Proceedings of the Institute of Mathematics and Mechanics,
National Academy of Sciences of Azerbaijan
Volume 48, Special Issue, 2022, Pages 131–152
https://doi.org/10.30546/2409-4994.48.2022.131152

AN EFFICIENT NUMERICAL METHOD BASED ON CUBIC B-SPLINE FOR TIME DEPENDENT PROBLEM WITH SMALL PARAMETER

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Abstract. This work is devoted to the development of a Galerkin-type approximation of the solution of parabolic reaction-diffusion problems, utilizing cubic B-Spline functions and a finite difference scheme. An error estimate for the semi discrete weak Galerkin scheme is established. A Von Neumann stability study of the proposed fully discrete Crank Nicolson scheme is also performed. In addition, examples are used to validate the proposed approximation. The numerical results produced demonstrate the procedure's efficacy and are in good agreement with the exact solution.

1. Introduction

Many incidents can be mathematically modelled by the system of the reaction-diffusion equation. These early models date from the first decades of the twentieth century under the influence of various applications of a wide variety of phenomena, not only in natural sciences but also in engineering and economy, such as: gaze dynamics, some biological models, cellular processes, ecology, propagation industrial processes, catalytic transport of contaminants in the environment population dynamics, and the environmental spread of flames and chemical and other reactions. Most of these, at first glance, are phenomena that have a common denominator: the presence of diffusion (allowing the spread of an epidemic or a chemical substance), and reaction (which is the specific way in which different phases of chemical components react); they are generically called reaction-diffusion systems [11]. Some important aspects of reaction diffusion problems such that those pertaining to non integer time derivatives [17], and fractional non-linear reaction-diffusion problems with periodic conditions or with initial conditions in [18, 1], as well as two-dimensional space fractional diffusion equation arising in transport phenomena [34].
The vast applications of these systems make them attractive models for research studies. Since there are no analytical solutions for them nor do they exist in some special cases, although they provide exact solutions, it becomes difficult to employ

2010 *Mathematics Subject Classification.* 65N30, 65N22, 65N06, 65D07.
Key words and phrases. Parabolic problem, Finite differences, Galerkin method, Cubic B-splines, Finite Difference Scheme.

them for complex situations , mentioned reductions and exact solutions of Lotka-Volterra and more complex reaction-diffusion systems with delays [28]. Thus, a new homotopy perturbation Method (NHPM) as a proficient semi-analytical method for solving the linear and non-linear reaction-diffusion equations was treated in [23]. In so doing, numerical methods for these systems are widely used and popular in the development of systems that have evolved into computational capabilities; and although they provide approximate solutions, they are sufficiently accurate for engineering purposes [21, 22].

Different numerical methods for the solution of the reaction-diffusion problems have been used such as finite difference method [26], finite element method [3], finite volume, spectral method [31], finite point method [33], trigonometric quintic B-spline collocation algorithm [36], exponential cubic B-spline collocation algorithm [19], monotone iterative technique combined to finite element method [17], stabilized discontinuous Galerkin method extended to non linear convection reaction diffusion problems in [35], and so on.

There is a class of reaction-diffusion equations that includes small parameters ε multiplied with the highest order derivative known as the singularly perturbed problem. These equation models can be used in many areas of science and engineering such as fluid dynamics, quantum mechanics, control theory, chemical science, and so much interest has been given to the numerical resolution of the singularly perturbed problems. We can cite the method using the equidistribution principle for semi-linear and non-linear singular perturbation problems [4, 5, 24]. The domain decomposition approaches via the equidistribution principle for a Spatio-temporal evolution problem can be found in [13]. In addition, the singularly perturbed parabolic reaction-diffusion and convection-diffusion problems are analyzed respectively in [16, 27].

B-spline functions have emerged as powerful and popular tools in the field of approximation theory for this kind of problem. They were introduced by Schoenberg in 1946 [9, 32]. These functions are well-conditioned, at least for orders ≤ 20. Such bases are also local in the sense that at every point only a fixed number(equal to the order) of B-splines is non-zero [12, 8].

Lots of splines-based numerical methods have been developed to solve partial differential equations. B-spline collocation methods have been used by Goh et al. [15] to study one-dimensional heat and advection-diffusion equations. Besides, Dag and Saka [10] applied B-spline collocation to study equal width equations, whereas [2] utilized quadratic B-Spline functions for the approximation of the wave equation. Therefore, Mittal [25] used B-spline to get numerical solutions of coupled reaction-diffusion systems, while Caglar [7] used the B-spline method for solving a linear system of second-order boundary value problems.

Mainly, B-spline collocation has gained popularity for solving nonlinear Volterra integro-differential equations for example [6]. In this paper, we have presented an efficient numerical method to approach the solution of the parabolic reaction-diffusion problem that uses the modified cubic B-spline Galerkin method. We solve this system by using a finite schema higher order.

The remaining work of the paper is arranged as follows: in Section 2, we formulate the problem to be solved. Section 3 shows the development of the B-spline collocation method followed by a semi-discrete weak Galerkin approximation with its temporal evolution via the Crank-Nicolson scheme in section 4. The fifth section focuses on deriving error estimates for the semi-discrete Galerkin scheme. Furthermore, a Von Neumann method has been utilized to prove the unconditional stability of the fully Crank Nicolson scheme in section 6. The performance of the proposed method has been tried on three problems whose explicit solutions are given, and the approximate solutions generated by the suggested method are then compared to analytical solutions in section 7. Section 8 summarizes this work with a brief conclusion.

2. Statement of the problem

We consider the parabolic reaction-diffusion problem with homogeneous boundary conditions

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) - \alpha \dfrac{\partial^2 u}{\partial x^2}(t,x) + \mu(x)u(t,x) = f(t,x) \quad a < x < b; \;\; t > 0 \\ \\ u(t,a) = u(t,b) = 0 \\ u(0,x) = u_0 = g(x) \end{cases} \tag{2.1}$$

where $0 < \alpha \ll 1$ is a small parameter, and μ, f and g are sufficiently smooth functions with $\mu(x) \geq \beta > 0$ for all $x \in [a,b]$.
Multiplying the equation of problem (2.1) by weight function $v \in H_0^1(a,b)$, and integrating by parts on $[a,b]$, we obtain

$$\int_a^b \frac{\partial u}{\partial t} v(x)\,\mathrm{d}x + \alpha \int_a^b \frac{\partial u}{\partial x}\frac{dv}{dx}\,\mathrm{d}x + \int_a^b \mu(x)u(t,x)v(x)\,\mathrm{d}x = \int_a^b f(t,x)v(x)\,\mathrm{d}x. \tag{2.2}$$

We introduce the scalar product in $L^2(a,b)$ defined by

$$\langle u(t), v\rangle_{L^2} = \int_a^b u(t,x)v(x)\,\mathrm{d}x$$

and the bilinear form $a(u(t),v): \; H^1(a,b) \times H^1(a,b) \longrightarrow \mathbb{R}$, such that

$$a(u(t),v) = \alpha \int_a^b \frac{\partial u}{\partial x}(t,x)\frac{\mathrm{d}v}{\mathrm{d}x}(x)\,\mathrm{d}x + \int_a^b \mu(x)u(t,x)v(x)\,\mathrm{d}x. \tag{2.3}$$

Hence, the weak formulation of the problem (2.2) is

$$\begin{cases} \quad \text{Find} \;\; u^N(t) \in H_0^1(a,b), \;\; \text{such that} \\ \dfrac{\mathrm{d}}{\mathrm{d}t}\langle u^N(t), v\rangle_{L^2} + a(u^N(t),v) = \langle f(t), v\rangle_{L^2}, \end{cases} \tag{2.4}$$

with an initial condition $u^N(0) = g(x)$.
Under suitable continuity and coercivity conditions on the symmetric bilinear form a(u,v) the existence and uniqueness of the solution $u(t,x)$ of the problem (2.1) will be proved.
We exhibit the following theorem that ensures them

Theorem 2.1. *Given a final time $T > 0$, an initial data $g \in L^2(a,b)$, if the symmetric bilinear form $a(,)$ is continuous and coercive, then there exists a unique solution $u(t) \in H_0^1(a,b)$ of the problem (2.4).*

Proof. It is clear that the form $a(.,.)$ given by (2.3) is bilinear and symmetric form, we only need to check their continuity and coercivity.
For boundedness, use Cauchy Schwartz inequality and the continuity of $\mu(x)$ to the bilinear form (2.3), so

$$\begin{aligned} |a(u(t),v)| &= \left|\alpha \int_a^b \frac{\partial u}{\partial x}(t,x)\frac{\mathrm{d}v}{\mathrm{d}x}(x)\,\mathrm{d}x + \int_a^b \mu(x)u(t,x)v(x)\,\mathrm{d}x\right| \\ &\leq \alpha\|\frac{\partial u}{\partial x}\|_{L^2(a,b)}\|\frac{dv}{dx}\|_{L^2(a,b)} + \beta^{'}\|u\|_{L^2(a,b)}\|v\|_{L^2(a,b)} \end{aligned}$$

so, the Poincaré's inequality states

$$|a(u(t),v)| \leq M\|u\|_{H_0^1(a,b)}\|v\|_{H_0^1(a,b)}$$

where, $M = (\alpha + \beta^{'}\eta)$, which is a positive constant.
For coercivity, it is obvious that

$$a(u(t),u(t)) \geq C\|u\|^2_{H_0^1(a,b)},$$

while β is a positive constant and $\mu(x) \geq \beta > 0$, and by supposing $C = \min\{\alpha,\beta\}$, so $a(.,.)$ is coercive as required.
Hence, all the hypotheses of the above theorem are fulfilled on $H_0^1(a,b)$, there exists a unique solution $u(t,x)$ of (2.4). □

In the next section, we are going to discuss the numerical solution of the equation (2.4) by using the B spline method.

3. B-spline collocation method

Now, we subdivide the interval $[a,b]$ into N sub-intervals of length h by the knots x_i such that $a = x_0 < x_1 < \cdots < x_N = b$, and we take as a basis for the functions defined on $[a,b]$, the set of cubic B-splines $\{B_{-1}, B_0, \cdots, B_{N+1}\}$, defined by

$$B_n(x) = \frac{1}{h^3}\begin{cases} (x-x_{n-2})^3; & [x_{n-2},x_{n-1}] \\ (x-x_{n-2})^3 - 4(x-x_{n-1})^3; & [x_{n-1},x_n] \\ (x_{n+2}-x)^3 - 4(x_{n+1}-x)^3; & [x_n,x_{n+1}] \\ (x_{n+2}-x)^3; & [x_{n+1},x_{n+2}] \\ 0; & \text{otherwise} \end{cases} \tag{3.1}$$

where $h = x_n - x_{n-1};\ \ n = \overline{1,N}$, in our case, h is uniform so $h = (b-a)/N$(for more details see [14, 29]).
Note that the cubic splines $B_n(x)$ and its first and second derivatives vanish outside the interval $[x_{n-2}, x_{n+2}]$. The values of $B_n(x)$ and its first and second derivatives $B_n'(x)$ at the knots are given by

x	x_{n-2}	x_{n-1}	x_n	x_{n+1}	x_{n+2}
$B_n(n)$	0	1	4	1	0
$B_n'(x)$	0	$3/h$	0	$-3/h$	0
$B_n''(x)$	0	$6/h^2$	$-12/h^2$	$6/h^2$	0

Thus, the approximation of the solution can be expressed as a function of B_n according to the formula

$$U^N(x,t) = \sum_{n=-1}^{N+1} c_n(t) B_n(x), \tag{3.2}$$

where c_n are coefficients to be determined.
As each spline covers four intervals $x_{n-2} \le x_{n+2}$, so that four B_{n-1}, B_n, B_{n+1} and B_{n+2} cover each finite element $[x_n, x_{n+1}]$, all other splines are zero in this region.
Then, from the equation (3.2), the values U_n of the solution at the knots x_n and the values U'_n and U''_n of the derivatives at the knots x_n are given by

$$U_n = U(x_n) = c_{n-1} + 4c_n + c_{n+1} \tag{3.3}$$

$$U'_n = U'(x_n) = \frac{3}{h}(c_{n+1} - c_{n-1}). \tag{3.4}$$

$$U''_n = U'(x_n) = \frac{6}{h^2}(c_{n-1} - 2c_n + c_{n+1}), \tag{3.5}$$

and from boundary conditions, we have

$$c_{-1} = -4c_0 - c_1 \quad \text{and} \quad c_{N+1} = -4c_N - c_{N-1}.$$

Hence, the approximation (3.2) takes the following form

$$U^N(x) = \sum_{n=0}^{N} c_n \widetilde{B}_n(x), \tag{3.6}$$

where

$$\begin{aligned}
\widetilde{B}_0(x) &= B_0(x) - 4B_{-1}(x) \\
\widetilde{B}_1(x) &= B_1(x) - B_{-1}(x) \\
\widetilde{B}_n(x) &= B_n(x); \qquad \forall \quad n = \overline{2, N-2} \\
\widetilde{B}_{N-1}(x) &= B_{N-1}(x) - B_{N+1}(x) \\
\widetilde{B}_N(x) &= B_N(x) - 4B_{N+1}(x).
\end{aligned}$$

Thus, there remains $(N+1)$ unknowns c_n $(n = \overline{0, N})$ to be determined.

4. The semi discrete weak Galerkin scheme

Let $\mathrm{S}_3^h(a,b)$ be the space of all cubic spline functions; we also define $\mathrm{S}_0^h(a,b) = \mathrm{S}_3^h(a,b) \cap H_0^1(a,b)$ which is a finite dimensional subspace of $H_0^1(a,b)$ parametrized by h where $0 < h < 1$, and for all $v \in H_0^1(a,b)$, there exists an element $S^N v \in \mathrm{S}_0^h(a,b)$ satisfying $\lim\limits_{N\to\infty} \|S^N v - v\| = 0$
We mention that S^N is a spline projection operator such that

$$\begin{cases} S^N : H_0^1(a,b) & \longrightarrow \mathrm{S}_3^h(a,b) \\ \qquad\quad u(t) & \longmapsto S^N u(t). \end{cases} \tag{4.1}$$

Then, the semi discrete weak Galerkin scheme (SDWG.) for the problem (2.4)

$$\begin{cases} \text{Find } u^N(t) \in \mathrm{S}_0^h(a,b), \text{ such that} \\ \frac{\mathrm{d}}{\mathrm{d}t}\langle u^N(t), v\rangle_{L^2} + a(u^N(t), v) = \langle f(t), v\rangle_{L^2}, \\ u^N(0) = u_0^N = S^N u_0 \in \mathrm{S}_0^h(a,b), \text{ is an approximation of } u_0. \end{cases} \tag{4.2}$$

Obviously, the set of $\left\{\widetilde{B}_n\right\}_{n=0}^{N}$ consists of $N+1$ linearly independent elements; they form a basis of $\mathrm{S}_0^h(a,b)$.
Our approximation consists of replacing in SDWG. (4.2) with $U^N(x)$ given by formula (3.6), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\sum_{n=0}^{N} c_n \int_a^b v(x)\widetilde{B}_n(x)\mathrm{d}x + \alpha\sum_{n=0}^{N} c_n \int_a^b v'(x)\widetilde{B}'_n(x)dx + \sum_{n=0}^{N} c_n \int_a^b \mu(x)v(x)\widetilde{B}_n(x)\mathrm{d}x = \int_a^b v(x)f(t,x)\mathrm{d}x$$

and by replacing $v(x)$ by $\widetilde{B}_k(x)$; $(k = \overline{0,N})$ the problem then becomes for all $k = \overline{0,N}$

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\sum_{n=0}^{N} c_n \int_a^b \widetilde{B}_n(x)\widetilde{B}_k(x)\mathrm{d}x + \alpha\sum_{n=0}^{N} c_n \int_a^b \widetilde{B}'_n(x)\widetilde{B}'_k(x)dx \\ + \sum_{n=0}^{N} c_n \int_a^b \mu(x)\widetilde{B}_n(x)\widetilde{B}_k(x)\mathrm{d}x = \int_a^b \widetilde{B}_k(x)f(t,x)\mathrm{d}x \end{aligned} \tag{4.3}$$

which is written in the matrix form

$$\frac{d}{dt}\mathscr{A}_1\boldsymbol{c} = \boldsymbol{F(t)} - \alpha\mathscr{A}_2\boldsymbol{c} - \mathscr{A}_3\boldsymbol{c}, \tag{4.4}$$

where

$$\boldsymbol{c} = (c_0, c_1, c_2, \cdots, c_N)^T$$

$$\boldsymbol{F(t)} = (F_0, F_1, \cdots, F_N)^T$$

$$\text{with} \quad F_n = \int_a^b \widetilde{B}_n(x)f(t,x)dx; \quad n = 0, \dots, N.$$

The matrices $\mathscr{A}_1$, $\mathscr{A}_2$ and $\mathscr{A}_3$ are $(N+1)\times(N+1)$ sparse matrices, such that

$$\mathscr{A}_{1nk} = \int_a^b \widetilde{B}_n(x)\widetilde{B}_k(x)\mathrm{d}x \quad ; \quad \mathscr{A}_{2nk} = \int_a^b \widetilde{B}'_n(x)\widetilde{B}'_k(x)\mathrm{d}x \quad ;$$

$$\mathscr{A}_{3nk} = \int_a^b \mu(x)\widetilde{B}_n(x)\widetilde{B}_k(x)\mathrm{d}x.$$

To solve the obtained system of ordinary differential equations (4.4), we can propose a scheme of Crank Nicolson. For this, the time domain $[0,T]$ is discretized by a step Δt. We note by c_n^j the value of the solution $\boldsymbol{c}$ at node x_n and at time

t_j and we write the scheme of Crank Nicolson as follows

$$\left(\mathscr{A}_1 + \frac{\Delta t}{2}(\alpha\mathscr{A}_2 + \mathscr{A}_3)\right) c_n^{j+1} = \frac{\Delta t}{2}(F(t_{j+1}) + F(t_j)) + \left(\mathscr{A}_1 - \frac{\Delta t}{2}(\alpha\mathscr{A}_2 + \mathscr{A}_3)\right) c_n^j, \tag{4.5}$$

where $c_n^0 = (g(x))_n$ and $(g(x))_n$ is the value of $g(x)$ in each node x_n of the discretization of $[a, b]$.
From the initial condition $U^N(x, 0)$ on the function $u(x, t)$, we determine the initial vector c^0. We firstly rewrite the equation (3.6) for the initial condition

$$U^N(x, 0) = g(x) = \sum_{n=0}^{N} c_n^0 \widetilde{B}_n(x);$$

so, we may write the initial condition as

$$g(x) = \sum_{n=0}^{N} g_n \widetilde{B}_n(x),$$

where the g_n must be determined. To do this we require $U^N(x, 0)$ to satisfy the following conditions

- It will agree with the initial condition $u(x, 0)$ at the knots x_n ; $n = 0, 1, \dots, N$ leading to $N - 1$ conditions.
- The first derivative of the approximate initial condition will agree with that of the exact initial condition at both ends of the range leading to two further conditions.

This leads to a matrix equation of the form $MG = N$, where

$$M = \begin{pmatrix} 12/h & 6/h & 0 & \dots & 0 \\ 1 & 4 & 1 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & 1 & 4 & 1 \\ 0 & \dots & 0 & -6/h & -12/h \end{pmatrix}, \; N = \begin{pmatrix} u'(x_0) \\ u_0(x_1) \\ \vdots \\ u_0(x_{N-1}) \\ u'(x_N) \end{pmatrix} \text{ and } G = \begin{pmatrix} g_0 \\ g_1 \\ \vdots \\ g_{N-1} \\ g_N \end{pmatrix}$$

To solve this matrix equation, we can use the Gauss elimination method. The initial vector c^0 is thus determined.

5. A priori estimate of the solution of SDWG. scheme

Before analyzing the accuracy and establishing the original convergence, we start by proving a priori bound of the solution of (4.2).

Theorem 5.1. *Assure that $f \in L^2(a, b)$; the semi discrete approximation $u^N(t)$ defined in* (4.2) *satisfies the following bound*

$$\|u^N(t)\|_{L^2}^2 \le C \left(\|u^N(0)\|_{L^2(a,b)}^2 + \int_0^t \|f(s)\|_{L^2(a,b)}^2 \,\mathrm{d}s \right) \tag{5.1}$$

where C is a positive constant.

Proof. Recalling the semi discrete form (4.2), and taking $v = u^N(t)$, we find

$$\frac{1}{2}\frac{d}{dt}\|u^N(t)\|^2_{L^2(a,b)} + a(u^N(t), u^N(t)) = \langle f(t), U^N(t)\rangle_{L^2(a,b)}.$$

It follows from the coercivity of $a(.,.)$ that

$$\frac{1}{2}\frac{d}{dt}\|u^N(t)\|^2_{L^2(a,b)} + C\|u^N(t)\|^2_{H^1_0(a,b)} \leq \langle f(t), U^N(t)\rangle_{L^2(a,b)}.$$

Hence, using the young inequality, we obtain

$$\frac{d}{dt}\|u^N(t)\|^2_{L^2(a,b)} + C\|u^N(t)\|^2_{L^2(a,b)} \leq \frac{1}{C}\|f(t)\|^2_{L^2(a,b)}.$$

By integrating the members of the last equation for t from 0 to t, we find

$$\|u^N(t)\|^2_{L^2(a,b)} + C\int_0^t \|u^N(s)\|^2_{L^2(a,b)}\,\mathrm{d}s \leq \|u^N(0)\|^2_{L^2(a,b)}\frac{1}{C}\int_0^t \|f(s)\|^2_{L^2(a,b)}\mathrm{d}s.$$

Finally, applying the Gronwall inequality [30] yields

$$\|u^N(t)\|^2_{L^2(a,b)} \leq \exp^{-Ct}\left\|u^N(0)\right\|^2_{L^2(a,b)}\frac{1}{C}\int_0^t \|f(s)\exp^{-C(t-s)}\|^2_{L^2(a,b)}\mathrm{d}s.$$

This proves the stability (in space) of the Galerkin approximation.
Concerning the convergence, we set an error function $e(t) = S^N u(t) - u^N(t)$, where S^N is a spline projection operator introduced by (5.2) and satisfies the property:

$$\|u(t) - S^N u(t)\| = \inf_{v\in \mathrm{S}^h_0(a,b)} \|u - v\| \leq Ch^4 \tag{5.2}$$

Then, for all $v \in H^1_0(a,b)$, the error function satisfies

$$\frac{d}{dt}\langle u(t) - S^N u(t), v\rangle_{L^2(a,b)} + a(u(t) - S^N u(t), v) = \frac{d}{dt}\langle e(t), v\rangle_{L^2(a,b)} + a(e(t), v)$$

This equation is easily obtained from (2.4) and (4.2) by writing $u^N(t) = S^N u(t) - e(t)$ in (4.2) as follows : for all $v \in \mathrm{S}^h_0(a,b)$

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle e(t), v\rangle_{L^2(a,b)} + a(e(t), v) = \frac{\mathrm{d}}{\mathrm{d}t}\langle u(t) - S^N u(t), v\rangle_{L^2(a,b)} + a(u(t) - S^N u(t), v),$$

By letting $v = e(t)$, we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|e(t)\|^2_{L^2(a,b)} + a(e(t), e(t)) = \frac{\mathrm{d}}{\mathrm{d}t}|\langle u(t) - S^N u(t), e(t)\rangle_{L^2(a,b)} + a(u(t) - S^N u(t), e(t))|.$$

According to the coercivity-condition, we find

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|e(t)\|^2_{L^2(a,b)} + C\|e(t)\|^2_{H^1_0(a,b)} \\ = \frac{d}{dt}|\langle u(t) - S^N u(t), e(t)\rangle_{L^2(a,b)} + a\left(u(t) - S^N u(t), e(t)\right)|\end{aligned} \tag{5.3}$$

The right side member can be written as follows

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}|\langle u(t) - S^N u(t), e(t)\rangle_{L^2(a,b)} + a\left(u(t) - S^N u(t), e(t)\right)| \\ &\leq \|u_t - S^N u_t\|^*_{H^1_0(a,b)}\|e(t)\|_{H^1_0(a,b)} + M\|u(t) - S^N u(t)\|_{H^1_0(a,b)}\|e(t)\|_{H^1_0(a,b)}.\end{aligned}$$

This, yields from the continuity of the bilinear form $a(,)$, and the definition of the norm of the dual space given in [30]. □

Then, applying the above property of the dual norm, and from (5.3), it follows

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|e(t)\|^2_{L^2(a,b)} + C\|e(t)\|^2_{H^1_0(a,b)} \\ \leq \left\{\gamma\|u_t - S^N u_t\|_{H^1_0(a,b)} + M\|u(t) - S^N u(t)\|_{H^1_0(a,b)}\right\}\|e(t)\|_{H^1_0(a,b)}.$$

Hence, the property of spline projection operator given in (5.2), and by posing $K = \max\{\gamma, M\}$ which is a positive constant independent of N, that holds

$$\frac{\mathrm{d}}{\mathrm{d}t}\|e(t)\|^2_{L^2(a,b)} + C\|e(t)\|^2_{H^1_0(a,b)} \leq K\left\{\|u_t - S^N u_t\|^2_{H^1_0(a,b)} + \|u(t) - S^N u(t)\|^2_{H^1_0(a,b)}\right\}.$$

Therefore, by integrating the two members of the last inequality from 0 to t, the following error bound yields

$$\|e(t)\|^2_{L^2(a,b)} + C\int_0^t \|e(s)\|^2_{H^1_0(a,b)}\mathrm{d}s \\ \leq \\ \|e(0)\|^2_{L^2(a,b)} + K\left\{\int_0^t \|(u_t - S^N u_t)(s)\|^2_{H^1_0(a,b)}\mathrm{d}s + \int_0^t \|(u - S^N u)(s)\|^2_{H^1_0(a,b)}\mathrm{d}s\right\} \tag{5.4}$$

For the boundedness of the norms

$$a(u - S^N u, u - S^N u) \geq C\|(u - S^N u)\|^2_1 \tag{5.5}$$

$$a(u - S^N u, u - v) \geq C\|(u - S^N u)\|^2_1, \quad \forall v \in \mathrm{S}^h_0(a,b).$$

Thus,

$$\langle u_t - S^N u_t, v\rangle_{H^1_0(a,b)} + a(u(t) - S^N u(t), v)_{H^1_0(a,b)} = 0,$$

$$\|(u - S^N u)\|_1 \leq \inf_{v\in \mathrm{S}^h_0(a,b)} \|u - v\| \leq Ch^3\|u\|_4,$$

yields

$$\|(u - S^N u)\| \leq Ch^4\|u\|_4 \tag{5.6}$$

By applying $\frac{\mathrm{d}}{\mathrm{d}t}$, on (5.5), it is analogous as above to see that

$$\|(u_t - S^N u_t)\| \leq Ch^4\|u_t\|_4. \tag{5.7}$$

The substitution of (5.6) and (5.7) on (5.8) leads to

$$\|e(t)\|^2_{L^2(a,b)} + C\int_0^t \|e(s)\|^2_{H^1_0(a,b)}\mathrm{d}s \\ \leq \|e(0)\|^2_{L^2(a,b)} + K\left\{\int_0^t Ch^4\|u_t\|^2_4\,\mathrm{d}s + \int_0^t Ch^4\|u\|^2_4\,\mathrm{d}s\right\} \tag{5.8}$$

Now, it is clear that $e(0) = 0$, so the error estimates yields

$$\|e(t)\|^2_{L^2(a,b)} + C\int_0^t \|e(s)\|^2_{H^1_0(a,b)}\,\mathrm{d}s \leq K(u)h^4, \tag{5.9}$$

here, $K(u(t))$ is a constant function of $u(t)$.

6. Stability analysis for Crank Nicolson scheme

Our stability method for the time discretization scheme (4.5) is based on Von Neumann theory. We begin by defining the amplitude factor of a typical Fourier mode as

$$\boldsymbol{c}_n^j = \hat{\boldsymbol{c}}^j \exp(inkh), \tag{6.1}$$

where k is a mode number, h is a spatial step that is required for the numerical scheme, and $i = \sqrt{-1}$ is the imaginary unit, and $\hat{\boldsymbol{c}}^j$ is the amplification factor. We assure that $\mu(x)$ is a local positive constant and equal to $\mu(x) = \beta$. Subsequently, the stability of time discretization for linear evolution equations can be expressed in terms of stability of the case as the right side and boundary data are zero. The scheme (4.5) will be replaced by the following system

$$\left(\mathscr{A}_1\left(1+\frac{\Delta t}{2}\beta\right)+\alpha\frac{\Delta t}{2}\mathscr{A}_2\right)\boldsymbol{c}_n^{j+1} = \left(\mathscr{A}_1\left(1+\frac{\Delta t}{2}\beta\right)+\alpha\frac{\Delta t}{2}\mathscr{A}_2\right)\boldsymbol{c}_n^{j} \tag{6.2}$$

The matrices $\mathscr{A}_1, \mathscr{A}_2$ couls been algebraically determined from (3.1), which are symmetric and have a septa-diagonal form. This implies that the general row for each matrix has the following form

$$\mathscr{A}_1 = (h/140)\begin{pmatrix}1 & 120 & 1191 & 2416 & 1191 & 120 & 1\end{pmatrix}^T, \tag{6.3}$$

$$\mathscr{A}_2 = (-1/10h)\begin{pmatrix}3 & 72 & 45 & 240 & 45 & 72 & 3\end{pmatrix}^T. \tag{6.4}$$

Then, for the i^{th} row, the vector $\boldsymbol{c}^j$ will be in the form

$$\begin{pmatrix}c_{n-3}^j & c_{n-2}^j & c_{n-1}^j & c_n^j & c_{n+1}^j & c_{n+2}^j & c_{n+3}^j\end{pmatrix}^T \tag{6.5}$$

Consequently, the substitution with (6.3), (6.4) and (6.5) into (6.2) leads to the following recurrence relationship

$$\begin{gathered} t_1c_{n-3}^{j+1}+t_2c_{n-2}^{j+1}+t_3c_{n-1}^{j+1}+t_4c_n^{j+1}+t_3c_{n+1}^{j+1}+t_2c_{n+2}^{j+1}+t_1c_{n+3}^{j+1} \\ = \\ \ell_1c_{n-3}^{j}+\ell_2c_{n-2}^{j}+\ell_3c_{n-1}^{j}+\ell_4c_n^{j}+\ell_3c_{n+1}^{j}+\ell_2c_{n+2}^{j}+\ell_1c_{n+3}^{j}, \end{gathered} \tag{6.6}$$

where

$$\begin{cases} t_1 = r_1 - s_1 & = (1+\frac{\Delta t}{2}\beta)\frac{h}{140} - \frac{3\alpha\Delta t}{20h} \\ t_2 = r_2 - s_2 & = (1+\frac{\Delta t}{2}\beta)\frac{120h}{140} - \frac{72\alpha\Delta t}{20h} \\ t_3 = r_3 - s_3 & = (1+\frac{\Delta t}{2}\beta)\frac{1191h}{140} - \frac{45\alpha\Delta t}{20h} \\ t_4 = r_4 - s_4 & = (1+\frac{\Delta t}{2}\beta)\frac{2416h}{140} - \frac{240\alpha\Delta t}{20h} \end{cases} \tag{6.7}$$

and,

$$\begin{cases} \ell_1 = k_1 - s_1 & = (1 - \dfrac{\Delta t}{2}\beta)\dfrac{h}{140} - \dfrac{3\alpha\Delta t}{20h} \\ \ell_2 = k_2 - s_2 & = (1 - \dfrac{\Delta t}{2}\beta)\dfrac{120h}{140} - \dfrac{72\alpha\Delta t}{20h} \\ \ell_3 = k_3 - s_3 & = (1 - \dfrac{\Delta t}{2}\beta)\dfrac{1191h}{140} - \dfrac{45\alpha\Delta t}{20h} \\ \ell_4 = k_4 - s_4 & = (1 - \dfrac{\Delta t}{2}\beta)\dfrac{2416h}{140} - \dfrac{240\alpha\Delta t}{20h} \end{cases} \tag{6.8}$$

By substituting (6.1) into the recurrence relationship (6.6), yields

$$\begin{gathered} \left(t_1\cos(3kh) + t_2\cos(2kh) + t_3\cos(kh) + \frac{t_4}{2}\right)\hat{\boldsymbol{c}}^{j+1} \\ = \\ \left(\ell_1\cos(3kh) + \ell_2\cos(2kh) + \ell_3\cos(kh) + \tfrac{\ell_4}{2}\right)\hat{\boldsymbol{c}}^{j} \end{gathered} \tag{6.9}$$

The necessary and sufficient condition for the error to remain bounded is that the modulus of the amplitude factor satisfies $|\hat{\boldsymbol{c}}^j| \leq 1$ such that $\hat{\boldsymbol{c}}^j$ will be given by

$$\hat{\boldsymbol{c}}^j = \frac{\ell_1\cos(3kh) + \ell_2\cos(2kh) + \ell_3\cos(kh) + \dfrac{\ell_4}{2}}{t_1\cos(3kh) + t_2\cos(2kh) + t_3\cos(kh) + \frac{t_4}{2}}. \tag{6.10}$$

Hence,

$$|\hat{\boldsymbol{c}}^j| = \left|\frac{(1 - \dfrac{\Delta t}{2}\beta)T - L}{(1 - \dfrac{\Delta t}{2}\beta)T + L}\right|. \tag{6.11}$$

We mention that

$$\begin{aligned} T &= \frac{h}{140}\left(\cos(3kh) + 120\cos(2kh) + 1191\cos(kh) + \frac{2416}{2}\right) \\ L &= \frac{\alpha\Delta t}{20h}\left(3\cos(3kh) + 72\cos(2kh) + 45\cos(kh) + \frac{240}{2}\right) \end{aligned} \tag{6.12}$$

It is clear that $|\hat{\boldsymbol{c}}^j| < 1$; this follows from $(1 + \frac{\Delta t}{2}\beta) > (1 - \frac{\Delta t}{2}\beta)$, since $(T, L > 0)$. Moreover, the numerical scheme (6.2) is unconditionally stable.

7. Numerical results

To illustrate the performance of the numerical method developed in this paper, three examples are considered. All computations are done by taking several values for α and for different values for N. We take $\mu(x) = \beta = 1$ for the first example and $\mu(x) = \beta = 0.1$ for the second example. On the third example, $\mu(x) = \mathrm{e}^{x^2+1} \geq \mathrm{e}^2 > 0$.

The numerical solutions obtained at the nodes are compared with the exact solutions. To measure the difference between numerical and analytical solutions, the L_∞ and L_2 error are used which defined respectively by

$$ME = \|u^{\text{exact}} - U^N\|_\infty \simeq \max_j |u_j^{\text{exact}} - U_j^N| \tag{7.1}$$

$$\|u^{\text{exact}} - U^N\|_2 = \left(h \sum_{j=0}^{N} |u_j^{\text{exact}} - U_j^N|^2 \right)^{\frac{1}{2}}. \tag{7.2}$$

Then, the convergence order is computed approximately by the following formulas

$$Order_x = \frac{log_{10}\left(|u^{exact} - U_{h_j}^N| / |u^{exact} - U_{h_{j+1}}^N|\right)}{log_{10}\left(h_j/h_{j+1}\right)} \tag{7.3}$$

$$Order_t = \frac{log_{10}\left(|u^{exact} - U_{\Delta t_j}^N| / |u^{exact} - U_{\Delta t_{j+1}}^N|\right)}{log_{10}\left(\Delta t_j/\Delta t_{j+1}\right)} \tag{7.4}$$

Example 7.1. Consider the diffusion-reaction problem with homogeneous boundary conditions (2.1), with $\mu(x) = \beta > 0$. The right-hand side source term and initial condition are calculated from the exact solution

$$u(x,t) = \cos(\frac{\pi x}{2}) exp(-(\frac{\alpha \pi^2}{4} + \beta)t)\,; \quad x \in [-1,1] \text{ and } t \in [0,1]. \tag{7.5}$$

$$f(x,t) = 0 \quad \text{and} \quad u(x,0) = \cos(\frac{\pi x}{2})$$

At first, as it can be seen from the Table 1, the approximation error decreases as a function of Δt, at different points of $x \in [-1,1]$ at time $t = 1$, for $N = 16$ and $\alpha = 10^{-3}$.

x	$\Delta t = 1/40$	$\Delta t = 1/80$	$\Delta t = 1/160$
-0.75	7.366253e − 06	1.839357e − 06	4.577209e − 07
-0.25	1.779027e − 05	4.447162e − 06	1.111598e − 06
0	1.925619e − 05	4.813716e − 06	1.203328e − 06
0.5	1.361541e − 05	3.403033e − 06	8.501027e − 07

TABLE 1. Approximation error as a function of Δt for $N = 16$ and at $t = 1$.

On the Figure 1, we show the representation of the analytic and approximative solution obtained using the cubic B-splines approximation method by taking $\alpha = 10^{-3}$, $N = 16$ and $\Delta t = 1/160$, and on the right side, there is the absolute error for the same data.

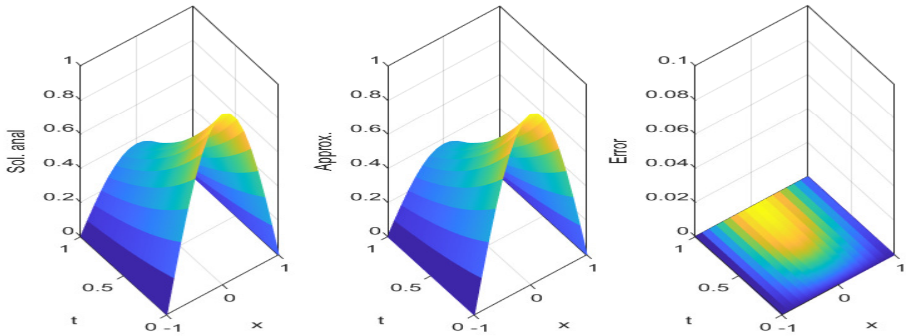

FIGURE 1. Representation of the solution (7.5) on the left, its approximation on the middle and the absolute error on the right for $N = 16$.

The Figure 2 shows that the numerical solution reproduces satisfactorily the behaviour of the exact solution of the test problem at time $t = 0.5$ and for $\alpha = 10^{-3}$, $N = 16$ and $\Delta t = 1/160$. The same result is obtained on the Figure 3 for $x = 0.5$ and same data.

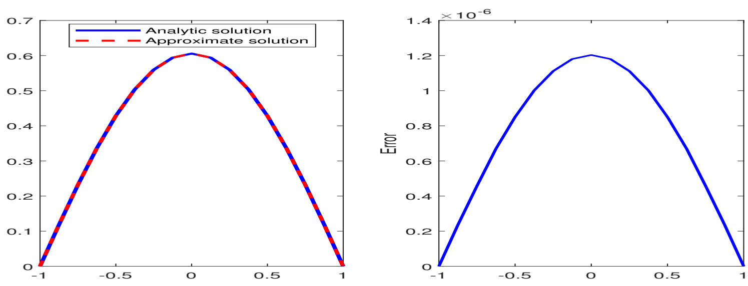

FIGURE 2. Comparison of the solution (7.5) and its approximate on the left and the approximation error on the right for $t = 0.5$

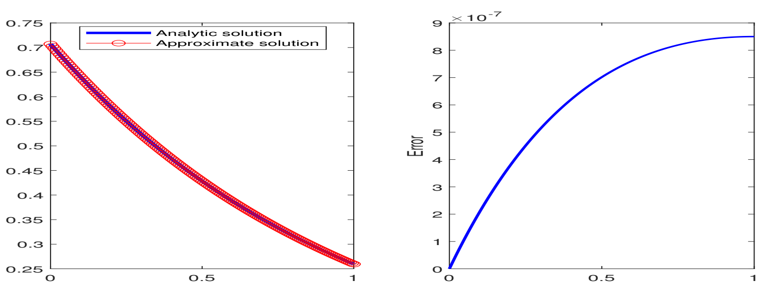

FIGURE 3. Comparison of the solution (7.5) and its approximation on the left and the error obtained on the right for $x = 0.5$

We turn to the convergence behaviour of the approximate solution. For temporal discretization, we choose N big enough so that the spatial discretization error is

negligible and allows Δt to vary. The results are presented in Table 2.
The error $\|U - U^N\|$ in L_∞ and L_2 norms at the selected point $x = 0.75$ is reported in Table 2.

α	Δt_j	L_2 norm	Order$_t$	ME	Order$_t$
10^{-3}	1/20	5.619320e − 05	−	7.117631e − 05	−
	1/40	1.390339e − 05	2.014959	1.779027e − 05	2.000309
	1/80	3.457669e − 06	2.007565	4.447162e − 06	2.000132
	1/160	8.620184e − 07	2.004009	1.111598e − 06	2.000250
10^{-6}	1/20	5.587293e − 05	−	7.082709e − 05	−
	1/40	1.382408e − 05	2.014966	1.770318e − 05	2.000293
	1/80	3.438065e − 06	2.007515	4.425569e − 06	2.000074
	1/160	8.572757e − 07	2.003765	1.106378e − 06	2.000019

TABLE 2. Order of convergence at $x = 0.75$ and $N = 16$

As expected, the temporal accuracy is of order 2; this is consistent with theoretical results.

Example 7.2. Consider the following reaction-diffusion problem with homogeneous boundary conditions (2.1), with $\mu(x) = \beta = 0.1 > 0$. The right-hand side source term and initial condition are calculated from the exact solution

$$u(x,t) = \sin(\pi x)exp(-\beta t)\,; \quad x \in [-1,1] \text{ and } t \in [0,1]. \tag{7.6}$$

So

$$f(x,t) = \alpha\pi^2 \sin(\pi x)exp(-\beta t) \quad \text{and} \quad u(x,0) = \sin(\pi x).$$

The variations of the absolute error of the approximation proposed by the cubic B-splines, for different values of N and different values of $x \in [-1,1]$ at time $t = 1$ are presented on the Table 5 for $\alpha = 10^{-3}$ (and 10^{-5}) and for $\beta = 0.1$.

x	$N = 16$ and $\Delta t = 1/80$		$N = 32$ and $\Delta t = 1/160$	
	$\alpha = 10^{-3}$	$\alpha = 10^{-6}$	$\alpha = 10^{-3}$	$\alpha = 10^{-6}$
−0.75	4.414444e − 07	7.903417e − 09	1.567505e − 08	2.060440e − 09
−0.50	3.624473e − 07	1.139890e − 08	1.576167e − 08	2.926101e − 09
−0.25	2.235418e − 07	8.094502e − 09	1.101364e − 08	2.069509e − 09
0.0	3.053113e − 16	2.393918e − 15	4.371503e − 16	6.768891e − 15
0.25	2.235418e − 07	8.094503e − 09	1.101363e − 08	2.069556e − 09
0.50	3.624472e − 07	1.139891e − 08	1.576166e − 08	2.926118e − 09
0.75	4.414444e − 07	7.903420e − 09	1.567506e − 08	2.060455e − 09

TABLE 3. Variations of the approximate error at $t = 1$ for different values of x, as a function of N, Δt and α and for $\beta = 0.1$.

On the Figure 4, we show the representation of the analytic and approximative solution with the absolute error obtained using the cubic B-splines approximation method by taking $\alpha = 10^{-6}$, $\beta = 0.1$ and $N = 64$ and $\Delta t = 1/160$.

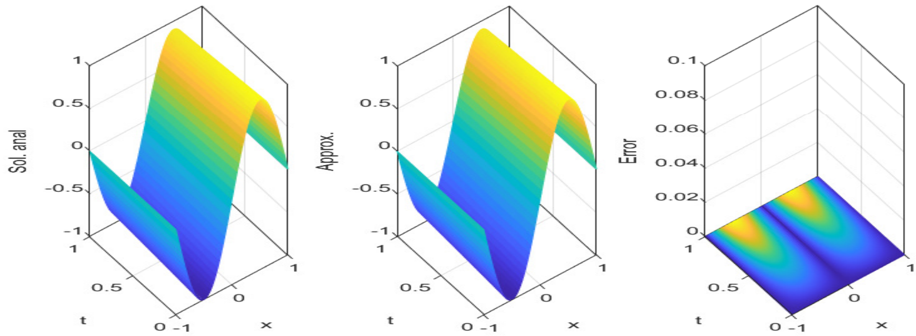

FIGURE 4. Representation of the solution (7.6), on the left, its approximation on the middle and the absolute error on the right for $N = 64$ and $\Delta t = 1/160$.

The Figure 5 shows that numerical solution reproduces satisfactorily the behaviour of the exact solution of the test problem at time $t = 0.75$ and for $\alpha = 10^{-6}$, $\beta = 0.1$ and $N = 64$ and $\Delta t = 1/160$.

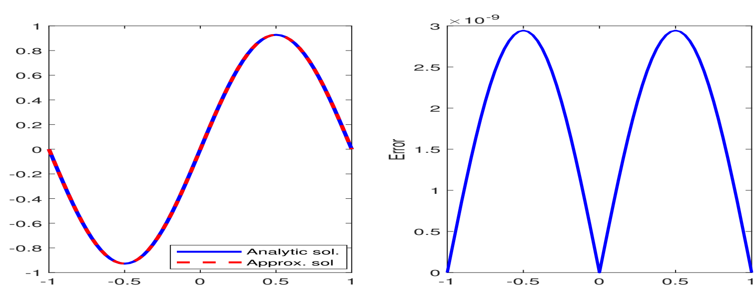

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FIGURE 5. Comparison of the solution (7.6) and its approximation on the left and the error obtained on the right for $t = 0.75$

Next, we test for the spatial discretization, we set the step time Δt small enough, say $\Delta t = 0.001$, so that the error of temporal discretization is negligible, and varies $N \leq 128$.

α	h_j	L_2 norm	Order_x	ME	Order_x
10^{-4}	0.25	5.973040e − 07	−	7.391323e − 07	−
	0.125	3.477656e − 08	4.102278	4.315066e − 08	4.098378
	0.0625	2.179679e − 09	3.995928	3.336495e − 09	3.692977
	0, 03125	1.209498e − 10	4.171635	3.526697e − 10	3.241944
10^{-1}	0.25	3.377211e − 04	−	3.713574e − 04	−
	0.125	1.935578e − 05	4.124996	1.954682e − 05	4.247803
	0.0625	1.181268e − 06	4.034356	1.181785e − 06	4.047894
	0, 03125	7.333561e − 08	4.009679	7.333847e − 08	4.010254

TABLE 4. Order of convergence at $t = 1$ and $\Delta t = 0.001$

So, it is seen that the numerical spatial convergence rates are ≤ 4, that ensures the theoretical convergence rates.

Example 7.3. Consider the diffusion-reaction problem with homogeneous boundary conditions (2.1), with $\mu(x) = \mathrm{e}^{x^2+1}$. The right-hand side source term and initial condition are calculated from the exact solution

$$u(x,t) = (t+1)\sin(\pi x)\,; \quad x \in [0,1] \text{ and } t \in [0,1] \tag{7.7}$$

So

$$f(x,t) = (1 + \alpha\pi^2(t+1) + (t+1)\mathrm{e}^{x^2+1})\sin(\pi x) \quad \text{and} \quad u(x,0) = \sin(\pi x).$$

The variations of the absolute error of the approximation proposed by the cubic B-splines, for different values of N and different values of $x \in [0,1]$ at time $t = 1$ are presented on the Table 5 for $\alpha = 10^{-3}$ (and 10^{-6}) and for $\beta = 0.1$.

x	$N = 16$ and $\Delta t = 1/40$		$N = 32$ and $\Delta t = 1/80$	
	$\alpha = 10^{-3}$	$\alpha = 10^{-6}$	$\alpha = 10^{-3}$	$\alpha = 10^{-6}$
0.125	1.598428e − 06	1.581830e − 06	9.626818e − 08	5.981152e − 09
0.25	2.879463e − 06	2.875897e − 06	1.779239e − 07	1.778510e − 07
0.375	3.770157e − 06	3.768921e − 06	2.338019e − 07	2.337255e − 07
0.5	4.105923e − 06	4.105682e − 06	2.547410e − 07	2.546983e − 07
0.625	3.819571e − 06	3.820194e − 06	2.368077e − 07	2.368056e − 07
0.75	2.949909e − 06	2.950052e − 06	1.821176e − 07	1.821500e − 07
0.875	1.650031e − 06	1.642207e − 06	9.924024e − 08	9.926550e − 08

x	$N = 64$ and $\Delta t = 1/160$	
	$\alpha = 10^{-3}$	$\alpha = 10^{-6}$
0.125	5.984026e − 09	5.981105e − 09
0.25	1.109477e − 08	1.110001e − 08
0.375	1.458736e − 08	1.458247e − 08
0.5	1.589384e − 08	1.589113e − 08
0.625	1.477490e − 08	1.477460e − 08
0.75	1.136142e − 08	1.136236e − 08
0.875	6.166137e − 09	6.167550e − 09

TABLE 5. Variations of the approximate error at $t = 1$ for different values of x, as a function of Δt, N and α.

On the Figure 6, we show the representation of the analytic and approximate solution obtained using the cubic B-splines approximation method by taking $\alpha = 10^{-3}$, and $N = 64$ and $\Delta t = 1/80$.

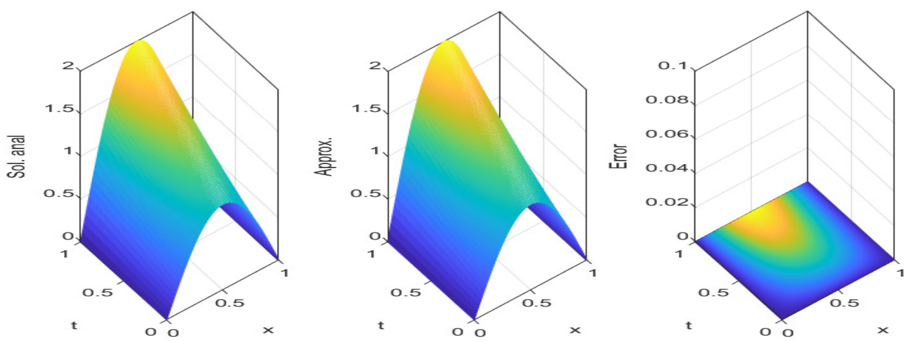

FIGURE 6. Representation of the solution (7.7) on the left, its approximation on the middle and the absolute error on the right for $N = 64$ and $\Delta t = 1/80$.

The Figure 7 shows that numerical solutions reproduce satisfactorily the behavior of the exact solution of the test problem at time $t = 0.5$ and for $\alpha = 10^{-3}$, $N = 64$ and $\Delta t = 1/160$.

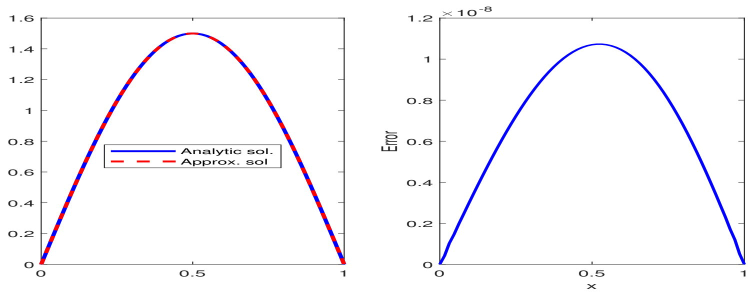

FIGURE 7. Comparison of the solution (7.7) and its approximation on the left and the approximate error on the right for $t = 0.5$

The L_2 norm error of the approximation proposed by the cubic B-splines, for different values of N and different values of Δt at time $t = 1$ is presented on the Tables $6 - 8$ for different values of α. We observe the uniform convergence of the numerical solution for all examples treated. We begin the table with $N = 16$ and the time step $\Delta t = 1/5$, and we multiply N bye two and divide Δt by two.

8. Conclusion

In this paper, a collocation method based on a combination of Galerkin-type approximation and cubic B-spline functions has been given for solving parabolic reaction-diffusion equations with small parameters; under initial and Dirichlet boundary conditions. The method is presented together as well as its evolution in time, i.e: the finite element method is applied for the spatial approximation and time discretization is based on the Crank Nicolson scheme. Moreover, an error estimation method is given to improve the approximate solutions.
Numerical examples are included which enhances its applicability and efficiency. We have shown in Table 2 that the temporal accuracy is of order 2. From the three tests problem treated; it is seen that the method has almost the uniform convergence, i,e: independent of perturbation parameter α, and our claim is very supported by the obtained results.

α	Number of intervals $N/$time step size Δt					
	$16/\frac{1}{5}$	$16/\frac{1}{10}$	$32/\frac{1}{20}$	$32/\frac{1}{40}$	$64/\frac{1}{80}$	$64/\frac{1}{160}$
10^0	4.343882e − 03	1.084500e − 03	2.709998e − 04	6.774166e − 05	1.693489e − 05	4.233689e − 06
10^{-2}	1.292975e − 03	3.221479e − 04	8.046889e − 05	2.011297e − 05	5.027977e − 06	1.256978e − 06
10^{-4}	1.232219e − 03	3.070504e − 04	7.670016e − 05	1.917114e − 05	4.792542e − 06	1.198120e − 06
10^{-6}	1.231615e − 03	3.069003e − 04	7.666269e − 05	1.916178e − 05	4.790202e − 06	1.197535e − 06
10^{-8}	1.231609e − 03	3.068988e − 04	7.666232e − 05	1.916169e − 05	4.790178e − 06	1.197529e − 06

TABLE 6. L_2 norm error as a function of N, Δt and α, for $t = 1$ for Example 7.1

α	Number of intervals $N/$time step size Δt					
	$16/\frac{1}{5}$	$16/\frac{1}{10}$	$32/\frac{1}{20}$	$32/\frac{1}{40}$	$64/\frac{1}{80}$	$64/\frac{1}{160}$
10^0	3.056214e − 05	3.072946e − 05	1.863637e − 06	1.878062e − 06	1.157830e − 07	1.166783e − 07
10^{-2}	8.017342e − 07	2.337094e − 06	1.937637e − 08	1.336222e − 07	4.815952e − 10	8.201748e − 09
10^{-4}	2.982793e − 06	7.217484e − 07	1.865149e − 07	4.521494e − 08	1.166016e − 08	2.829923e − 09
10^{-6}	3.015964e − 06	7.537171e − 07	1.884884e − 07	4.710762e − 08	1.178052e − 08	2.944248e − 09
10^{-8}	3.016297e − 06	7.540389e − 07	1.885083e − 07	4.712679e − 08	1.178173e − 08	2.945428e − 09

TABLE 7. L_2 norm error as a function of N, Δt and α, for $t = 1$ for Example 7.2

α	Number of intervals $N/$time step size Δt					
	$16/\frac{1}{5}$	$16/\frac{1}{10}$	$32/\frac{1}{20}$	$32/\frac{1}{40}$	$64/\frac{1}{80}$	$64/\frac{1}{160}$
10^0	2.943839e − 06	2.942682e − 06	1.828421e − 07	1.828436e − 07	1.141010e − 08	1.141011e − 08
10^{-2}	2.910010e − 06	2.905630e − 06	1.803571e − 07	1.803400e − 07	1.125219e − 08	1.125213e − 08
10^{-4}	2.905739e − 06	2.901316e − 06	1.800583e − 07	1.800411e − 07	1.123327e − 08	1.123320e − 08
10^{-6}	2.905723e − 06	2.901300e − 06	1.800560e − 07	1.800387e − 07	1.123308e − 08	1.123302e − 08
10^{-8}	2.905723e − 06	2.901300e − 06	1.800560e − 07	1.800387e − 07	1.123308e − 08	1.123302e − 08

TABLE 8. L_2 norm error as a function of N, Δt and α, for $t = 1$ for Example 7.3

Acknowledgments

K.L. Redouane and N. Arar acknowledge support from Directorate General for Scientific Research and Technological Development, Algeria.

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Received: March 23, 2022; Revised: July 9, 2022; Accepted: August 11, 2022