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NONLINEAR FRACTIONAL ORDER NEUTRAL-TYPE STOCHASTIC INTEGRO-DIFFERENTIAL SYSTEM WITH ROSENBLATT PROCESS - A CONTROLLABILITY EXPLORATION

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Abstract. We investigate the controllability analysis of nonlinear fractional order neutral-type stochastic integro-differential system with non-Gaussian process. We stress out the stochastic term of our system driven by the uncomplicated non-Gaussian Hermite process known as the Rosenblatt process, which is named after by Murray Rosenblatt who first devised this introduced concept. This process is self-similar with consistent accretion and beside emerged as restriction in the non-central limit theorem, and it exists in the second wiener chaos. The necessary and sufficient conditions for the controllability are verified by employing fixed point techniques. At end, we present illustrative examples to clarify the abstract results.

1. Introduction

For several centuries ago, the investigators and technologists are always enthusiastic to work on a real-world problem to understand natural phenomenon. Many works are derived by modeling real life problems and try to get solutions for them and then apply the obtained results to real-life that used to live in a better way. Mathematical modeling is one of the best tools to solve this type of situations to investigate the solutions with some accuracies; it is used in different branches of sciences and engineering. To model a more complex natural phenomenon with more accurate solutions, it is needed to employ modified approaches like a complex system instead of using integer-order derivative, we can replace it by non-integer one. By using a fractional order derivative, we can study the queries of any complex natural phenomenon, fractional differential equations far-reaching applications towards physical phenomena such as fluid dynamics, etc. Recently, the subject of fractional calculus theory and its applications have been arising a considerable interest due to its ability to model many practical systems [6, 7, 11, 12, 20]. By using fractional derivatives, we can reveal the modifications in an interval. The fractional by-product is in non-indigenous nature, it makes

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fractional derivatives appropriate to mimic extra bodily phenomena along with earthquake vibrations, polymers,..., etc., see [17, 24].

The applied science has showed that many phenomena have been modeled with the aid of using fractional differential equations coincide with a few uncertainties. There are many fluctuations in the environments and also there are intrinsic and extrinsic noises available in the field. The necessity to clear up veritable issues for greater unique answers, it is recommended the view of stochastic fractional differential equations [5, 13]. Stochastic differential equations of fractional order play an emerging role rather than the integer-order systems and subsequently. The stochastic process is a probability distribution over a space of paths, the theory of stochastic processes was reconciled decades ago. Applications of stochastic processes as virtual identity can be found in numerous disciplines such as control theory, traffic engineering and renewal theory. The mathematical theory of stochastic analysis was developed by Ito. It is regulated via sense of means of boundary and preliminary conditions, however, currently, no longer foreordained via way of means of them. Each time the equation is solved beneath equal preliminary and boundary conditions, the answer takes exclusive numerical values although a particular sample emerges as the answer technique is repeated many times. It has huge applications in various research areas, including environment, finance, and medicine, etc. For important works of fractional stochastic systems and their applications, we may refer to [15, 18, 19, 23, 25].

Here the stochastic process is taken as the uncomplicated non-Gaussian Hermite process known as the *Rosenblatt process*. The most popular self-similar process is the fractional Brownian motion, which is also the only Hermite Gaussian process. Rosenblatt's process was introduced in 1961 by M. Rosenblatt in the work Independence and dependence [3]. Although outlined throughout the past 60s associated with the later 70s because of their look within the non-central limit theorem, the systematic analysis of Rosenblatt processes has solely been developed during the last 10 years, intended by their specific properties (self-similarity, stationarity of the increments, long-range dependence) since they are non-Gaussian and self-similar with stationary increments. A self-similar object is strictly or about such a region of itself. Self-similar processes are invariant in distribution beneath appropriate scaling. Among the applications of the Rosenblatt processes in statistics or economics, corresponding to the Rosenblatt distribution conjointly seems to be the straight line distribution of an estimation of the long-range dependence parameter. More details about this process can be found in [3, 21].

The Hermite process $(Z_H^k(t))$ is in a multiple Wiener-Ito stochastic integral with respect to Brownian motion $B(y)_{y\in R}$ is given as

$$(Z_H^k(t)) = C(H,k)\int_{\mathbb{R}^k}\int_0^t \Big(\prod_{j=1}^k (s-y_i)_+^{-(\frac{1}{2}+\frac{1-H}{k})}\Big) ds dB(y_1)...dB(y_k),$$

where $x_+ = \max(x,0)$, the constant $C(H,k)$ is positive, it is H-self similar for any $c > 0$, and it has stationary increments. In the above integral, when $k = 1$ the process is the fractional Brownian motion with Hurst parameter $H \in (\frac{1}{2}, 1)$. For $k \geq 2$ the process is not Gaussian. In particular, when $k = 2$, the process is known as the Rosenblatt process. Because of Gaussianity, numerous practical applications of fractional brownian motion process have been widely studied.

But in concrete situations when the Gaussianity is not plausible, one can use a Hermite process living in a higher chaos. More precisely, the Rosenblatt process is not Gaussian, but it has self-similar, stationary of increments and large range dependence property like fractional Brownian motion.

Controllability is an important aspect of mathematical control theory which was introduced by Kalman [10]. The concept of controllability denotes the ability to move the state of the dynamical control system from an initial state to the desired final state by using a suitable control function. In the last years, different aspects of controllability for ordinary as well as fractional dynamic systems, for both deterministic and stochastic structures, have been studied by many researchers [1, 2, 4, 8, 9, 14, 16, 22, 23].

This work is involved with nonlinear fractional-order neutral-type stochastic integro-differential system with Rosenblatt process, the controllability is deceased in the accessible source of studies. Our main contributions are highlighted as follows:

- We have developed a solution for the controllability problem of non-linear fractional order neutral type stochastic integro-differential system with Rosenblatt process.
- We take the terms in the system as a bounded linear operators instead of a matrix, which produces the same results as a matrix.
- The illustration the results on stochastic systems bounded linear operators are more competent.
- We take the stochastic term as driven by the Rosenblatt process which is non-Gaussian and has the properties like self-similarity, stationarity of the increments and has long range dependence.
- We intend to bring new lights to the Rosenblatt process, since many real-life phenomena are modeled by fractional Brownian motion a only Gaussian Hermite process, when the property of Gaussianity is failed one can use Rosenblatt process.
- We define the controllability Grammian operator, which is defined by the Mittag-Leffler function to prove the controllability results.
- By employing Banach contraction principle to prove the controllability criteria instead of semigroup theory which does not applicable to obtain the results on controllability.
- We have provided a numerical example to illustrate the theory.
- Generally speaking, both the Riemann-Liouville and the Caputo fractional operators do not possess neither semigroup nor commutative properties, which are inherent to the derivatives on integer order.

The paper is organized as follows. In Section 2, we review some essential facts from stochastic analysis and fractional calculus that are used to obtain our main results. In Section 3, we formulate a suitable solution representation and controllability criteria of Linear system. Then, we will extend the investigation to nonlinear system to be controllable in Section 4. Finally, in Section 5, we give appropriate examples to illustrate the given theory. We end with Section 6 of conclusions to our results of this research.

2. Preface

In this section we give some basic definitions and properties which are useful to establish our theoretical results.

2.1. Rosenblatt process.

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ be a filtered probability space consists of a probability space $(\Omega, \mathcal{F}, P)$ and a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ contained in $\mathcal{F}$, the filtered probability space is said to satisfy the usual conditions namely

- The probability space $(\Omega, \mathcal{F}, P)$ is complete.
- $\mathcal{F}_0$ contains all $A \in \mathcal{F}$ such that $P(A) = 0$.
- $\mathcal{F}_t = \mathcal{F}_{t+}, \forall t \in J$, where $\mathcal{F}_{t+}$ is the intersection of all $\mathcal{F}_s$ where $s > t$, i.e. the filtration is right continuous.

Suppose that $\{Z_H(t), t \in [0, b]\}$ is the one-dimensional Rosenblatt process with Hurst parameter $H \in (\frac{1}{2}, 1)$. That is, $Z_H(t)$ is a Non-Gaussian process with covariance function

$$E(Z_H(t), Z_H(s)) = \tfrac{1}{2}(|s|^{2H} + |t|^{2H} - |s-t|^{2H}).$$

Moreover, the Rosenblatt process with Hurst parameter $H > \frac{1}{2}$ has the representation as (see [21]):

$$Z_H(t) = d(H) \int_0^t \int_0^t \{ \int_{Y_1 \vee Y_2}^t \frac{\partial K^{H'}}{\partial u}(u, y_1) \frac{\partial K^{H'}}{\partial u}(u, y_2) du \} dB(Y_1) dB(Y_2)$$

Where $t \leq s$, $\{B(t), t \in [0, b]\}$ is a Brownian motion, and $K^H(t, s)$ is the kernel given by

$$K^H(t, s) = c_H s^{\frac{1}{2}-H} \int_s^t (u-s)^{H-\frac{3}{2}} u^{H-\frac{1}{2}} du$$

for $t > s$, where

$$c_H = \sqrt{\frac{H(2H-1)}{\Gamma(2-2H, H-\frac{1}{2})}},$$

and $\Gamma(.,.)$ denotes the Gamma function.

We put $K_H(t, s) = 0$ if $t \leq s$. Let $\{Z_n(t)\}_{n\in N}$ denote a sequence of two-sided one dimensional Rosenblatt process mutually independent on $(\Omega, \mathcal{F}, P)$. Consider a K-valued stochastic process $Z_Q(t)$ given by the series $Z_Q(t) = \sum_{n=1}^{\infty} Z_n(t) Q^{\frac{1}{2}} e_n, t \geq 0$. Moreover, if Q is a non-negative self-adjoint trace class operator, then, this series converges in the space K, that is, it holds that $Z_Q(t) \in L^2(\Omega, K)$. Then, the above $Z_Q(t)$ is a K-valued Q-Rosenblatt process with covariance operator Q.

Definition 2.1. [19] Let $L(K, Y)$ represents the space of all bounded linear operators from K to Y, and $Q \in L(K, Y)$ represents a non-negative self-adjoint operator in separable Hilbert spaces K and Y. Let $L_2^0 = L_2(Q^{\frac{1}{2}}K, Y)$ be the space of all Hilbert-Schmidt operators from $Q^{\frac{1}{2}}K$ into Y, where L_2^0 is a separable Hilbert space, equipped with the norm $\|\omega\|^2_{L_2^0} = \|\omega Q^{\frac{1}{2}}\|^2 = Tr(\omega Q \omega^*)$.

Definition 2.2. [19] Let $\omega : [o, b] \to L_2(Q^{\frac{1}{2}}K, V)$ such that

$$\sum_{n=1}^{\infty} \|K_H^*(\phi Q^{\frac{1}{2}} e_n)\|_{L^2([0,b],H)} < \infty.$$

Then, for $t \geq 0$, its stochastic integral with respect to the Rosenblatt process $Z_Q(t)$ is defined as

$$\begin{aligned}\int_0^t \omega(s)dZ_Q(s) &= \sum_{n=1}^{\infty}\int_0^t \omega(s)Q^{\frac{1}{2}e_n dZ_n(s)} \\ &= \sum_{n=1}^{\infty}\int_0^t\int_0^t (K_H^*(\phi Q^{\frac{1}{2}}e_n))(y_1,y_2)dB(y_1)dB(y_2).\end{aligned}$$

Let Y and U_2 be Separable Hilbert Spaces. We define,

- $X := L_2(\Omega, \mathcal{F}_b, Y)$, Which is the Hilbert space of all $\mathcal{F}_b$- measurable square integrable random variables with values in Y.
- H is a closed subspace of $H : J \to L_2(\mathcal{F}, Y)$ consisting of all $\mathcal{F}_t$- measurable processes with values in Y and endowed with the norm

$$\|\Psi\|_H^2 = \sup_{t\in J} E\|\Psi\|^2,$$

  where E denotes expectation with respect to P.
- $U := L_2(J, U_2)$, which is a Hilbert space of all square integrable and $\mathcal{F}_t$-measurable processes with values in U_2.

2.2. Fractional calculus.

Definition 2.3. [17] For $n \in \mathbb{N}$, the Euler gamma function $\Gamma : \mathbb{C}-\{0,-1,-2,...\} \to \mathbb{C}$, for complex arguments with positive real part it is defined as

$$\Gamma(z) = \int_0^{\infty} e^{-t}t^{z-1}dt, Rez > 0.$$

Definition 2.4. [17] Let $[a,b]$ be a finite interval on the real axis $\mathbb{R}$. The $Riemann-Liouville\ fractional$ $integral$ of order $\alpha > 0, n-1 < \alpha \leq n$ and $n \in \mathbb{N}$ is defined as

$$I_{0+}^{\alpha}g(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}g(s)ds,$$

where $\Gamma(.)$ is the Euler gamma function and $g(t)$ a suitable function.

Definition 2.5. [17] The Caputo fractional derivative of order $\alpha > 0,\ n-1 < \alpha < n$ is defined as

$$({}^C D_{0+}^{\alpha}g)(t) = \frac{1}{\Gamma(n-\alpha)}\int_0^t (t-s)^{n-\alpha-1}g^{(n)}(s)ds,$$

where the function $g(t)$ has absolutely continuous derivatives up to order $(n-1)$. If $0 < \alpha < 1$, then

$$({}^C D_{0+}^{\alpha}g)(t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{g'(s)}{(t-s)^{\alpha}}ds.$$

Note: The Caputo fractional derivative is equivalent to the composition of the same operators ((n-α)-fold integration and n^{th} order differentiation) ${}^{C}D^{\alpha}g = I^{n-\alpha}D^{n}g$.
In particular, for $0 < \alpha < 1$, $(I^{\alpha}_{0+}\,{}^{C}D^{\alpha}_{0+})g(t) = g(t) - g(0)$.

Definition 2.6. [12] Let A be a bounded linear operator, the Mittag-Leffler function is given by,

$$E_{\alpha,\beta}(A) = \textstyle\sum_{k=0}^{\infty} \frac{A^k}{\Gamma(k\alpha+\beta)}.$$

In particular, for $\beta = 1$,

$$E_{\alpha,1}(A) = E_{\alpha}(A) = \textstyle\sum_{k=0}^{\infty} \frac{A^k}{\Gamma(k\alpha+1)}.$$

Lemma 2.1. [15] *Suppose that A is a bounded linear operator defined on a Banach space, and assume that $\|A\| < 1$. Then $(I - A)^{-1}$ is linear and bounded. Also*

$$(I - A)^{-1} = \sum_{k=0}^{\infty} A^k.$$

Definition 2.7. [2] If X is a Banach space and $T : X \to X$ is a contraction mapping then T has a unique fixed point.

3. Main results

Consider the neutral linear stochastic fractional integro-differential equation as

$$\begin{aligned} {}^{C}D^{\alpha}[y(t) + g(t, y(t))] &= Ay(t) + h(t, y(t), \int_0^t l(t, s, y(s))ds) + Bu(t) \\ &\quad + f(t)dZ_H(t),\ t \in J := [0, b], \\ y(0) &= \Psi_0, \end{aligned} \tag{3.1}$$

- Where, ${}^{c}D^{\alpha}$ represents the caputo derivatives of order $0 < \alpha < 1$,
- $y(.)$ takes the value in a real separable Hilbert space Y with inner product $< ., . >$ and norm $|.|_Y$,
- $A : Y \to Y$ is a bounded linear operator,
- $u(.)$ the control function belongs to the space $L^2(J, U)$,
- $B : U \to H$ is a linear bounded operator,
- $g : J \times Y \to Y$ is continuous,
- $h : J \times Y \times Y \to Y$ and $l : J \times Y \times Y \to Y$ are the continuous functions,
- $z_H(t)$ is a Rosenblatt process with Hurst parameter $H \in (\frac{1}{2}, 1)$ and $t \in J = [0, b]$ on a real separable space $(K, \|.\|_K, < ., . >_K)$,
- $f(t)$ is a Hilbert-Schmidt operator for all $t \in J$, and ψ_0 is the initial function.

Assumption A_1 [15]: For solution representation of the system (3.1), we consider this assumption. The operator $A \in \mathbb{L}(Y)$ commutes with the fractional integral operator I^{α} on Y and

$$\|A\|^2 \leq \frac{(2\alpha - 1)(\Gamma(\alpha))^2}{T^{2\alpha}}.$$

Lemma 3.1. *For* $0 < \alpha < 1$*, and* $f : J \to L_2^0$ *is continuous and bounded, then prove that the solution of the system* (3.1) *can be represented as*

$$\begin{aligned} y(t) &= E_\alpha(At^\alpha)[\Psi_0 + g(0,\Psi_0)] - g(t,y(t)) + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(A(t-s)^\alpha) \\ &\quad \times [Ag(s,y(s)) + h(s,y(s), \int_0^s l(s,\tau,y(\tau))d\tau) + Bu(s)]ds \\ &\quad + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(A(t-s)^\alpha) \times f(s) dZ_H(s). \end{aligned}$$

Proof. Taking I^α on both sides of (3.1) using assumption A_1 and lemma 2.1, we can get the solution of (3.1) as,

$$\begin{aligned} y(t) &= (\Psi_0 + g(0,\Psi_0)) - g(t,y(t)) + I^\alpha Ay(t) + I^\alpha Bu(t) \\ &\quad + I^\alpha h(t,y(t), \int_0^t l(t,s,y(s))ds) + I^\alpha f(t) dZ_H(t). \\ y(t)[I - I^\alpha A] &= (\Psi_0 + g(0,\Psi_0)) - g(t,y(t)) + I^\alpha Bu(t) \\ &\quad + I^\alpha h(t,y(t), \int_0^t l(t,s,y(s))ds) + I^\alpha f(t) dZ_H(t). \\ y(t) &= [I - I^\alpha A]^{-1}\{(\Psi_0 + g(0,\Psi_0)) - g(t,y(t)) + I^\alpha Bu(t) \\ &\quad + I^\alpha h(t,y(t), \int_0^t l(t,s,y(s))ds) + I^\alpha f(t) dZ_H(t))\}. \\ y(t) &= \sum_{k=0}^\infty [(I^\alpha A)^k]\{(\Psi_0 + g(0,\Psi_0)) - g(t,y(t)) + I^\alpha Bu(t) \\ &\quad + I^\alpha h(t,y(t), \int_0^t l(t,s,y(s))ds) + I^\alpha f(t) dZ_H(t))\}. \end{aligned}$$

$$\begin{aligned} y(t) &= \sum_{k=0}^\infty [(I^\alpha A)^k]\{\Psi_0 + g(0,\Psi_0)\} - g(t,y(t)) + \sum_{k=1}^\infty [(I^\alpha A)^k] g(t,y(t)) \\ &\quad + \sum_{k=0}^\infty [(I^\alpha A)^k] I^\alpha Bu(t) + \sum_{k=0}^\infty [(I^\alpha A)^k] I^\alpha h(t,y(t), \int_0^t l(t,s,y(s))ds) \\ &\quad + \sum_{k=0}^\infty [(I^\alpha A)^k] I^\alpha f(t) dZ_H(t)). \end{aligned}$$

$$\begin{aligned} y(t) = E_\alpha(At^\alpha)[\Psi_0 + g(0,\Psi_0)] - g(t,y(t)) + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(A(t-s)^\alpha) \\ [Ag(s,y(s)) + h(s,y(s), \int_0^s l(s,\tau,y(\tau))d\tau) + Bu(s)]ds \\ + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(A(t-s)^\alpha) f(s) dZ_H(s). \quad (3.2) \end{aligned}$$

□

Definition 3.1. Define the operator $K_b : U \to Y$ as

$$k_b u = \int_0^b (b-s)^{\alpha-1} E_{\alpha,\alpha}(A(b-s)^\alpha) Bu(s) ds$$

Clearly, the adjoint operator k_b^* of K_b, $K_b^* : Y \to U$ as

$$(k_b^* y)(t) = (b-t)^{\alpha-1} B^* E_{\alpha,\alpha}(A^*(b-t)^\alpha) E\{x/\mathcal{F}_t\}.$$

Definition 3.2. The controllability grammian operator $M_b : Y \to Y$

$$M_b(y) = \int_0^b (b-s)^{2\alpha-2} E_{\alpha,\alpha}(A(b-s)^\alpha) BB^* E_{\alpha,\alpha}(A^*(b-s)^\alpha) E\{x/\mathcal{F}_s\} ds.$$

Here $*$ represents adjoint operator.

Lemma 3.2. *The operator $M_b = K_b K_b^*$ is well defined and bounded for any $\alpha \in (\frac{1}{2}, 1]$.*

Proof. The proof of this Lemma is obvious, It is clear that the grammian operator M_b is linear and bounded for all $\alpha \in (\frac{1}{2}, 1]$. □

Definition 3.3. [16] The stochastic fractional system (3.1) is said to be completely controllable on the interval J if for every $y_1 \in Y$, there exists a control $u \in U$ such that the solution $y(t)$ given in (3.2) satisfies $y(b) = y_1$.

Theorem 3.1. *Let us assume that A_1 is satisfied, then the linear system* (3.1) *is completely controllable.*

Proof. Using assumption A_1 we obtain the solution of (3.1) as in (3.2).
Let y_1 be an arbitrary point in Y. Since the linear operator M_b is invertible, we define the control as

$$u(t) = (b-t)^{(\alpha-1)} B^* E_{\alpha,\alpha}(A^*(b-t)^\alpha) E\{M_b^{-1}(y_1 - E_\alpha(At^\alpha)[\Psi_0 + g(0,\Psi_0)] + g(t,y(t))$$
$$- \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(A(t-s)^\alpha)[Ag(s,y(s)) + h(s,y(s), \int_0^s l(s,\tau,y(\tau))d\tau)]ds$$
$$- \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(A(t-s)^\alpha) f(s) dZ_H(s))/\mathcal{F}_s\}. \quad (3.3)$$

Substituting (3.3) in (3.2) we get

$$\begin{aligned} y(t) \quad = \quad & E_\alpha(At^\alpha)[\Psi_0 + g(0,\Psi_0)] - g(t,y(t)) + \int_0^t (t-s)^{2\alpha-2} E_{\alpha,\alpha}(A(t-s)^\alpha) BB^* \\ & \times E_{\alpha,\alpha}(A^*(t-s)^\alpha) E\{M_b^{-1}(y_1 - E_\alpha(At^\alpha)[\Psi_0 + g(0,\Psi_0)] + g(t,y(t)) \\ & - \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(A(t-s)^\alpha)[Ag(s,y(s)) + h(s,y(s), \int_0^s l(s,\tau,y(\tau))d\tau)]ds \\ & - \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(A(t-s)^\alpha) f(s) dZ_H(s)/\mathcal{F}_s\} \\ & + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(A(t-s)^\alpha)[Ag(s,y(s)) + h(s,y(s), \int_0^s l(s,\tau,y(\tau))d\tau)]ds \\ & + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(A(t-s)^\alpha) f(s) dZ_H(s) \end{aligned}$$

Evaluating $y(t)$ given in the above equation at $t = b$ we obtain

$$\begin{aligned}
y(b) &= E_\alpha(Ab^\alpha)[\Psi_0 + g(0,\Psi_0)] - g(b, y(b)) \\
&+ M_b M_b^{-1} E\{(y_1 - E_\alpha(Ab^\alpha)[\Psi_0 + g(0,\Psi_0)] \\
&+ g(b, y(b)) - \int_0^b (b-s)^{\alpha-1} E_{\alpha,\alpha}(A(b-s)^\alpha) \\
&\quad \times [Ag(s, y(s)) + h(s, y(s), \int_0^s l(s,\tau,y(\tau))d\tau)]ds \\
&- \int_0^b (b-s)^{\alpha-1} E_{\alpha,\alpha}(A(b-s)^\alpha) f(s) dZ_H(s))/\mathcal{F}_s\} \\
&+ \int_0^b (b-s)^{\alpha-1} E_{\alpha,\alpha}(A(b-s)^\alpha) \\
&\quad \times [Ag(s, y(s)) + h(s, y(s), \int_0^s l(s,\tau,y(\tau))d\tau)]ds \\
&+ \int_0^b (b-s)^{\alpha-1} E_{\alpha,\alpha}(A(b-s)^\alpha) f(s) dZ_H(s) \\
&= y_1
\end{aligned}$$

Since y_1 is an arbitrary point in Y, we infer from the above that $u(t)$ defined in (3.3) steers the system to all points in Y. Thus the proof is completed. □

4. Controllability Criteria of Nonlinear System

Consider the corresponding non-linear system for (3.1), as

$$\begin{aligned}
{}^C D^\alpha[y(t) + g(t, y(t))] &= Ay(t) + h(t, x(t), \int_0^t l(t,s,x(s))ds) + Bu(t) \\
&\quad + f(t, y_t) dZ_H(t)),\ t \in J := [0,b], \\
y(0) &= \Psi_0, \qquad (4.1)
\end{aligned}$$

- Where, ${}^C D^\alpha$ represents the caputo derivatives of order $0 < \alpha < 1$,
- $y(.)$ takes the value in a real separable Hilbert space Y with inner product $< .,. >$ and norm $|.|_Y$,
- $A : Y \to Y$ is a bounded linear operator,
- $u(.)$ the control function belongs to the space $L^2(J, U)$,
- $B : U \to H$ is a linear bounded operator,
- $g : J \times Y \to Y$ is continuous,
- $h : J \times Y \times Y \to Y$ and $l : J \times Y \times Y \to Y$ are the continuous functions,
- $z_H(t)$ is a Rosenblatt process with Hurst parameter $H \in (\frac{1}{2}, 1)$ and $t \in J = [0,b]$ on a real separable space $(K, \|.\|_K, < .,. >_K)$,
- $y_t \in \beta$ (where β is the abstract phase space, for details see [19]),
- $f : J \times \beta \to L_2^0$, where $L_2^0 = L_2(Q^{\frac{1}{2}}K, Y)$ be the space of all Hilbert-Schmidt operators from $Q^{\frac{1}{2}}K$ into Y,
- ψ_0 is the initial function.

The solution of (4.1) is given by,

$$\begin{aligned} y(t) &= E_\alpha(At^\alpha)[\Psi_0 + g(0,\Psi_0)] - g(t,y(t)) + \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(A(t-s)^\alpha) \\ &\quad \times[Ag(s,y(s)) + h(s,y(s),\int_0^s l(s,\tau,y(\tau))d\tau) + Bu(s)]ds \\ &+ \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(A(t-s)^\alpha)f(s,y_s)dZ_H(s) \end{aligned}$$

To prove the controllability results for the nonlinear system (4.1), we consider the following assumptions.

Assumption A_2: Assume that there exists constants $V_i > 0$ for $i = 1,2,...,8$ and W_1, W_2 such that

$$\begin{aligned} \|h(t,y_1,x_1) - h(t,y_2,x_2)\|^2 &\le V_1(\|y_1-y_2\|^2 + \|x_1-x_2\|^2) \\ \|f(t,y_{t_1}) - f(t,y_{t_2})\|^2 &\le V_2(\|y_{t_1}-y_{t_2}\|^2) \\ \|l(t,s,y_1) - l(t,s,y_2)\|^2 &\le V_3\|y_1-y_2\|^2 \\ \|g(t,y_1) - g(t,y_2)\|^2 &\le V_4\|y_1-y_2\|^2 \end{aligned}$$

$$\begin{aligned} V_5 &= \sup_{t\in J}\|f(t,0)\| \\ V_6 &= \sup_{t\in J}\|h(t,0,0)\|, \\ V_7 &= \sup_{t\in J}\|\int_0^t l(t,s,0)ds\| \\ V_8 &= \sup_{t\in J}\|g(t,0)\| \\ W_1 &= \sup_{0\le t\le b}\|E_\alpha(At^\alpha)\|^2 \\ W_2 &= \sup_{0\le t\le b}\|E_{\alpha,\alpha}(At^\alpha)\|^2 \end{aligned}$$

Assumption A_3: Let $\delta = \frac{8b^{2\alpha}W_2}{(2\alpha-1)}(V_2b^{-1}+V_2+V_1+V_1V_3b)$ be such that $0 \le \delta < 1$.

Theorem 4.1. *If the assumption* (A_1)*-*(A_3) *are satisfied and if the linear fractional dynamical system* (3.1) *is controllable, then the non-linear fractional dynamical system* (4.1) *is controllable.*

Proof. Let y_1 be an arbitrary point in Y. Define the operator ϕ on Y by

$$\begin{aligned} \phi y(t) &= E_\alpha(At^\alpha)[\Psi_0 + g(0,\Psi_0)] - g(t,y(t)) + \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(A(t-s)^\alpha) \\ &\quad \times[Ag(s,y(s)) + h(s,y(s),\int_0^s l(s,\tau,y(\tau))d\tau) + Bu(s)]ds \\ &+ \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(A(t-s)^\alpha)f(s,y_s)dZ_H(s). \end{aligned}$$

Since the linear system (3.1) corresponding to the nonlinear system (4.1) is controllable. We have, M_b is invertible and so we can define the control variable u as

$$\begin{aligned} u(t) &= (b-t)^{\alpha-1}B^*E_{\alpha,\alpha}(A^*(b-t)^\alpha)E\{M_b^{-1}(y_1 - E_\alpha(Ab^\alpha)[\Psi_0 + g(0,\Psi_0)] \\ &+ g(t,y(t)) - \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(A(t-s)^\alpha) \\ &\quad \times[Ag(s,y(s)) + h(s,y(s),\int_0^s l(s,\tau,y(\tau))d\tau)]ds \\ &- \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(A(t-s)^\alpha)f(s,y_s)dZ_H(s))/\mathcal{F}_s\}. \end{aligned}$$

Clearly, $\phi(y(b)) = y_1$, To show ϕ has a fixed point, our claim is ϕ maps Y into itself. Provided we can obtain a fixed point of the nonlinear operator ϕ.

$$\begin{aligned} \sup_{t\in J} E\|u(t)\|^2 &\leq 4\|K_b^*\|^2\|M_b^{-1}\|^2[E\|y_1\|^2 + W_1E\|\Psi_0 + g(0,\Psi_0)\|^2 \\ &\quad +V_7 + W_2\frac{Zb^{2(\alpha-1)}}{2\alpha-1} + W_2\frac{Z^1b^{2(\alpha-1)}}{2(\alpha-1)-1}] \\ &= T_1 < \infty \end{aligned}$$

Where

$$\begin{aligned} Z &= (V_1V_3 + V_1/\sup_{t\in J} E\|y(t)\|^2 + V_5 + V_1V_6) < \infty \\ Z^1 &= (V_2 + V_7/\sup_{t\in J} E\|y(t)\|^2 + V_4 + V_8) < \infty. \end{aligned}$$

Further from the assumptions, we have

$$\begin{aligned} \sup_{t\in J} \|\Phi y(t)\|^2 &\leq 4W_1E\|\Psi_0 + g(0,\Psi_0)\|^2 + V_7 + 4W_2T_1\|B\|^2\frac{b^{2\alpha}}{2\alpha-1} \\ &\quad +4W_2Z\frac{b^{2\alpha}}{2\alpha-1} + 4W_2Z^1\frac{b^{2\alpha-1}}{2\alpha-1} < \infty. \end{aligned}$$

Now, for $y_1, y_2 \in Y$, we have

$$
\begin{aligned}
& \sup_{t\in J} E\|\Phi y_1(t) - \Phi y_2(t)\|^2 \\
= \; & \sup_{t\in J} E\| \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(A(t-s)^\alpha) B K_b^* M_b^{-1} \\
& \{\int_0^b (b-\theta)^{\alpha-1} E_{\alpha,\alpha}(A(b-\theta)^\alpha)[h(\theta, y_1(\theta), \int_0^\theta l(\theta,\tau,y_1(\tau))d\tau) \\
& -h(\theta, y_2(\theta), \int_0^\theta l(\theta,\tau,y_2(\tau))d\tau)d\theta + \int_0^b (b-\theta)^{\alpha-1} E_{\alpha,\alpha}(A(b-\theta)^\alpha) \\
& \times[Ag(\theta, y_1(\theta)) - Ag(\theta, y_2(\theta))]d\theta] + \int_0^b (b-\theta)^{\alpha-1} E_{\alpha,\alpha}(A(b-\theta)^\alpha) \\
& [f(\theta, y_{t_1}(\theta)) - f(\theta, y_{t_2}(\theta))]dz_H(\theta)\}ds + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(A(b-s)^\alpha) \\
& [h(s, y_1(s), \int_0^s l(s,\tau,y_1(\tau))d\tau) - h(s, y_2(s), \int_0^s l(s,\tau,y_2(\tau))d\tau)]ds \\
& + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(A(b-s)^\alpha)[Ag(s, y_1(s)) - Ag(s, y_2(s))]ds \\
& + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(A(b-s)^\alpha)[f(s, y_{t_1}(s)) - f(s, y_{t_2}(s))]dz_H(s)\|^2 \\
\leq \; & \frac{8b^{2(\alpha)} W_2}{2\alpha - 1}(V_2 b^{-1} + V_2 + V_1 + V_1 V_3 b) \sup_{t\in J} E\|y_1(t) - y_2(t)\|^2 \\
\leq \; & \delta \|y_1 - y_2\|^2.
\end{aligned}
$$

Using Assumption A_3 we conclude that ϕ is a contraction mapping and hence there exists a unique fixed point $y_1 \in Y$ for ϕ. Any fixed point ϕ satisfied $y(b) = y_1$, for an arbitrary $y_1 \in Y$. Therefore the given system (4.1) is completely controllable on J. □

5. Examples

Next, we provide an illustration to the above theoretical results.

Example 5.1. Consider the fractional order linear stochastic system with Rosenblatt Process,

$$
{}^C D^{\frac{2}{3}} \left(y(t) + \begin{pmatrix} e^{-t} sinty_1(t) \\ (e^t+1) sinty_2(t) \end{pmatrix} \right) = Ay(t) + Bu(t) + (1 + \int_0^1 2y(s))ds) + f(t) dZ_H(t),
$$

$$
\Psi_0 = 0, \quad (5.1)
$$

where $\alpha = \frac{2}{3}$, $y(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix}$, for $t \in [0, 1]$,
$A = \begin{pmatrix} -2 & 1 \\ 1 & 1 \end{pmatrix}$; $B = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$; $f(t) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $g(t, y(t)) = \begin{pmatrix} e^{-t} sinty_1(t) \\ (e^t+1) sinty_2(t) \end{pmatrix}$.
Here $y(t)$ is the state variable, and $u(t)$ is the control variable. We apply Theorem

3.1 to prove that the system (5.1) is controllable on [0,1].
In this example, the solution is given by

$$\begin{aligned} y(t) &= E_{\frac{2}{3}}(At^{\frac{2}{3}})[\Psi_0 + g(0,\Psi_0)] - g(t,y(t)) \\ &+ \int_0^t (t-s)^{\frac{2}{3}-1} E_{\frac{2}{3},\frac{2}{3}}(A(t-s)^{\frac{2}{3}}) \\ &\quad \times [Ag(s,y(s)) + h(s,y(s),\int_0^s l(s,\tau,y(\tau))d\tau) + Bu(s)]ds \\ &+ \int_0^t (t-s)^{\frac{2}{3}-1} E_{\frac{2}{3},\frac{2}{3}}(A(t-s)^{\frac{2}{3}}) f(s) dZ_H(s). \end{aligned}$$

We have the control of the system (5.1) as

$$\begin{aligned} u(t) &= (1-t)^{\frac{2}{3}-1} B^* E_{\frac{2}{3},\frac{2}{3}}(A^*(1-t)^{\frac{2}{3}}) E\{M_b^{-1}(y_1 - E_{\frac{2}{3}}(Ab^{\frac{2}{3}})[\Psi_0 + g(0,\Psi_0)] \\ &+ g(t,y(t)) - \int_0^t (t-s)^{\frac{2}{3}-1} E_{\frac{2}{3},\frac{2}{3}}(A(t-s)^{\frac{2}{3}}) \\ &\quad \times [Ag(s,y(s)) + h(s,y(s),\int_0^s l(s,\tau,y(\tau))d\tau)]ds \\ &- \int_0^t (t-s)^{\frac{2}{3}-1} E_{\frac{2}{3},\frac{2}{3}}(A(t-s)^{\frac{2}{3}}) f(s) dZ_H(s)/\mathcal{F}_s\}. \end{aligned}$$

By computation, we have the controllability grammian operator as

$$M_1(y) = \int_0^1 (1-s)^{2(\frac{2}{3})-2} E_{\frac{2}{3},\frac{2}{3}}(A(1-s)^{\frac{2}{3}}) BB^* E_{\frac{2}{3},\frac{2}{3}}(A^*(1-s)^{\frac{2}{3}}) E\{x/\mathcal{F}_s\} ds.$$

$$\begin{aligned} M_1(y) &= \begin{pmatrix} 349.4871 & -99.9949 \\ -99.9949 & 29.8226 \end{pmatrix} \\ &= 10422.6139 - 9998.9800 \\ &= 423.6339 > 0, \end{aligned}$$

which is positive definite. Hence by Theorem 3.1, the system (5.1) given in this example is completely controllable on $[0,1]$.

Example 5.2. Consider the fractional order non-linear stochastic system with Rosenblatt Process,

$${}^C D^{\frac{4}{5}} \left(y(t) + \begin{pmatrix} e^{-t} sinty_1(t) \\ (e^t+1) sinty_2(t) \end{pmatrix} \right) = Ay(t) + Bu(t) + (1 + \int_0^1 2y(s))ds) + f(t,y_t) dZ_H(t),$$

$$\Psi_0 = 0, \quad (5.2)$$

where $\alpha = \frac{4}{5}$, $y(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix}$, for $t \in [0,1]$,

$A = \begin{pmatrix} 0 & -\frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}$; $B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$; $f(t,y_t) = \begin{pmatrix} \frac{1}{1+t} \\ \frac{e^{-siny_2}}{1+t} \end{pmatrix}$ and $g(t,y(t)) = \begin{pmatrix} e^{-t} sinty_1(t) \\ (e^t+1) sinty_2(t) \end{pmatrix}$

Here $y(t)$ is the state variable, and $u(t)$ is the control variable. We apply Theorem

4.1 to prove that the system (5.2) is controllable on [0,1]. In this example, the solution is given by

$$\begin{aligned} y(t) &= E_{\frac{4}{5}}(At^{\frac{4}{5}})[\Psi_0 + g(0, \Psi_0)] - g(t, y(t)) + \int_0^t (t-s)^{\frac{4}{5}-1} E_{\frac{4}{5},\frac{4}{5}}(A(t-s)^{\frac{4}{5}}) \\ & \quad \times [Ag(s, y(s)) + h(s, y(s), \int_0^s l(s, \tau, y(\tau))d\tau)]ds \\ &+ \int_0^t (t-s)^{\frac{4}{5}-1} E_{\frac{4}{5},\frac{4}{5}}(A(t-s)^{\frac{4}{5}}) f(s, y_s) dZ_H(s). \end{aligned}$$

We have the control of the system (5.2) as

$$\begin{aligned} u(t) &= (1-t)^{\frac{4}{5}-1} B^* E_{\frac{4}{5},\frac{4}{5}}(A^*(1-t)^{\frac{4}{5}}) E\{M_b^{-1}(y_1 - E_{\frac{4}{5}}(Ab^{\frac{4}{5}})[\Psi_0 + g(0, \Psi_0)] \\ &+ g(t, y(t)) - \int_0^t (t-s)^{\frac{4}{5}-1} E_{\frac{4}{5},\frac{4}{5}} \\ & \quad \times (A(t-s)^{\frac{4}{5}})[Ag(s, y(s)) + h(s, y(s), \int_0^s l(s, \tau, y(\tau))d\tau)]ds \\ &- \int_0^t (t-s)^{\frac{4}{5}-1} E_{\frac{4}{5},\frac{4}{5}}(A(t-s)^{\frac{4}{5}}) f(s, y_s) dZ_H(s)/\mathcal{F}_s\}. \end{aligned}$$

By computation, we have the controllability grammian operator as

$$M_1(y) = \int_0^1 (1-s)^{2(\frac{4}{5})-1} E_{\frac{4}{5},\frac{4}{5}}(A(1-s)^{\frac{4}{5}}) BB^* E_{\frac{4}{5},\frac{4}{5}}(A^*(1-s)^{\frac{4}{5}}) E\{x/\mathcal{F}_s\} ds.$$

$$\begin{aligned} M_1(y) &= \begin{pmatrix} 0.4754 & -0.5709 \\ -0.5709 & 1.3249 \end{pmatrix} \\ &= 0.62985 - 0.32592 \\ &= 0.30393 > 0, \end{aligned}$$

which is positive definite. We also obtain the value of δ in Assumption A_3 to be $\delta = 0.6295 < 1$. All the assumption of Theorem 4.1 are verified and hence the system (5.2) is completely controllable on $[0, 1]$.

6. Conclusion

We examined the controllability analysis for both linear and nonlinear fractional order neutral-type stochastic integro-differential system with non-Gaussian process, named as Rosenblatt process. We formulate a set of necessary and sufficient conditions for our introduced systems to be controllable by employing standard techniques. The fractional Brownian motion is the foremost studied method within the class of Hermite processes due to its vital importance in modeling. Our main interest during this work, from the stochastic calculus purpose of view, was to consider the non-Gaussian Rosenblatt process. Although it received a smaller attention than the half Brownian motion, however this method remains of much interests in sensible applications as results of its self-similarity, stationarity of increments and long vary dependence. Truly the terribly giant utilization of the fractional Brownian motion in application (hydrology, telecommunications) are due to these properties.

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