Proceedings of the Institute of Mathematics and Mechanics,
National Academy of Sciences of Azerbaijan
Volume 48, Special Issue, 2022, Pages 39–55
https://doi.org/10.30546/2409-4994.48.2022.3955

NUMERICAL SOLUTION OF SINGULAR STOCHASTIC INTEGRAL EQUATIONS OF ABEL'S TYPE USING OPERATIONAL MATRIX METHOD

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Abstract. This paper proposes the orthogonal shifted Legendre collocation method to obtain an approximate solution for Abel-type singular linear stochastic integral equation. Our proposed method converts the singular linear stochastic equation to a system of algebraic equations. The shifted Legendre polynomials are used to obtain the operational matrices for such a conversion. The main advantage of the proposed method is that a sparse, tridiagonal orthogonal matrix is obtained, thus making process simple and efficient. The convergence and error analysis of the proposed method are discussed. Numerical examples prove the applicability and the efficiency of our proposed method.

1. Introduction

In 1823, N. H. Abel [26], a Norwegian Mathematician, made one of the earliest applications of integral equations to a physical problem in Mechanics. Later, many physical problems such as classical simple harmonic oscillator problems and quantum simple harmonic problems were modelled using integral equations [5]. In recent years, various types of deterministic and stochastic integral equations have become inevitable to represent a variety of physical problems arising in plasma physics, elasticity theory, stereology, spectroscopy, scattering theory and astrophysics. Notably, stochastic integral equations have piqued the interest of researchers more than deterministic integral equations, as the real-time scenario can be more satisfactorily modelled using stochastic models. The study on various deterministic integral equations of Volterra type, Fredholm type, Volterra-Fredholm type, integro-differential equations, etc., can be seen in [12, 17, 9, 19, 20, 21, 31, 4, 3]. Also, various problems modelled using stochastic integral equations and stochastic integro-differential equations can be found in [18, 22, 6, 16].

Most of the deterministic and stochastic integral equations of various types are either challenging or time-consuming to solve. Varied applications of integral equations necessitate the development of new methodologies to solve such equations. In the past several decades, researchers have developed various efficient

2010 *Mathematics Subject Classification.* 60H20; 60H35; 45E10; 11B83.

Key words and phrases. Stochastic integral equations, Abel's type equations, Shifted Legendre polynomials.

numerical techniques for the same. Wavelet techniques using Haar wavelets, Legendre wavelets, Boubaker wavelets [1, 10, 30], polynomial approximation methods using Legendre polynomials [24] and operational matrix methods using block pulse function [23] are a few among many.

An integral equation is singular not only when the range of integration is infinite but also when the kernel has singularities within the range of integration. Such integral equations possess very unusual properties and occur frequently in mathematical physics. The numerical solution of one such two-dimensional weakly singular stochastic integral equation can be seen in [28]. The application of a new collocation technique based on Berstein polynomials can be seen in [11]. [7, 32] also throw a light on a few different numerical techniques to solve stochastic integral equations. In particular, Samadyar and Farshid Mirzaee [29] have applied the orthonormal Bernoulli polynomials collocation approach to solving such singular stochastic integral equations of Abel-type. Also, the application of Boubaker wavelets operational matrix of integration for such an equation can be seen in [30].

Not only motivated by the above works but also due to the limited research works on singular stochastic integral equations, in this article, we attempt to find an approximate solution of the following singular stochastic Itõ-Volterra integral equation of Abel-type:

$$X(t) = X_0 + \lambda_1 \int_0^t \frac{X(s)}{(t-s)^\alpha} ds + \lambda_2 \int_0^t k(t,s)X(s)dB(s),\ t \in [0,1], \qquad (1.1)$$

where X_0, λ_1 and λ_2 are constants and $0 < \alpha < 1$. The functions $X(t)$ and $k(t,s)$ are stochastic processes. They are defined on the probability space triplet $(\Omega, \mathcal{F}, P)$. Also, $B(t)$ is the Brownian motion process defined on the same space and $X(t)$, the unknown function to be determined, is the solution of the singular stochastic Itõ-Volterra integral equation of Abel-type. Our proposed methodology has the following advantages:

- Shifted Legendre polynomials have orthogonal property. This property plays a vital role in obtaining the operational matrices and thereafter in the proposed methodology, and is more convenient than the other non - orthogonal polynomials.
- The proposed method involves operational matrices which help in reducing the problem considered to an algebraic system of equations. The system of equations thus obtained is solved, using a well-known numerical technique.
- The proposed method provides a more accurate solution and is easy to implement, as it involves sparse matrices.
- The convergence analysis discussed ensures the efficiency, whereas the error analysis ensures the accuracy of the proposed method.
- The theorem on time complexity ensures the validity and applicability of the proposed technique.
- Unlike other papers in the literature, an approximation is considered for the kernel function which paves a way for more accurate results.

Even though the proposed method has the above-mentioned salient features that overcome the limitations of the other methods discussed in the literature, the limitations of our proposed method lie in the value of N. For higher values of N, the amount of computational work is slightly higher.

In this paper, we use the shifted version of Legendre polynomials called shifted Legendre polynomials to find an approximate solution to our problem under consideration. The orthogonal property of this polynomial produces triangular, tri-diagonal and diagonal matrices at different instances. It is very advantageous to use this novel method as this is very simple and betterment can be realized in solving such singular stochastic equations because of the salient features of the shifted Legendre polynomial. The orthogonal property, together with the operational matrix of integration and stochastic integration converts the problem into simultaneous algebraic equations. By solving the system of algebraic equations by any known method, we arrive at the numerical solution of the problem considered. The theoretical analysis is also carried out to ensure the convergence of the proposed approximation technique. It has also been proved that the error function reduces to zero for larger values of the parameters involved. The applicability, validity, accuracy and efficiency of the proposed technique are tested with some numerical examples and the solution quality can be realized through various figures. The error curve is also plotted which justifies that fluctuations in error fall within the error bound discussed in the theoretical analysis.

The overview of this paper comprises the following. The fundamental definitions and theorems required for our subsequent study are given in the Mathematical background followed by the fundamentals of shifted Legendre polynomials and their properties. The various operational matrices required for the proposed method are also derived. In the next section, we give a detailed presentation of the convergence theorems and error estimates. The accuracy and applicability of the scheme are tested on a few examples and comparative results are also presented in the section on numerical examples. The superiority of this method is also highlighted in that section. The final section has the concluding remarks.

2. Mathematical Background

In this section, we provide the fundamental definitions of stochastic calculus and information about our subsequent study [8, 14, 25]. We start by defining Brownian motion, which is a fundamental example of a stochastic process. The underlying probability space $(\Omega, \mathcal{F}, P)$ can be constructed on the space $\Omega = C_0(R_+)$ of continuous real-valued functions on R_+ starting at 0. Next, we introduce the idea of Hilbert space and Banach space, where the concept of defining a norm has been established in the probability space $(\Omega, \mathcal{F}, P)$. The idea of the convergence of a sequence $\{X_n\}$ in the given space, where the function is defined, is also discussed. The basic properties of It$\hat{o}$ integral and It$\hat{o}$ isometry are also elucidated for our subsequent development.

Definition 2.1. [15] Let $(\Omega, \mathcal{F}, P)$ be a probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$. A (standard) one-dimensional Brownian motion is a real-valued continuous $\{\mathcal{F}_t\}$-adapted process $\{B_t\}_{\ t\geq 0}$ with the following properties:
(i) $B_0 = 0$ a.s.;
(ii) for $0 \leq s < t < \infty$, the increment $(B_t - B_s)$ is normally distributed with

mean zero and variance $(t-s)$;
(iii) for $0 \leq s < t < \infty$, the increment $(B_t - B_s)$ is independent of $\{\mathcal{F}_s\}$.

Definition 2.2. [14] Let $p \geq 2$ and $L^p(\Omega, H)$ be the collection of all strongly measurable random variables and if $\|V\|_{L^p} = \{E\,|V|^p\}^{1/p} = \left(\int_\Omega |V|^p\, dP\right)^{1/p}$, for each $V \in L^P(\Omega, H)$ then $L^p(\Omega, H)$ is a Banach space.

Definition 2.3. [14] Let $A, B \in [0,T] \to \mathbb{R}$ and if $A(t) \leq \lambda + \int_0^t B(s)A(s)ds$ for $t \in [0,T]$ then
$A(t) \leq \lambda \left(\int_0^t B(s)ds\right)$ for all $t \in [0,T]$ with $\lambda \geq 0$.

Definition 2.4. [13] The sequence $\{X_n\}$ converges to X in L^2 if $E(|X_n|^2) < \infty$ and $E(\|X_n - X\|)^2 \longrightarrow 0$ when $n \to \infty$.

Definition 2.5. [25] The Itô integral of $f \in v(s,T)$ is defined by $\int_s^T f(t,w)dB(t)(w)$ $= lim_{n\to\infty} \int_s^T \varphi_n(t,w)dB$, where φ_n is the sequence of elementary functions such that $E\left(\int_s^T (f-\varphi_n)^2 dt\right) \to 0$ as $n \to \infty$.

Lemma 2.1. [25] *The Itõ isometry of $f \in v(s,T)$ is given by*
$E\left((\int_s^T (f(t,w)dB(t)(w))^2\right) = E\left(\int_s^T (f^2(t,w)dt)\right)$.

3. Shifted Legendre Polynomials

3.1. Preliminaries and properties. The Legendre polynomials, $P_n(z)$, are the solutions of Legendre's Differential Equations [2]. The orthogonal property of Legendre polynomials is defined as $\int_{-1}^1 P_n(z)P_m(z)dz = \frac{2}{2n+1}\delta_{nm}$, where δ_{nm} is the Kronecker delta. The shifted Legendre polynomials $L_n(t)$ are derived from $P_n(z)$ by replacing z with $(2t-1)$, which in turn refines the interval to $[0,1]$. The orthogonal property of $L_n(t)$ with Kronecker delta in $[0,1]$ is defined by $\int_0^1 L_n(t)L_m(t)dt = \frac{1}{2n+1}\delta_{nm}$. Then,
(i) the recurrence relation of $L_n(t)$ is defined as

$$L_{i+1}(t) = \frac{(2i+1)(2t-1)}{i+1}L_i(t) - \frac{i}{i+1}L_{i-1}(t), i = 1, 2..., \tag{3.1}$$

where $L_0(t) = 1$ and $L_1(t) = 2t-1$.
(ii) The analytic form of the shifted Legendre polynomials $L_n(t)$ of degree n is given by

$$L_n(t) = \sum_{i=0}^{n} (-1)^{n+i} \frac{(n+i)!}{(n-i)!} \frac{t^i}{(i!)^2}. \tag{3.2}$$

Note that $L_n(0) = (-1)^n$ and $L_n(1) = 1$.
(iii) The shifted Legendre vector L(t) is

$$L(t) = [L_0(t) \quad L_1(t) \quad . \quad . \quad . \quad L_N(t)]^T. \tag{3.3}$$

(iv) the matrix form of L(t) which is of degree N can be represented as

$$\begin{bmatrix} 1 & 0 & \cdots & 0 \\ (-1)^{1+0}\frac{(1+0)!}{(1-0)!(0!)^2} & (-1)^{1+1}\frac{(1+1)!}{(1-1)!(1!)^2} & \cdots & 0 \\ (-1)^{2+0}\frac{(2+0)!}{(2-0)!(0!)^2} & (-1)^{2+1}\frac{(2+1)!}{(2-1)!(1!)^2} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ (-1)^{N+0}\frac{(N+0)!}{(N-0)!(0!)^2} & (-1)^{N+1}\frac{(N+1)!}{(N-1)!(1!)^2} & \cdots & (-1)^{N+N}\frac{(N+N)!}{(N-N)!(N!)^2} \end{bmatrix} \begin{bmatrix} 1 \\ t \\ t^2 \\ \vdots \\ t^N \end{bmatrix} \tag{3.4}$$

Thus

$$L(t) = DY(t). \tag{3.5}$$

The dual matrix Q_1 is

$$\begin{aligned} Q_1 &= \int_0^1 L(t)L^T(t)dt = \int_0^1 DY(t)(DY(t))^T dt, \\ &= D\left(\int_0^1 Y(t)Y^T(t)dt\right) D^T, \\ &= DHD^T, \end{aligned} \tag{3.6}$$

where H, a Hilbert matrix of order $(N+1)$ is given by

$$H = \int_0^1 Y(t)Y^T(t)dt = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \cdots & \frac{1}{N+1} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \cdots & \frac{1}{N+2} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \cdots & \frac{1}{N+3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{N+1} & \frac{1}{N+2} & \frac{1}{N+3} & \cdots & \frac{1}{2N+1} \end{bmatrix}.$$

Theorem 3.1. [23] *Any arbitrary function $u(t) \in L^2[0,1]$ can be approximated in terms of $L_n(t)$ as*

$$u(t) = \sum_{n=0}^{\infty} u_n L_n(t), \tag{3.7}$$

from which the coefficients u_j are given by

$$u_j = (2j+1)\int_0^1 u(x)L_j(x)dx, j = 0, 1, ... \tag{3.8}$$

If $u(t)$ is approximated by the first $(N+1)$ terms, then

$$u(t) \simeq \sum_{n=0}^{N} u_n L_n(t) = U^T L(t) = L^T(t)U,$$

where U is the shifted Legendre coefficient vector given by

$$U = [u_0 \quad u_1 \quad . \quad . \quad . \quad u_N]^T.$$

We approximate the kernel function by truncating the Taylor series of degree N in the form

$$k(t,s) = \sum_{m=0}^{N}\sum_{n=0}^{N} k_{mn} t^m s^n,$$

where $k_{mn} = \frac{1}{m!n!}\frac{\partial^{m+n}k(0,0)}{\partial t^m \partial s^n}$, $n, m = 0, 1, ..., N$.
The matrix form of the above expression is given by $k(t,s) = Y(t)KY^T(s)$. Additionally, the kernel function $k(t,s)$ can be expanded approximately by $L_m(s)$ and $L_n(t)$ of degree N in the form

$$k_N(t,s) = \sum_{m=0}^{N}\sum_{n=0}^{N} L_{k_{mn}} L_m(t) L_n(s),$$

and the matrix form of $k(t,s)$ in terms of $L(t)$ and $L^T(s)$ is

$$k(t,s) = L(t)K_L L^T(s), K_L = L_{k_{mn}}.$$

3.2. Operational Matrices. In the subsequent parts of this section, we construct operational matrices as follows. We define the product matrix $Q(t)$, as

$$Q(t) = L(t)L^T(t), \tag{3.9}$$

where $Q(t)$ is an $(N+1)$ order matrix. Let $U = [u_0 \quad u_1 \quad . \quad . \quad . \quad u_N]^T$, then

$$Q(t)U \simeq \hat{U}L(t). \tag{3.10}$$

$\hat{U}$ is the product operational matrix of shifted Legendre polynomials, which is calculated as

$$Q(t)U = D\left[\sum_{i=0}^{N} u_i L_i(t) \quad \sum_{i=0}^{N} u_i t L_i(t) \quad . \quad . \quad . \quad \sum_{i=0}^{N} u_i t^n L_i(t)\right]^T. \tag{3.11}$$

By approximating each $t^k L_i(t)$ by $L^T(t)C_{k,i}$, we get

$$C_{k,i} = [C_0^{k,i} \quad C_1^{k,i} \quad . \quad . \quad . \quad C_N^{k,i}]^T.$$

From Eq.(7),

$$\int_0^t t^k L_i(t)L(t)dt \simeq \left[\int_0^t L(t)L^T(t)dt\right] C_{k,j} = Q_1 C_{k,j}.$$

Therefore, for each i and k , we get

$$\begin{aligned} C_{k,i} &\simeq Q_1^{-1}\int_0^t t^k L(t)L_i(t)dt, \\ &= Q_1^{-1}\left[\int_0^t t^k L_0(t)L_i(t)dt \quad \int_0^t t^k L_1(t)L_i(t)dt \quad . \quad . \quad . \quad \int_0^t t^k L_N(t)L_i(t)dt\right]^T. \end{aligned}$$

Now the term $\sum_{i=0}^{N} u_i t^k L_i(t)$ can be computed as follows:

$$\begin{aligned}
\sum_{i=0}^{N} u_i t^k L_i(t) &\simeq \sum_{i=0}^{N} u_i L^T(t) C_{k,i}, \\
&= \sum_{i=0}^{N} u_i \sum_{j=0}^{N} L_j(t) C_j^{k,i}, \\
&= \sum_{j=0}^{N} L_j(t) \sum_{j=0}^{N} u_i C_j^{k,i}, \\
&= L^T(t) \begin{bmatrix} \sum_{i=0}^{N} u_i C_0^{k,i} & \sum_{i=0}^{N} u_i C_1^{k,i} & . & . & . & \sum_{i=0}^{N} u_i C_N^{k,i} \end{bmatrix}^T, \\
&= L^T(t) [C_{k,0} \quad C_{k,1} \quad . \quad . \quad . \quad C_{k,N}]^T U.
\end{aligned}$$

Thus,

$$\sum_{i=0}^{N} u_i t^k L_i(t) \simeq L^T(t) \hat{C}_k. \tag{3.12}$$

where $\hat{C}_k = [C_{k,0} \quad C_{k,1} \quad . \quad . \quad . \quad C_{k,N}]U$, $k = 0, 1, 2 \dots N$.
Let $\hat{L} = [\hat{C_0} \quad \hat{C_1} \quad \dots \quad \hat{C_N}]$ be a matrix. From (3.11) and (3.12), we obtain $\hat{U} = D\hat{L}^T$.
The integrals of $L_n(s)$ are evaluated with the aid of recurrence property of $L_n(t)$

$$\int_0^t L_n(s) ds = \frac{1}{2(2n+1)} [L_{n+1}(t) - L_{n-1}(t)]. \tag{3.13}$$

Therefore,

$$\int_0^t L(s) ds = PL(t) - \frac{1}{2(2n+1)} L_{n+1}(t), \tag{3.14}$$

where P is the matrix, which denotes the integration matrix of polynomials, given by

$$P = \begin{bmatrix}
\frac{1}{2} & \frac{1}{2} & 0 & 0\dots & 0 & 0 \\
\frac{-1}{6} & 0 & \frac{1}{6} & 0\dots & 0 & 0 \\
0 & \frac{1}{10} & 0 & \frac{1}{10}\dots & 0 & 0 \\
0 & 0 & \frac{-1}{14} & 0\dots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0\dots & 0 & \frac{1}{2(2m-3)} \\
0 & 0 & 0 & 0\dots & \frac{-1}{2(2m-3)} & 0
\end{bmatrix}. \tag{3.15}$$

The integration of the vector $L(t)$ can be approximated from (3.14)

$$\int_0^t L(s) ds \simeq PL(t). \tag{3.16}$$

Hence any function $f(t)$ is approximated as

$$\int_0^t f(s) ds \simeq \int_0^t F^T L(s) ds = F^T PL(t). \tag{3.17}$$

3.3. Stochastic operational matrix of shifted Legendre polynomials. For the vector $L(t)$, we define it's Itõ integral with stochastic operational matrix of integration P_s as

$$\int_0^t L(s)dW(s) = P_s L(t), \tag{3.18}$$

where P_s is the stochastic operational matrix of integration of order $(N+1) \times (N+1)$. It is computed as follows:

$$\int_0^t L(s)dW(s) = \int_0^t DX(s)dW(s), \tag{3.19}$$

$$= D\left[\int_0^t dW(s) \quad \int_0^t sdW(s) \quad . \quad . \quad . \quad \int_0^t s^N dW(s)\right]^T,$$

$$= D\left[W(t)Y(t) - \left[0 \quad \int_0^t dW(s) \quad . \quad . \quad . \quad N\int_0^t s^{N-1}dW(s)\right]^T\right],$$

$$= D\vartheta(t) = D(\lambda_i), i = 0, 1, ..., N,$$

where $\lambda_i = t^i W(t) - \int_0^t s^{i-1}W(s)ds$, $i = 0, 1, ..., N$.
Evaluating the integral for each i, we get,
$\lambda_i = t^i W(t) - \frac{t^i}{4}(2(\frac{t}{2})^{i-1}W(\frac{t}{2}) + t^{i-1}W(t)) = [(1-\frac{i}{4})W(t) - \frac{i}{2}W(\frac{t}{2})]t^i$.
We assume that $W(0.5)$ and $W(0.25)$ are the approximate value of $W(t)$ and $W(\frac{t}{2})$ respectively for any value of $t \in [0,1]$. Hence,

$$D\vartheta(t) = D\ \Gamma_s \begin{pmatrix} 1 \\ t \\ \vdots \\ t^N \end{pmatrix},$$

where

$$\Gamma_s = \begin{pmatrix} W(0.5) & 0 & \dots & 0 \\ 0 & \frac{3}{4}W(0.5) - \frac{1}{2}W(0.25) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & (1-\frac{N}{4})W(0.5) - \frac{N}{2^N}W(0.25) \end{pmatrix}.$$

Hence, $D\vartheta(t) = D\Gamma_s Y(t) = D\Gamma_s D^{-1}L(t) = P_s L(t)$, where $P_s = D\Gamma_s D^{-1}$.
By using (3.7) and (3.16), the Itõ integral of any function $u(t)$ is defined as

$$\int_0^t u(s)dW(s) = \int_0^t U^T L(s)dW(s) = U^T P_s L(t). \tag{3.20}$$

3.4. Proposed method. The stochastic processes $X(t)$ and $k(t,s)$ are approximated as follows:

$$X(t) = X^T L(t) \text{ (or) } L^T(t)X,$$
$$k(t,s) = L^T(t)KL(s) \text{ (or) } L^T(s)KL(t),$$

where X and K are the shifted Legendre coefficient matrices corresponding to $X(t)$ and $k(t,s)$ respectively. Substituting the above approximations in (1.1),

$$X^T L(t) = X_0 + \lambda_1 \int_0^t \frac{X^T L(s)}{(t-s)^\alpha} ds + \lambda_2 \int_0^t L^T(t) K L(s) L^T(s) X dB(s). \quad (3.21)$$

Let $I_1 = \lambda_1 \int_0^t \frac{X^T L(s)}{(t-s)^\alpha} ds$ and $I_2 = \lambda_2 \int_0^t L^T(t) K L(s) L^T(s) X dB(s)$.

Consider I_1,

$$\begin{aligned} I_1 &= \lambda_1 \int_0^t \frac{X^T L(s)}{(t-s)^\alpha} ds, \\ &= \lambda_1 X^T \int_0^t \frac{L(s)}{(t-s)^\alpha} ds, \text{ where } L(s) = \begin{bmatrix} 1 & s & s^2 & . & . & . & s^N \end{bmatrix}^T. \end{aligned}$$

$$\int_0^t \frac{L(s)}{(t-s)^\alpha} ds = \left[\int_0^t \frac{1}{(t-s)^\alpha} ds \quad \int_0^t \frac{s}{(t-s)^\alpha} ds \quad . \quad . \quad . \quad \int_0^t \frac{s^N}{(t-s)^\alpha} ds \right],$$

where, $\int_0^t \frac{s^n}{(t-s)^\alpha} ds = \frac{\Gamma(n+1)\Gamma(1-\alpha)}{\Gamma(n+2-\alpha)} t^{n+1-\alpha}$.

Therefore, I_1 becomes,

$$\begin{aligned} \lambda_1 \int_0^t \frac{X^T L(s)}{(t-s)^\alpha} ds &= \lambda_1 X^T \left[\frac{\Gamma(1)\Gamma(1-\alpha)}{\Gamma(2-\alpha)} t^{1-\alpha} \cdots \frac{\Gamma(N+1)\Gamma(1-\alpha)}{\Gamma(N+2-\alpha)} t^{N+1-\alpha} \right]^T, \\ &= \lambda_1 X^T R L(t) (say). \end{aligned}$$

Consider I_2,

$$\begin{aligned} \lambda_2 \int_0^t L^T(t) K L(s) L^T(s) X dB(s) &= \lambda_2 L^T(t) K \int_0^t L(s) L^T(s) X dB(s), \\ &= \lambda_2 L^T(t) K \int_0^t Q(s) X dB(s), \\ &= \lambda_2 L^T(t) K \hat{X} \int_0^t L(s) dB(s), \\ &= \lambda_2 L^T(t) K \hat{X} P_s L(s), \\ &= \lambda_2 \hat{M} L(t) (say). \end{aligned}$$

Substituting the above, (3.21) becomes,

$$X^T L(t) = X_0 + \lambda_1 X^T R L(t) + \lambda_2 \hat{M} L(t). \quad (3.22)$$

By collocating (3.22) at $(N+1)$ points defined on $t_i = \frac{2i+1}{2(N+1)}$, $i = 0, 1, 2, ..., N$, we arrive at the following $(N+1)$ linear algebraic system of equations:

$$X^T L(t_i) = X_0 + \lambda_1 X^T R L(t_i) + \lambda_2 \hat{M} L(t_i), \; i = 0, 1, 2, ..., N. \quad (3.23)$$

Solving the above linear algebraic system of equations using an appropriate well-known numerical method, the approximate solution of (1.1) is obtained.

4. Theoretical Analysis

Let the error function $e_N(t) = (X(t) - X_N(t))$, where $X_N(t)$ is the N^{th} degree approximation of the exact solution $X(t)$. The error bound and convergence theorem are as follows:

Theorem 4.1. *Let $f_N(t)$ be the shifted Legendre approximation of an arbitrary function $f(t)$, then the error bound is given by $\|f(t) - f_N(t)\| \leq C\hat{F}2^{-N},\ t \in [0,1],\ \hat{F} = \sup_t \left\|f^N(t)\right\|_{L^2}, C$ being a constant.*

Proof.

$$\begin{aligned}
\|f(t) - f_N(t)\|^2 &= \int_0^1 (f(t) - f_N(t))^2\, dt, \\
&\leq \int_0^1 \left(\frac{1}{N!2^N}\hat{F}dt\right)^2, \\
&\leq \left(\frac{1}{N!2^N}\hat{F}\right)^2, \\
&= \left(C\hat{F}2^{-1}\right)^2, \text{ where } C = \frac{1}{N!} \text{ and } \hat{F} = \sup_t \left\|f^N(t)\right\|.
\end{aligned}$$

Theorem 4.2. *Let $k_N(t,s)$ be the shifted Legendre approximation of an arbitrary function $k(t,s)$, then the error bound is given by $\|k(t,s) - k_N(t,s)\| \leq \hat{C}\hat{K}2^{-2N},\ (t,s) \in [0,1] \times [0,1],\ \hat{K} = \sup_{(t,s)} \left\|\frac{\partial^{2N} k(t,s)}{\partial t^N \partial s^N}\right\|_{L^2}, \hat{C}$ being a positive constant.*

Proof. The proof of this theorem is similar to the proof of the previous theorem.

Theorem 4.3. *Consider the singular stochastic Itõ-Volterra integral equation of Abel-type denoted by (1.1). Let $X_N(t)$ be the approximate solution obtained by using shifted Legendre polynomial approximation. Furthermore, assume the following:*
(I) $X_N(t) \leq A_1,\ t \in [0,1]$
(II) $k_N(t,s) \leq A_2,\ (t,s) \in [0,1] \times [0,1]$
(III) $G(N) < 1$, then we have,

$$\|X(t) - X_N(t)\| \leq \frac{|\lambda_1| \left|\frac{(X(\epsilon) - X_N(\epsilon))}{1-\alpha}\right| t^{(1-\alpha)} + |\lambda_2| \|B(t)\| A_1 \lambda(N)}{1 - |\lambda_2| \|B(t)\| (\lambda(N) + A_2)}$$

and $X_N(t) \to X(t)$in L^2 when $E\left(|e_N(t)|^2\right) \to 0$ where $\lambda(N) = \hat{C}\hat{K}2^{-2N}$ and $G(N) = |\lambda_2| \|B(t)\| (\lambda(N) + A_2)$.

Proof. We know that,

$$\begin{aligned}
X(t) &= X_0 + \lambda_1 \int_0^t \frac{X(s)}{(t-s)^\alpha} ds + \lambda_2 \int_0^t k(t,s) X(s) dB(s), \\
X_N(t) &= X_0 + \lambda_1 \int_0^t \frac{X_N(s)}{(t-s)^\alpha} ds + \lambda_2 \int_0^t k_N(t,s) X_N(s) dB(s).
\end{aligned}$$

Therefore,

$$e_N(t) = \lambda_1 \int_0^t \frac{X(s) - X_N(s)}{(t-s)^\alpha} ds + \lambda_2 \int_0^t \left(k(t,s)X(s) - k_N(t,s)X_N(s)\right) dB(s). \tag{4.1}$$

Let

$$J_1 = \lambda_1 \int_0^t \frac{X(s) - X_N(s)}{(t-s)^\alpha} ds,$$
$$J_2 = \lambda_2 \int_0^t \left(k(t,s)X(s) - k_N(t,s)X_N(s)\right) dB(s).$$

To simplify the above expressions, we state the result as mentioned in [27]

If $h : [p,q] \to \mathbb{R}$ is a continuous function and f is an integrable function that its sign does not change on the interval $[p,q]$, then there exists a constant $\epsilon \in (p,q)$ such that $\int_p^q h(x)f(x)dx = h(\epsilon)\int_p^q f(x)dx$.

By the above result,

$$\begin{aligned} J_1 &= \lambda_1 \left(X(\epsilon) - X_N(\epsilon)\right) \int_0^t \frac{1}{(t-s)^\alpha} ds, \\ &= \lambda_1 \frac{\left(X(\epsilon) - X_N(\epsilon)\right)}{1-\alpha} t^{(1-\alpha)}, \\ \text{therefore, } \|J_1\| &\le |\lambda_1| \left| \frac{\left(X(\epsilon) - X_N(\epsilon)\right)}{1-\alpha} \right| t^{(1-\alpha)}. \end{aligned}$$

and

$$J_2 = \lambda_2 \int_0^t \left(k(t,s)X(s) - k_N(t,s)X_N(s)\right) dB(s),$$
$$\|J_2\| \le |\lambda_2| \, \|B(t)\| \left(\lambda(N) \|e_N(t)\| + A_2 \|X(t) - X_N(t)\| + A_1\lambda(N)\right).$$

Using the above inequalities in (4.1) ,

$$\begin{aligned} \|e_N(t)\| &\le |\lambda_1| \left| \frac{\left(X(\epsilon) - X_N(\epsilon)\right)}{1-\alpha} \right| t^{(1-\alpha)} \\ &+ |\lambda_2| \, \|B(t)\| \left(\lambda(N) \|e_N(t)\| + A_2 \|e_N(t)\| + A_1\lambda(N)\right) \\ \|e_N(t)\| &\le \frac{|\lambda_1| \left| \frac{(X(\epsilon) - X_N(\epsilon))}{1-\alpha} \right| t^{(1-\alpha)} + |\lambda_2| \, \|B(t)\| \, A_1\lambda(N)}{1 - |\lambda_2| \, \|B(t)\| \left(\lambda(N) + A_2\right)} \\ \text{also, } \|e_N(t)\| &= E\left(|e_N(t)|^2\right). \end{aligned}$$

By Gronwall inequality, $E\left(|e_N(t)|^2\right) \to 0$. Hence, the theorem is proved.

4.1. Time complexity. This proposed method deals with matrix multiplication and solving a system of equations. The key steps involved in the calculation of time complexity of the proposed approach are the construction of the approximation vector, computation of various matrices like $D, P, \Gamma_s, P_s, R, \hat{M}$. This approach also involves the process of initialization of λ_1 and λ_2 values. Finally, generating and solving the system of equations in terms of connection coefficients according to (3.23) and handling them in terms of traditional numerical methods.

Theorem 4.4. *Suppose that N and k are the degree of the approximate function and the number of simulations respectively, then the time complexity of this proposed method is $O(k(N+1)^2)$.*

Proof. The key steps involved in the proposed approach are as given below:
Step 1: Construct the approximate vector $L(x), L(t)$.
Step 2: Compute the matrices $D, H, K_L, P, \Gamma_s, P_s, R, \hat{M}$.
Step 3: Initialize λ_1 and λ_2.
Step 4: Generate and solve the system of algebraic equations according to (3.23).
This proposed method has 2 major steps of computation. Step 2 computes various matrix multiplications which require $O\left((N+1)^2\right)$ time. Step 4 computes the system of equations and displays the approximate solution numerically. They require $O\left((N+1)^2\right)$ time. These steps are executed k times. Hence, the overall time complexity of this proposed method is $O\left(k\,(N+1)^2\right)$. □

5. Numerical Analysis

Two examples are considered to illustrate the reliability, effectiveness and efficiency of our proposed method. The numerical calculations were performed by running a code written using MAPLE software and MATLAB.

Example 1:

Consider the linear singular stochastic Itõ - Volterra integral equation: [29]

$$X(t) = \frac{1}{18} - \int_0^t \frac{X(s)}{(t-s)^\alpha} ds - \int_0^t sin(s)X(s)dB(s), \ t \in [0,1]. \tag{5.1}$$

The exact solution for the above problem is

$$X(t) = \frac{1}{18} exp\left(-\frac{t^{1-\alpha}}{1-\alpha} - \frac{1}{4}t + \frac{1}{8}sin(2t) - \int_0^t sin(s)dB(s)\right).$$

Figure.1 is the error graph obtained for $\alpha = 0.125, 0.25$ for $N = 4, 8$. Tables 1 and 2 depict the exact solution, and the comparison of approximate solution using the proposed methodology(SLP) with Bernoulli polynomials method (BPM) and Boubaker Wavelet method (BWM) with $\alpha = 0.5$ for $N = 4$ and $N = 8$ respectively.

TABLE 1. Error comparison for Example 1($\alpha = 0.5$ and N=4).

	N=4				Absolute error		
t_i	Exact	SLP	BPM[29]	BWM[30]	SLP	BPM	BWM
0.0	0.0555	0.0556	0.0453	0.0496	0.0001	0.0102	0.0060
0.1	0.029	0.0295	0.0379	0.0267	0.0005	0.0089	0.0023
0.2	0.022	0.0227	0.0323	0.0219	0.0004	0.0100	0.0004
0.3	0.017	0.0185	0.0281	0.0245	0.0010	0.0106	0.0070
0.4	0.015	0.0156	0.0249	0.0154	0.0005	0.0098	0.0003
0.5	0.014	0.0133	0.0223	0.0216	0.0007	0.0083	0.0076
0.6	0.010	0.0115	0.0200	0.0177	0.0013	0.0098	0.0075
0.7	0.014	0.0101	0.0180	0.0118	0.0044	0.0035	0.0027
0.8	0.012	0.0090	0.0163	0.0154	0.0039	0.0033	0.0025
0.9	0.008	0.0082	0.0148	0.0050	0.0000	0.0066	0.0032

TABLE 2. Error comparison for Example 1($\alpha = 0.5$ and N=8).

N=8					Absolute error		
t_i	Exact	SLP	BPM	BWM	SLP	BPM	BWM
0.0	0.0555	0.0556	0.0492	0.0523	0.0001	0.0063	0.0033
0.1	0.0300	0.0295	0.0385	0.0313	0.0006	0.0085	0.0012
0.2	0.0220	0.0227	0.0322	0.0291	0.0002	0.0097	0.0066
0.3	0.0187	0.0187	0.0277	0.0245	0.0000	0.0090	0.0058
0.4	0.0134	0.0156	0.0242	0.0154	0.0022	0.0108	0.0020
0.5	0.0111	0.0133	0.0213	0.0132	0.0022	0.0103	0.0021
0.6	0.0134	0.0115	0.0190	0.0148	0.0019	0.0056	0.0014
0.7	0.0100	0.0107	0.0170	0.0130	0.0000	0.0063	0.0023
0.8	0.0074	0.0090	0.0154	0.0054	0.0016	0.0080	0.0020
0.9	0.0059	0.0082	0.0140	0.0060	0.0023	0.0081	0.0001

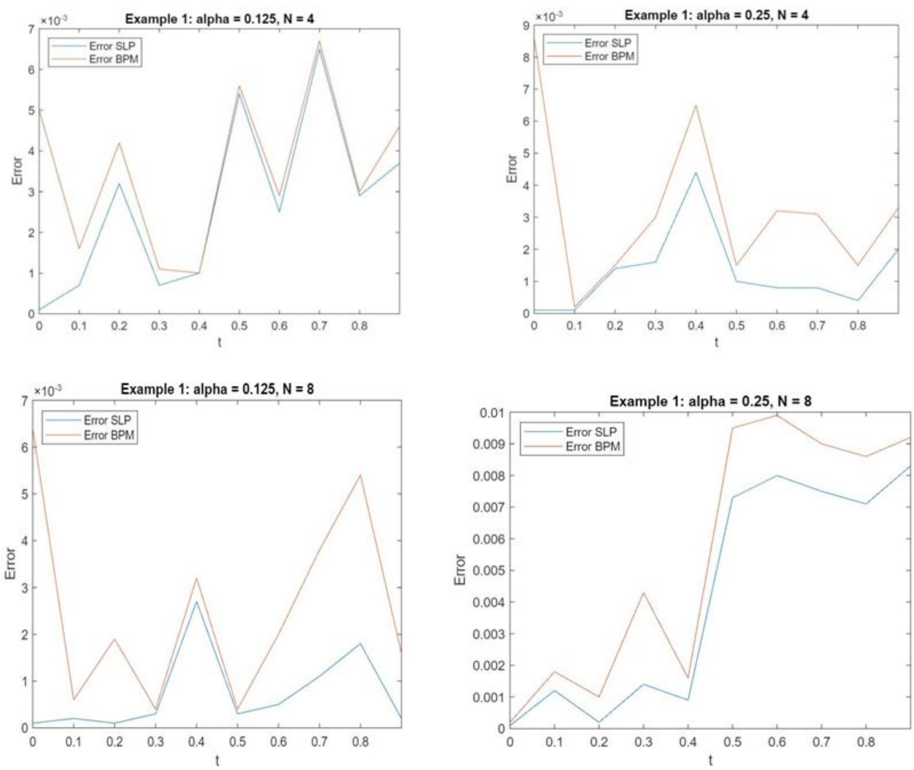

FIGURE 1. Error graphs for Example 1.

Example 2:
Consider the linear singular stochastic Itõ - Volterra integral equation:[29]

$$X(t) = \frac{1}{36} - \int_0^t \frac{X(s)}{(t-s)^\alpha} ds - \int_0^t e^x sin(s)X(s)dB(s), \ t \in [0,1]. \qquad (5.2)$$

The exact solution for the above problem is

$$X(t) = \frac{1}{36}exp\left(-\frac{t^{1-\alpha}}{1-\alpha} - \frac{1}{4}te^{2t} + \frac{1}{8}e^{2t}sin(2t) - \int_0^t e^x sin(s)dB(s)\right)$$

Figure.2 is the error graph obtained for $\alpha = 0.125, 0.25$ for $N = 4, 8$. Tables 3 and 4 depict the exact solution, and the comparison of approximate solution using the proposed methodology (SLP) with Bernoulli polynomials method (BPM) and Boubaker Wavelet method (BWM) with $\alpha = 0.5$ for $N = 4$ and $N = 8$ respectively.

Table 3. Error comparison for Example 2($\alpha = 0.5$ and N=4).

	N=4				Absolute error		
t_i	Exact	SLP	BPM[29]	BWM[30]	SLP	BPM	BWM
0.0	0.0277	0.0277	0.0227	0.0267	0.0000	0.0050	0.0011
0.1	0.0144	0.0148	0.0190	0.0140	0.0004	0.0046	0.0004
0.2	0.0111	0.0113	0.0162	0.0119	0.0002	0.0051	0.0008
0.3	0.0086	0.0092	0.0140	0.0092	0.0006	0.0055	0.0006
0.4	0.0073	0.0077	0.0124	0.0066	0.0004	0.0051	0.0007
0.5	0.0070	0.0065	0.0110	0.0064	0.0005	0.0040	0.0006
0.6	0.0043	0.0055	0.0097	0.0017	0.0012	0.0054	0.0026
0.7	0.0091	0.0050	0.0086	0.0088	0.0041	0.0005	0.0003
0.8	0.0079	0.0052	0.0076	0.0078	0.0027	0.0003	0.0001
0.9	0.0028	0.0037	0.0066	0.0016	0.0009	0.0038	0.0012

Table 4. Error comparison for Example 2($\alpha = 0.5$ and N=8).

	N=8				Absolute error		
t_i	Exact	SLP	BPM	BWM	SLP	BPM	BWM
0.0	0.0277	0.0277	0.0254	0.0264	0.0000	0.0023	0.0014
0.1	0.0146	0.0148	0.0200	0.0156	0.0002	0.0054	0.0010
0.2	0.0115	0.0113	0.0171	0.0125	0.0002	0.0056	0.0010
0.3	0.0097	0.0093	0.0152	0.0098	0.0004	0.0054	0.0001
0.4	0.0090	0.0078	0.0137	0.0086	0.0012	0.0047	0.0004
0.5	0.0077	0.0067	0.0126	0.0066	0.0010	0.0049	0.0011
0.6	0.0075	0.0058	0.0118	0.0074	0.0017	0.0043	0.0001
0.7	0.0106	0.0050	0.0112	0.0102	0.0056	0.0006	0.0004
0.8	0.0099	0.0052	0.0109	0.0093	0.0047	0.0010	0.0006
0.9	0.0083	0.0055	0.0108	0.0061	0.0028	0.0026	0.0022

6. Conclusion

In this paper, shifted Legendre orthogonal polynomial approximation method using operational matrices has been developed to solve singular stochastic integral equations of Abel's type. The proposed method, compared to the other methods such as the Bernoulli polynomial collocation method and the Boubaker wavelets method, has less computational error. The numerical calculations performed by running a code written in MAPLE and the graphs demonstrate the better efficiency and accuracy of shifted Legendre polynomial approximation and the resultant operational matrices. The theoretical analysis manifests that the error approaches zero for a higher degree of approximation. Hence, wherever the

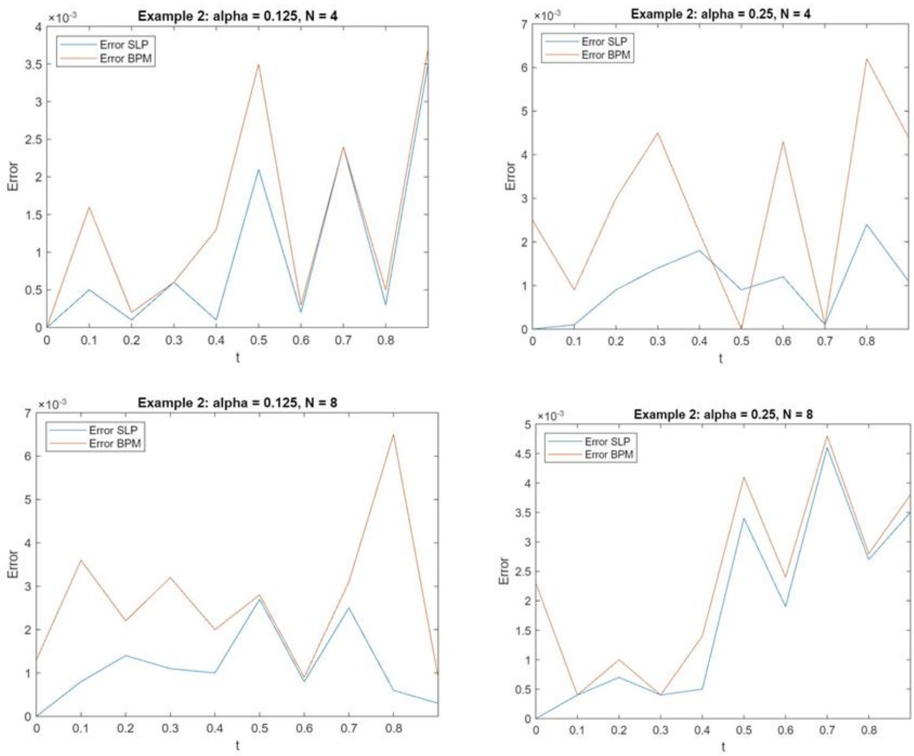

FIGURE 2. Error graphs for Example 2.

exact solution of a problem considered is not available, our proposed method, a powerful tool, can be used, not only to obtain an efficient numerical solution of singular stochastic integral equations but also, in the future, to stochastic partial differential equations of higher orders.

References

[1] I. Aziz, New algorithms for the numerical solution of nonlinear Fredholm and Volterra integral equations using Haar wavelets, *J. Comput. Appl. Math.* **239** (2013), 333-345.

[2] H.D. Azodi, Using shifted Legendre scaling functions for solving fractional biochemical reaction problem, *Caspian J. Math.Sci.* **7(1)** 2018, 88-101.

[3] A.H. Bhrawy, E. Tohidi and F. Soleymani, A new Bernoulli matrix method for solving high-order linear and nonlinear Fredholm integro-differential equations with piecewise intervals, *Appl Math Comput.* **219(2)** (2012), 482-497.

[4] A. Borhanifar and K. Sadri, A new operational approach for numerical solution of generalized functional integro-differential equations, *J. Comput. Appl. Math.* **279** (2015), 80-96.

[5] T.L. Chow, *Mathematical Methods for Physicists: A concise introduction*, Cambridge University Press, 2000.

[6] K. Dareiotis and J.M. Leahy, Finite difference schemes for linear stochastic integro-differential equations, *Stoch Proc Appl.* **126(10)** (2016), 3202-3234.

[7] A. Dezhbord, T. Lotfi and K. Mahdiani, A new efficient method for cases of the singular integral equation of the first kind, *J. Comput. Appl. Math.* **296** (2016), 156-169.

[8] T.C. Gard, *Introduction to Stochastic Differential Equations*, Marcel Dekker, New York, 1998.

[9] E. Gokmen, G. Yuksel and M. Sezer, A numerical approach for solving Volterra type functional integral equations with variable bounds and mixed delays, *J. Comput. Appl. Math.* **311** (2017), 354-363.

[10] M.H. Heydari, M.R. Hooshmandasl, A. Shakiba and C. Cattani, Legendre wavelets Galerkin method for solving nonlinear stochastic integral equations, *Nonlinear Dynam.* **85(2)** (2016) 1185-1202.

[11] O.R. Işik, M. Sezer and Z. Güney, Bernstein series solution of a class of linear integro−differential equations with weakly singular kernel, *Appl. Math. Comput.* **217(16)**(2011), 7009-7020.

[12] S. Karimi and M. Jozi, A new iterative method for solving linear Fredholm integral equations using the least squares method, *Appl. Math. Comput.* **250** (2015), 744-758.

[13] M. Khodabin, K. Maleknejad and T. Damercheli, Approximate solution of the stochastic Volterra integral equations via expansion method, *Int. J. Ind. Math.* **6** (2014), 41-48.

[14] F. Klebaner, *Introduction to Stochastic Calculus with Applications*, Second Edition, Imperial college Press, 2005.

[15] X. Mao, *Stochastic Differential Equations and applications*, Second Edition, Horwood Publishing, Chichester. UK 2007.

[16] H. Mei, G. Yin and F. Wu, Properties of stochastic integro-differential equations with infinite delay: regularity, ergodicity, weak sense Fokker-Planck equations, *Stoch Proc Appl.* **126(10)** (2016), 3102-3123.

[17] F. Mirzaee and N. Samadyar, Numerical solution based on two-dimensional orthonormal Bernstein polynomials for solving some classes of two-dimensional nonlinear integral equations of fractional order, *Appl. Math. Comput.* **344** (2019), 191-203.

[18] F. Mohammadi, A wavelet-based computational method for solving stochastic Itõ-Volterra integral equations, *J. Comput. Phys.*,**298** (2015), 254-265.

[19] R. Mokhtari and E. Babolian, A new approach for solving Volterra integral equations using the reproducing kernel method, *Int. J. Ind. Math.* **9(1)** 2017, 21-26.

[20] F. Mirzaee and E. Hadadiyan, Numerical solution of Volterra-Fredholm integral equations via modification of hat functions, *Appl Math Comput.* **280** (2016), 110-123.

[21] F. Mirzaee and S.F. Hoseini, Application of Fibonacci collocation method for solving Volterra-Fredholm integral equations, *Appl. Math. Comput.* **273** (2016), 637-644.

[22] F. Mirzaee, N. Samadyar and S.F. Hoseini, Euler polynomial solutions of nonlinear stochastic Itõ-Volterra integral equations, *J. Comput. Appl. Math.* **330** (2018), 574-585.

[23] F. Mirzaee and N. Samadyar, Application of operational matrices for solving system of linear Stratonovich Volterra integral equation, *J. Comput. Appl. Math.* **320** (2017), 164-175.

[24] S. Nemati, P.M. Lima and Y. Ordokhani, Numerical solution of a class of two-dimensional nonlinear Volterra integral equations using Legendre polynomials, *J. Comput. Appl. Math.* **242** (2013), 53-69.

[25] B. Oksendal, *Stochastic Differential Equations, an introduction with Applications*, Fifth Edition, Springer-Verlag, New York, 1998.

[26] O. Ore and N. Henrik Abel: *Mathematician Extraordinary*, Chelsea Pub Co.1957.

[27] W. Rudin, *Principles of Mathematical Analysis*, 3rd ed. New York (NY): McGraw-Hill Book Company, 1976.
[28] N. Samadyar and F. Mirzaee, Numerical solution of two-dimensional weakly singular stochastic integral equations on non-rectangular domains via radial basis functions, *Eng. Anal. Bound. Elem.* **101** (2019), 27-36.
[29] N. Samadyar and F. Mirzaee, Orthonormal Bernoulli polynomials collocation approach for solving stochastic Itõ-Volterra integral equations of Abel type, *Int. J. Numer. Model.* **33** (2020), e2688.
[30] S. Shiralashetti and L. Lamani, Numerical solution of stochastic Abel's integral equations using Boubaker wavelets stochastic operational matrix of integration, *Stochastic Modeling and Applications* **24(1)** (2020), 55-68.
[31] N.H. Sweilam, A.M. Nagy, I.K. Youssef and M.M. Mokhtar, New spectral second kind Chebyshev wavelets scheme for solving systems of integro-differential equations, *Int. J. Appl. Comput. Math.* **3(2)** (2017), 333-345.
[32] A.M. Wazwaz and R. Rach, Two reliable methods for solving the Volterra integral equation with a weakly singular kernel, *J. Comput. Appl. Math.* **302** (2016), 71-80.

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Received: March 7, 2022; Revised: July 6, 2022; Accepted: July 10, 2022