j^{2q}}{r^2}\Bigg[(\frac{t^r-s^r}{r})^{a_2-1}E_{a_2,a_2}\left(-\lambda_j(\frac{t^r-s^r}{r})^{a_2}\right)\\
&\qquad\qquad - (\frac{t^r-s^r}{r})^{a_1-1}E_{a_1,a_1}\left(-\lambda_j(\frac{t^r-s^r}{r})^{a_1}\right)\Bigg]^2|\langle\Theta\varphi_j\rangle|^2\\
&= \sum_{j=1}^{\infty}\frac{\lambda_j^{2q}}{r^2}\left[\mathcal{M}_2(c,d,\epsilon,\theta,T)\lambda_j^{\theta-1}(\frac{t^r-s^r}{r})^{c\theta-\epsilon-1}\Big[(a_2-a_1)^\epsilon + (a_2-a_1)\Big]\right]^2|\langle\Theta\varphi_j\rangle|^2
\end{aligned}$$

(3.13)

Hence, we get the desired estimate

$$\begin{aligned}
&\Big\|\mathcal{N}_{a_2}(\frac{t^r-s^r}{r})\Theta - \mathcal{N}_{a_1}(\frac{t^r-s^r}{r})\Theta\Big\|_{\mathbb{H}^q(\Omega)}\\
&\le \frac{\mathcal{M}_2(c,d,\epsilon,\theta,T)}{r(\frac{t^r-s^r}{r})^{-c(1+p-q)+\epsilon+1}}\Big[(a_2-a_1)^\epsilon + (a_2-a_1)\Big]\|\Theta\|_{\mathbb{H}^p(\Omega)}. \qquad (3.14)
\end{aligned}$$

The proof is completed. □

Theorem 3.1. *Let c, a, a_1, a_2 and d be positive constants such that $0 < c < a_1 \le a \le a_2 < d < c\big(\frac{1}{q-p} - 1\big) < 1$ and $b > 0$ is a large enough positive number. For $f \in \mathbb{H}^p(\Omega)$, if there exist two positive numbers ϵ, ϑ satisfying $0 < dr(q-p) < \vartheta < rc(1-q+p) < 1$ and $\epsilon < \min\{c(1-q+p) - \frac{\vartheta}{r}, \frac{\vartheta}{r} - d(q-p)\}$, then the nonlinear* *integral equation (1.1) has a unique mild solution* $\mathrm{u}_a(x,t) \in \mathbb{C}^{\vartheta}_{\varrho}((0,T],\mathbb{H}^q(\Omega))$ *and the following estimate holds*

$$\|\mathrm{u}_a\|_{\mathbb{C}^{\vartheta}_{\varrho}((0,T],\mathbb{H}^q(\Omega))} \le \mathcal{W}^{c,d,\varrho}_{q,p,\vartheta}(\|f\|_{\mathbb{H}^p(\Omega)}, C_{\mathscr{F}}), \tag{3.15}$$

where the constant $\mathcal{W}^{c,d,\varrho}_{q,p,\vartheta}$ *is defined in (3.32). In addition, suppose that* $\mathrm{u}_{a_1},\ \mathrm{u}_{a_2} \in \mathbb{C}^{\vartheta}_{\varrho}((0,T],\mathbb{H}^q(\Omega))$ *are two solutions to Problem (1.1) with respect to the fractional orders* a_1 *and* a_2 *respectively, we also obtain*

$$\|\mathrm{u}_{a_2} - \mathrm{u}_{a_1}\|_{\mathbb{C}^{\vartheta}_{\varrho}((0,T],\mathbb{H}^q(\Omega))} \le \frac{\mathscr{W}_1\Big[(a_2-a_1)^{\epsilon} + (a_2-a_1)\Big]}{1-\mathscr{W}_2}. \tag{3.16}$$

Proof. Now, we divided the proof into three parts.
Part 1. The existence and uniqueness of a solution $\mathrm{u} \in \mathbb{C}^{\vartheta}_{\varrho}((0,T],\mathbb{H}^q(\Omega))$ to the nonlinear integral equation (2.23) presented in this part. We begin by defining a mapping $\Psi : \mathbb{C}^{\vartheta}_{\varrho}((0,T],\mathbb{H}^q(\Omega)) \to \mathbb{C}^{\vartheta}_{\varrho}((0,T],\mathbb{H}^q(\Omega))$ by

$$\Psi\mathrm{u}_a := \mathcal{M}_a(\frac{t^r}{r})f + \int_0^t \mathcal{N}_a(\frac{t^r-s^r}{r})\mathscr{F}(\mathrm{u}(s))\mathrm{d}s^r. \tag{3.17}$$

The main argument is using the Banach fixed-point theorem to affirm that the equation $\Psi\mathrm{u}_a = \mathrm{u}_a$ has the unique solution u_a. Hence, we have to show that Ψ is a contraction on $\mathbb{C}^{\vartheta}_{\varrho}((0,T],\mathbb{H}^q(\Omega))$. Indeed, for any $\mathrm{u}_{a,1},\ \mathrm{u}_{a,2} \in \mathbb{C}^{\vartheta}_{\varrho}((0,T],\mathbb{H}^q(\Omega))$, we have

$$\Psi\mathrm{u}_{a,1} - \Psi\mathrm{u}_{a,2} = \int_0^t \mathcal{N}_a(\frac{t^r-s^r}{r})\Big[\mathscr{F}(\mathrm{u}_{a,1}(s)) - \mathscr{F}(\mathrm{u}_{a,2}(s))\Big]\mathrm{d}s^r \tag{3.18}$$

From the above equality, we deduce

$$\Big\|\Psi\mathrm{u}_{a,1} - \Psi\mathrm{u}_{a,2}\Big\|_{\mathbb{H}^q(\Omega)} \le \Bigg\|\int_0^t \mathcal{N}_a(\frac{t^r-s^r}{r})\Big[\mathscr{F}(\mathrm{u}_{a,1}(s)) - \mathscr{F}(\mathrm{u}_{a,2}(s))\Big]\mathrm{d}s^r\Bigg\|_{\mathbb{H}^q(\Omega)} \tag{3.19}$$

For $\varrho > 0$, using the result (3.10) and the assumption (3.1) we obtain

$$\begin{aligned}
&t^{\vartheta}\mathrm{e}^{-\varrho t^r}\Big\|\Psi\mathrm{u}_{a,1} - \Psi\mathrm{u}_{a,2}\Big\|_{\mathbb{H}^q(\Omega)} \\
&\le \overline{\mathcal{N}}_3(c,d)t^{\vartheta}\mathrm{e}^{-\varrho t^r}\int_0^t \Big(\frac{t^r-s^r}{r}\Big)^{c(1+p-q)-1}\big\|\mathscr{F}(\mathrm{u}_{a,1}(s)) - \mathscr{F}(\mathrm{u}_{a,2}(s))\big\|_{\mathbb{H}^p(\Omega)}\mathrm{d}s^r \\
&\le \overline{\mathcal{N}}_3(c,d)C_{\mathscr{F}}t^{\vartheta} \\
&\qquad \times \int_0^t (\frac{t^r-s^r}{r})^{c(1-q+p)-1}\mathrm{e}^{-\varrho(t^r-s^r)}s^{-\vartheta}s^{\vartheta}\mathrm{e}^{-\varrho s^r}\big\|\mathrm{u}_{a,1}(s) - \mathrm{u}_{a,2}(s)\big\|_{\mathbb{H}^q(\Omega)}\mathrm{d}s^r, \\
&\le \frac{\overline{\mathcal{N}}_3(c,d)C_{\mathscr{F}}}{r^{c(1-q+p)-1}}\big\|\mathrm{u}_{a,1} - \mathrm{u}_{a,2}\big\|_{\mathbb{C}^{\vartheta}_{\varrho}((0,T],\mathbb{H}^q(\Omega))}t^{\vartheta}\int_0^t (t^r-s^r)^{c(1-q+p)-1}s^{-\vartheta}\mathrm{e}^{-\varrho(t^r-s^r)}\mathrm{d}s^r. \\
&\le \frac{\overline{\mathcal{N}}_3(c,d)C_{\mathscr{F}}}{r^{c(1-q+p)-1}}\big\|\mathrm{u}_{a,1} - \mathrm{u}_{a,2}\big\|_{\mathbb{C}^{\vartheta}_{\varrho}((0,T],\mathbb{H}^q(\Omega))}t^{rc(1-q+p)} \\
&\qquad \times \int_0^t \Big(1-\Big(\frac{s}{t}\Big)^r\Big)^{c(1-q+p)-1}\Big(\frac{s}{t}\Big)^{-\vartheta}\mathrm{e}^{-\varrho t^r(1-(\frac{s}{t})^r)}\mathrm{d}\Big(\frac{s}{t}\Big)^r.
\end{aligned} \tag{3.20}$$

Using the transformation of the integral expression. Set $\xi = \left(\frac{s}{t}\right)^r$, we get

$$t^{\vartheta} e^{-\varrho t^r} \left\| \Psi u_{a,1} - \Psi u_{a,2} \right\|_{\mathbb{H}^q(\Omega)} \leq \frac{\overline{\mathcal{N}}_3(c,d) C_{\mathscr{F}}}{r^{c(1-q+p)-1}} \left\| u_{a,1} - u_{a,2} \right\|_{\mathbb{C}^{\vartheta}_{\varrho}((0,T],\mathbb{H}^q(\Omega))} t^{rc(1-q+p)} \times \int_0^1 \xi^{-\frac{\vartheta}{r}} (1-\xi)^{c(1-q+p)-1} e^{-\varrho t^r (1-\xi)} d\xi. \quad (3.21)$$

By setting $\epsilon_1 := -\frac{\vartheta}{r}, \epsilon_2 := c(1-q+p)-1$ then

$$\mathscr{C}^{\epsilon_2+1}_{\epsilon_1,\epsilon_2}(\varrho) := \sup_{t\in[0,T]} (t^r)^{c(1-q+p)} \int_0^1 \xi^{-\frac{\vartheta}{r}} (1-\xi)^{c(1-q+p)-1} e^{-\varrho t^r(1-\xi)} ds \xrightarrow{\varrho\to\infty} 0. \quad (3.22)$$

We get the following estimate

$$\left\| \Psi u_{a,1} - \Psi u_{a,2} \right\|_{\mathbb{C}^{\vartheta}_{\varrho}((0,T],\mathbb{H}^q(\Omega))} \leq \frac{\overline{\mathcal{N}}_3(c,d) C_{\mathscr{F}}}{r^{c(1-q+p)-1}} \left\| u_{1a,} - u_{a,2} \right\|_{\mathbb{C}^{\vartheta}_{\varrho}((0,T],\mathbb{H}^q(\Omega))} \mathscr{C}^{\epsilon_2+1}_{\epsilon_1,\epsilon_2}(\varrho). \quad (3.23)$$

With the assumption $0 < \vartheta < rc(1-q+p) < r < 1$, we see that

$$\begin{cases} \frac{-\vartheta}{r} > -1, \\ c(1-q+p) > c(1-q+p) - 1 > -1, \\ \frac{-\vartheta}{r} + c(1-q+p) - 1 > -1. \end{cases} \quad (3.24)$$

Using Lemma 2.6, we can state that

$$\mathscr{C}^{\epsilon_2+1}_{\epsilon_1,\epsilon_2}(\varrho) \xrightarrow{\varrho\to\infty} 0. \quad (3.25)$$

Therefore, we can choose ϱ_0 such that

$$\frac{\overline{\mathcal{N}}_3(c,d) C_{\mathscr{F}}}{r^{c(1-q+p)-1}} \mathscr{C}^{\epsilon_2+1}_{\epsilon_1,\epsilon_2}(\varrho_0) < 1. \quad (3.26)$$

Accordingly, we determine that Ψ is a contractive mapping in the space $\mathbb{C}^{\vartheta}_{\varrho}((0,T],\mathbb{H}^q(\Omega))$. Applying the Banach fixed point theorem, we confirm that the nonlinear integral equation (1.1) has a uniques solution $u_a \in \mathbb{C}^{\vartheta}_{\varrho}((0,T],\mathbb{H}^q(\Omega))$.

Part 2. Suppose that $f \in \mathbb{H}^p$. It follows easily from the use of (2.24) that

$$t^{\vartheta} e^{-\varrho t^r} \| u_a(\cdot,t) \|_{\mathbb{H}^q(\Omega)} \leq t^{\vartheta} e^{-\varrho t^r} \left\| \mathcal{M}_a\left(\frac{t^r}{r}\right) f \right\|_{\mathbb{H}^q(\Omega)} + t^{\vartheta} e^{-\varrho t^r} \left\| \int_0^t \mathcal{N}_a\left(\frac{t^r - s^r}{r}\right) \mathscr{F}(u_a(s)) ds^r \right\|_{\mathbb{H}^q(\Omega)} \quad (3.27)$$

By the similar techniques as in Part 1, we can easy find that $e^{-\varrho t^r} \leq 1$ for all $t > 0$. Using Lemma 3.1 with assumption condition (3.1) and $\vartheta > rd(q-p)$, we get

$$
\begin{aligned}
& t^{\vartheta} e^{-\varrho t^r} \|\mathrm{u}_a(\cdot,t)\|_{\mathbb{H}^q(\Omega)} \\
& \leq t^{\vartheta} e^{-\varrho t^r} (\frac{t^r}{r})^{-d(q-p)} \|f\|_{\mathbb{H}^p(\Omega)} \\
& \qquad + t^{\vartheta} e^{-\varrho t^r} \int_0^t \overline{\mathcal{N}}_3(c,d)(\frac{t^r-s^r}{r})^{c(1+p-q)-1} \|\mathscr{F}(\mathrm{u}_a(s))\|_{\mathbb{H}^p(\Omega)} \mathrm{d}s^r \\
& \leq \frac{\overline{\mathcal{M}}_2(c,d) T^{\vartheta-rd(q-p)}}{r^{-d(q-p)}} \|f\|_{\mathbb{H}^p(\Omega)} \\
& \qquad + \overline{\mathcal{N}}_3(c,d) t^{\vartheta} e^{-\varrho t^r} \int_0^t (\frac{t^r-s^r}{r})^{c(1+p-q)-1} \|\mathscr{F}(\mathrm{u}_a(s)) - \mathscr{F}(0)\|_{\mathbb{H}^p(\Omega)} \mathrm{d}s^r \\
& \leq \frac{\overline{\mathcal{M}}_2(c,d) T^{\vartheta-rd(q-p)}}{r^{-d(q-p)}} \|f\|_{\mathbb{H}^p(\Omega)} \\
& \qquad + \overline{\mathcal{N}}_3(c,d) t^{\vartheta} e^{-\varrho t^r} \int_0^t (\frac{t^r-s^r}{r})^{c(1+p-q)-1} C_{\mathscr{F}} \|(\mathrm{u}_a(s))\|_{\mathbb{H}^q(\Omega)} \mathrm{d}s^r \\
& \leq \frac{\overline{\mathcal{M}}_2(c,d) T^{\vartheta-rd(q-p)}}{r^{-d(q-p)}} \|f\|_{\mathbb{H}^p(\Omega)} \\
& + \overline{\mathcal{N}}_3(c,d) C_{\mathscr{F}} t^{\vartheta} \times \int_0^t (\frac{t^r-s^r}{r})^{c(1+p-q)-1} e^{-\varrho(t^r-s^r)} s^{-\vartheta} s^{\vartheta} e^{\varrho s^r} \|\mathrm{u}_a(s)\|_{\mathbb{H}^q(\Omega)} \mathrm{d}s^r .
\end{aligned} \tag{3.28}
$$

Using a similar evaluation technique as (3.2), we get

$$
\begin{aligned}
& t^{\vartheta} e^{-\varrho t^r} \|\mathrm{u}_a(\cdot,t)\|_{\mathbb{H}^q(\Omega)} \\
& \leq \frac{\overline{\mathcal{M}}_2(c,d) T^{\vartheta-rd(q-p)}}{r^{-d(q-p)}} \|f\|_{\mathbb{H}^p(\Omega)} \\
& + \frac{\overline{\mathcal{N}}_3(c,d) C_{\mathscr{F}}}{r^{c(1-q+p)-1}} \|\mathrm{u}_a\|_{\mathbb{C}^{\vartheta}_{\varrho}((0,T],\mathbb{H}^q(\Omega))} t^{rc(1-q+p)} \int_0^1 \xi^{-\frac{\vartheta}{r}} (1-\xi)^{c(1-q+p)-1} \mathrm{e}^{-\varrho t^r(1-\xi)} \mathrm{d}\xi .
\end{aligned} \tag{3.29}
$$

Therefore, we can deduce

$$
\begin{aligned}
\|\mathrm{u}_a\|_{\mathbb{C}^{\vartheta}_{\varrho}((0,T],\mathbb{H}^q(\Omega))} \leq & \frac{\overline{\mathcal{M}}_2(c,d) T^{\vartheta-rd(q-p)}}{r^{-d(q-p)}} \|f\|_{\mathbb{H}^p(\Omega)} \\
& + \frac{\overline{\mathcal{N}}_3(c,d) C_{\mathscr{F}}}{r^{c(1-q+p)-1}} \|\mathrm{u}_a\|_{\mathbb{C}^{\vartheta}_{\varrho}((0,T],\mathbb{H}^q(\Omega))} \mathscr{C}^{\epsilon_2+1}_{\epsilon_1,\epsilon_2}(\varrho).
\end{aligned} \tag{3.30}
$$

With the assumption $0 < dr(q-p) < \vartheta < rc(1-q+p) < 1$, we can obtain the following estimate

$$
\|\mathrm{u}_a\|_{\mathbb{C}^{\vartheta}_{\varrho}((0,T],\mathbb{H}^q(\Omega))} \leq \mathcal{W}^{c,d,\varrho}_{q,p,\vartheta}(\|f\|_{\mathbb{H}^p(\Omega)}, C_{\mathscr{F}}), \tag{3.31}
$$

where

$$
\mathcal{W}^{c,d,\varrho}_{q,p,\vartheta}(\|f\|_{\mathbb{H}^q(\Omega)}, C_{\mathscr{F}}) := \frac{\overline{\mathcal{M}_2(c,d)} T^{\vartheta-rd(q-p)} r^{d(q-p)} \|f\|_{\mathbb{H}^p(\Omega)}}{1 - \frac{\overline{\mathcal{N}}_3(c,d) C_{\mathscr{F}}}{r^{c(1-q+p)-1}} \mathscr{C}^{\epsilon_2+1}_{\epsilon_1,\epsilon_2}(\varrho)}. \tag{3.32}
$$

Part 3. The continuous dependency of mild solutions on the fractional order of the time derivative is investigated in this section. To find that, we first apply the triangle inequality.

$$t^{\vartheta}e^{-\varrho t^r}\|\mathrm{u}_{a_2}(\cdot,t)-\mathrm{u}_{a_1}(\cdot,t)\|_{\mathbb{H}^q(\Omega)}\le \mathcal{U}_1+\mathcal{U}_2+\mathcal{U}_3, \tag{3.33}$$

where

$$\mathcal{U}_1 := t^{\vartheta}e^{-\varrho t^r}\Big\|\big[\mathcal{M}_{a_2}(\frac{t^r}{r})-\mathcal{M}_{a_1}(\frac{t^r}{r})\big]f\Big\|_{\mathbb{H}^q(\Omega)}, \tag{3.34}$$

$$\mathcal{U}_2 := t^{\vartheta}e^{-\varrho t^r}\left\|\int_0^t\big[\mathcal{N}_{a_2}(\frac{t^r-s^r}{r})-\mathcal{N}_{a_1}(\frac{t^r-s^r}{r})\big]\mathscr{F}(\mathrm{u}_{a_2}(s))\mathrm{d}s^r\right\|_{\mathbb{H}^q(\Omega)}, \tag{3.35}$$

$$\mathcal{U}_3 := t^{\vartheta}e^{-\varrho t^r}\left\|\int_0^t\mathcal{N}_{a_1}(\frac{t^r-s^r}{r})\Big[\mathscr{F}(\mathrm{u}_{a_2}(s))-\mathscr{F}(\mathrm{u}_{a_1}(s))\Big]\mathrm{d}s^r\right\|_{\mathbb{H}^q(\Omega)}. \tag{3.36}$$

Estimating $\mathcal{U}_1$: Applying the evaluation (3.4) in Lemma 3.1 with the condition $\epsilon<\frac{\vartheta}{r}-d(q-p)$, we have

$$\begin{aligned}\mathcal{U}_1 &\le \frac{\mathcal{M}_1(c,d,\epsilon,\theta,T)}{r^{-d(q-p)-\epsilon}}\Big[(a_2-a_1)^{\epsilon}+(a_2-a_1)\Big]\mathrm{e}^{-\varrho t^r}t^{\vartheta-r[d(q-p)+\epsilon]}\|f\|_{\mathbb{H}^p(\Omega)}\\ &\le \frac{\mathcal{M}_1(c,d,\epsilon,\theta,T)T^{\vartheta-r[d(q-p)+\epsilon]}\|f\|_{\mathbb{H}^p(\Omega)}}{r^{-d(q-p)-\epsilon}}\Big[(a_2-a_1)^{\epsilon}+(a_2-a_1)\Big].\end{aligned} \tag{3.37}$$

Estimating $\mathcal{U}_2$: Similarly, applying the inequality (3.5) of Lemma 3.1, we also have

$$\begin{aligned}\mathcal{U}_2 \le{}& \frac{\mathcal{M}_2(c,d,\epsilon,\theta,T)}{r^{c(1+p-q)-\epsilon}}\Big[(a_2-a_1)^{\epsilon}+(a_2-a_1)\Big]\mathrm{e}^{-\varrho t^r}t^{\vartheta}\\ &\times\int_0^t(t^r-s^r)^{c(1-q+p)-\epsilon-1}\|\mathscr{F}(\mathrm{u}_{a_2}(s))-\mathscr{F}(0)\|_{\mathbb{H}^p(\Omega)}\mathrm{d}s.\end{aligned} \tag{3.38}$$

Using the assumption (3.1), we find that

$$\begin{aligned}\mathcal{U}_2 \le{}& \frac{C_{\mathscr{F}}\mathcal{M}_2(c,d,\epsilon,\theta,T)}{r^{c(1+p-q)-\epsilon}}\Big[(a_2-a_1)^{\epsilon}+(a_2-a_1)\Big]\\ &\times t^{\vartheta}\int_0^t s^{-\vartheta}(t^r-s^r)^{c(1-q+p)-\epsilon-1}\mathrm{e}^{-\varrho(t^r-s^r)}\Big[s^{\vartheta}\mathrm{e}^{-\varrho s^r}\|\mathrm{u}_{a_2}\|_{\mathbb{H}^q(\Omega)}\Big]\mathrm{d}s^r.\end{aligned} \tag{3.39}$$

Applying an estimate technique similar to (3.2), we obtain

$$\begin{aligned}\mathcal{U}_2 \le{}& \frac{C_{\mathscr{F}}\mathcal{M}_2(c,d,\epsilon,\theta,T)}{r^{c(1+p-q)-\epsilon}}\Big[(a_2-a_1)^{\epsilon}+(a_2-a_1)\Big]\\ &\times\|\mathrm{u}_2\|_{\mathbb{C}^{\vartheta}_{\varrho}((0,T],\mathbb{H}^q(\Omega))}(t^r)^{c(1-q+p)-\epsilon}\int_0^1\xi^{-\frac{\vartheta}{r}}(1-\xi)^{c(1-q+p)-\epsilon-1}\mathrm{e}^{-\varrho t^r(1-\xi)}\mathrm{d}\xi.\end{aligned} \tag{3.40}$$

Therefore, thanks to the choose condition $\epsilon<\min\{\frac{\vartheta}{r}-d(q-p),c(1-q+p)\}$ then

$$\begin{cases}\frac{-\vartheta}{r}>-1,\\ c(1-q+p)-\epsilon>c(1-q+p)-\epsilon-1>-1.\end{cases} \tag{3.41}$$

Using (3.3) and Lemma 2.6, we get

$$\mathcal{U}_2\le\frac{C_{\mathscr{F}}\mathcal{M}_2(c,d,\epsilon,\theta,T)}{r^{c(1+p-q)-\epsilon}}\Big[(a_2-a_1)^{\epsilon}+(a_2-a_1)\Big]\mathcal{W}^{c,d,\varrho}_{q,p,\vartheta}(\|f\|_{\mathbb{H}^p(\Omega)},C_{\mathscr{F}})\mathscr{C}^{\epsilon_2'+1}_{\epsilon_1,\epsilon_2'}(\varrho). \tag{3.42}$$

where $\epsilon_2' = c(1-q+p) - \epsilon - 1$

Estimating $\mathcal{U}_3$: From (3.36), the inequality (3.3) of Lemma 3.1 and the assumption (3.1), we also have

$$\begin{aligned}\mathcal{U}_3 &\leq \frac{\overline{\mathcal{N}}_3(c,d)}{r^{c(1-q+p)-1}} t^{\vartheta} \int_0^t (t^r - s^r)^{c(1+p-q)-1} \mathrm{e}^{-\varrho t^r} \|\mathscr{F}(\mathrm{u}_{a_2}(s)) - \mathscr{F}(\mathrm{u}_{a_1}(s))\|_{\mathbb{H}^p(\Omega)} \mathrm{d}s^r, \\ &\leq \frac{\overline{\mathcal{N}}_3(c,d) C_{\mathscr{F}}}{r^{c(1-q+p)-1}} t^{\vartheta} \\ &\qquad \times \int_0^t s^{-\vartheta} (t^r - s^r)^{c(1-q+p)-1} \mathrm{e}^{-\varrho(t^r - s^r)} \Big[s^{\vartheta} \mathrm{e}^{-\varrho s^r} \|\mathrm{u}_{a_2}(s) - \mathrm{u}_{a_1}(s)\|_{\mathbb{H}^q(\Omega)} \Big] \mathrm{d}s^r. \\ &\leq \frac{\overline{\mathcal{N}}_3(c,d) C_{\mathscr{F}}}{r^{c(1-q+p)-1}} \big\|\mathrm{u}_{a_1} - \mathrm{u}_{a_1}\big\|_{\mathbb{C}^{\vartheta}_{\varrho}((0,T],\mathbb{H}^q(\Omega))} \mathscr{C}^{\epsilon_2+1}_{\epsilon_1,\epsilon_2}(\varrho). \end{aligned} \tag{3.43}$$

Hence, from (3.37)-(3.43), we obtain the following result

$$\begin{aligned} & t^{\vartheta} \mathrm{e}^{-\varrho t^r} \|\mathrm{u}_{a_2}(\cdot,t) - \mathrm{u}_{a_1}(\cdot,t)\|_{\mathbb{H}^q(\Omega)} \\ &\leq \frac{\mathcal{M}_1(c,d,\epsilon,\theta,T) T^{\vartheta - r[d(q-p)+\epsilon]} \|f\|_{\mathbb{H}^p(\Omega)}}{r^{-d(q-p)-\epsilon}} \Big[(a_2 - a_1)^{\epsilon} + (a_2 - a_1) \Big] \\ &+ \frac{C_{\mathscr{F}} \mathcal{M}_2(c,d,\epsilon,\theta,T) \mathscr{C}^{\epsilon_2'+1}_{\epsilon_1,\epsilon_2'}(\varrho) \mathcal{W}^{c,d,\varrho}_{q,p,\vartheta}(\|f\|_{\mathbb{H}^p(\Omega)}, C_{\mathscr{F}})}{r^{c(1+p-q)-\epsilon}} \Big[(a_2 - a_1)^{\epsilon} + (a_2 - a_1) \Big] \\ &+ \frac{\overline{\mathcal{N}}_3(c,d) C_{\mathscr{F}} \mathscr{C}^{\epsilon_2+1}_{\epsilon_1,\epsilon_2}(\varrho)}{r^{c(1-q+p)-1}} \big\|\mathrm{u}_{a_1} - \mathrm{u}_{a_1}\big\|_{\mathbb{C}^{\vartheta}_{\varrho}((0,T],\mathbb{H}^q(\Omega))}. \end{aligned} \tag{3.44}$$

For simplicity, we use several notations below.

$$\begin{aligned} \mathscr{W}_1 &:= \frac{\mathcal{M}_1(c,d,\epsilon,\theta,T) T^{\vartheta - r[d(q-p)+\epsilon]} \|f\|_{\mathbb{H}^p(\Omega)}}{r^{-d(q-p)-\epsilon}} \\ &\qquad + \frac{C_{\mathscr{F}} \mathcal{M}_2(c,d,\epsilon,\theta,T) \mathscr{C}^{\epsilon_2'+1}_{\epsilon_1,\epsilon_2'}(\varrho) \mathcal{W}^{c,d,\varrho}_{q,p,\vartheta}(\|f\|_{\mathbb{H}^p(\Omega)}, C_{\mathscr{F}})}{r^{c(1+p-q)-\epsilon}}, \end{aligned} \tag{3.45}$$

$$\mathscr{W}_2 := \frac{\overline{\mathcal{N}}_3(c,d) C_{\mathscr{F}} \mathscr{C}^{\epsilon_2+1}_{\epsilon_1,\epsilon_2}(\varrho)}{r^{c(1-q+p)-1}}. \tag{3.46}$$

From all the above results, we obtain

$$\|\mathrm{u}_{a_2} - \mathrm{u}_{a_1}\|_{\mathbb{C}^{\vartheta}_{\varrho}((0,T],\mathbb{H}^q(\Omega))} \leq \frac{\mathscr{W}_1 \Big[(a_2 - a_1)^{\epsilon} + (a_2 - a_1) \Big]}{1 - \mathscr{W}_2}. \tag{3.47}$$

Theorem 3.1 has been proved. □

4. Conclusion.

In this research, we investigate a Hyper-Bessel operator-based time-fractional diffusion equation. First, we give some stability findings for the Mittag-Leffler function parameters. Then we focus on examining the continuity of the solution of the initial problem that corresponds to the fractional-order based on Mittag-Leffler function theories, the Banach fixed point theorem, and Sobolev embeddings.

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