Proceedings of the Institute of Mathematics and Mechanics,
National Academy of Sciences of Azerbaijan
Volume 48, Special Issue, 2022, Pages 153–167
https://doi.org/10.30546/2409-4994.48.2022.153167

# ON LEGENDRE WAVELETS FOR POISSON EQUATION IN THE FRAME OF COMPLEX SOLUTION

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**Abstract**. In this paper, we study the Poisson equation with a complex solution:

$$\Delta F(x,y) - \Gamma(x,y) = 0, \ \ (x,y) \in [0,1] \times [0,1]$$

$\Gamma(x,y)$ is a given complex function, $F(x;y) = f(x;y) + ig(x;y)$ is the unknown complex function,where $f$ and $g$ are complex functions twice differentiable on the interval $[0;1]$, $\Delta$ denotes the Laplacian operator, $i$ is the imaginary unit. The method is based on Legendre wavelets (LW), and the idea, is that the two integration and derivation matrices are mutually used to reduce the problem in order to study numerically linear algebraic system. Some illustrative example are presented to explain the efficiency and simplicity of the presented method.

## 1. Introduction

This work is based on the application of two-dimensional Legendre wavelets for the numerical resolution of the Poisson equation

$$\Delta F(x,y) - \Gamma(x,y) = 0, \ \ (x,y) \in [0,1] \times [0,1],$$

with the boundary conditions

$$\begin{cases} F(0,0) = 0 & \\ F(x,0) = h(x) & , \ x \in [0,1] \\ F(0,y) = k(y) & , \ y \in [0,1] \\ \dfrac{\partial F(x,1)}{\partial x} = \alpha & , \ x \in [0,1] \\ \dfrac{\partial F(1,y)}{\partial y} = \beta & , \ y \in [0,1], \end{cases}$$

where $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ denote the Laplacian operator, $F(x,y)$ is an unknown complex function to be determined, $\Gamma(x,y)$ is a given complex function, $h(x)$ and $k(y)$ are given functions of complex variables $x$, $y$ which are twice continuously differentiable on the interval $[0,1]$, and $\alpha = \alpha_1 + i\alpha_2$, $\beta = \beta_1 + i\beta_2$ are two known complex constants.

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2010 *Mathematics Subject Classification.* 65T60, 31A30.

*Key words and phrases.* Legendre wavelet, operational matrix of derivative, operational matrix of integrating.

In this direction, the derivation operational matrix for Legendre wavelets [4] is derived, and then this matrix was used to solve the above-mentioned problem and obtain a numerical solution. This approach has the main characteristic of reducing these problems to solving systems of algebraic equations which greatly simplify these problems.

This work is organized into five sections: in the second section, we described the Legendre wavelets and their properties. In the third section, we studied the convergence of the expansions of Legendre wavelets. In the fourth section, we described the application of the proposed method to solve the Poisson equation. In the fifth and last section, a conclusion was drawn.

## 2. Notations and Preliminaries

### 2.1. Wavelets and Legendre wavelets.

**2.1.1.** *Wavelets.* Nonlinear PDE represent a variety of models that play a major role for many real world problems in different field [3, 12], and for fractional case [13]. In recent times, works from several different fields of science and engineering are based on wavelets. These wavelets, are a family of functions formed from the dilation and translation of a single function called the mother wavelet $w(t)$. the parameter $b$ varies continuously which allows to have the following family of continuous wavelets as [1].

$$w_{a,b}(t) = |a|^{-\frac{1}{2}}\, w\left(\frac{t-b}{a}\right), \quad a, b \in \mathbb{R},\ a \neq 0. \tag{2.1}$$

For a restriction of parameters $a$ and $b$ to discrete values as $a = a_1^{-k}$, $b = nb_1a_1^{-k}$, where $a_1 > 1$ and $b_1 > 0$, we obtain the following family of discrete wavelets

$$w_{k,n}(t) = |a_1|^{\frac{k}{2}}\, w\left(a_1^k t - nb_1\right). \tag{2.2}$$

This family $w_{k,n}(t) \triangleq w_{kn}(t)$ constitutes a wavelet basis for $L^2(\mathbb{R})$ where it becomes orthonormal for $a_1 = 2$, $b_1 = 1$ and $n$, $k$ are positive integers.

**2.1.2.** *The Legendre wavelets.* The Legendre wavelets are defined on the interval $[0, 1)$ as [1] :

$$w_{nm}(t) = \begin{cases} \sqrt{m+\frac{1}{2}}\, 2^{\frac{k}{2}}\, G_m\left(2^k t - \tilde{n}\right) & \frac{\tilde{n}-1}{2^k} \leqslant t \leqslant \frac{\tilde{n}}{2^k} \\ 0 & \text{otherwise} \end{cases} \tag{2.3}$$

where $m = 0, 1, \ldots, M-1$, $M$ is a fixed positive integer, $n = 1, 2, \ldots, 2^{k-1}$, $k \in N$, and $\tilde{n} = 2n - 1$. The real numbers $a = 2^{\frac{k}{2}}$, $b = \tilde{n}2^{\frac{k}{2}}$ are the dilation and translation parameter successively. $G_m(t)$ denote the Legendre polynomials of degree $m$ defined on the interval $[-1, 1]$, that we can determine them by using the following recurrence [23] :

$$\begin{cases} G_0(t) = 1,\ G_1(t) = t \\ G_{m+2}(t) = \left(\frac{2m+3}{m+2}\right) t\, G_{m+1}(t) - \left(\frac{m+1}{m+2}\right) G_m(t)\ , \quad m \geq 0 \end{cases} \tag{2.4}$$

**2.2. Function approximation.** A function $h(t)$ defined over $[0,1)$ may be expanded as [9] and [1]

$$h(t) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} a_{nm} w_{nm}(t), \tag{2.5}$$

where $a_{nm} = \langle h(t), w_{nm}(t)\rangle$ is the inner product defined as $a_{nm} = \int_0^1 h(t) w_{nm}(t)\, dt$.

If the infinite series in Eq. (2.5) is truncated, then it can be rewritten as

$$h(t) \simeq \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1} a_{nm} w_{nm}(t) = A^T W(t), \tag{2.6}$$

where $T$ indicate transposition and $A$ and $W(t)$ are $\widehat{m} = 2^{k-1}M$ column vectors.

For simplicity, Eq. (2.6) can be written as

$$h(t) \simeq \sum_{i=1}^{\widehat{m}} a_i w_i(t) = A^T W(t), \tag{2.7}$$

where $a_i = a_{nm}$, $w_i(t) = w_{nm}(t)$. $i$ is an integer given by $i = M(n-1) + m + 1$ thus we have

$$A \triangleq [a_1, a_2, ...., a_{\widehat{m}}]^T, \quad W(t) \triangleq [w_1(t), w_2(t), ...., w_{\widehat{m}}(t)]^T. \tag{2.8}$$

Similarly, for any function $h(x,y)$ with two variables $x$, $y$ defined on $[0,1)\times[0,1)$ can be developed in a Legendre wavelet basis as [9] :

$$h(x,y) \simeq \sum_{i=1}^{\widehat{m}}\sum_{j=1}^{\widehat{m}} f_{ij} w_i(x) w_j(y) = W^T(x)\ F W(y), \tag{2.9}$$

where $F = [f_{ij}]$ and $f_{ij} = \langle W(x), \langle h(x,y), W(y)\rangle\rangle$ in which $\langle .,.\rangle$ denotes the inner product.

By taking the collocation points $t_i = \frac{2i-1}{\widehat{m}}$ $(i = 1, 2, \ldots, \widehat{m})$, in the $W(t)$, we define the Legendre wavelets matrix $\phi_{\widehat{m}\times\widehat{m}}$ as

$$\phi_{\widehat{m}\times\widehat{m}} \triangleq \left[W\left(\frac{1}{2\widehat{m}}\right), W\left(\frac{3}{2\widehat{m}}\right), \ldots, W\left(\frac{2\widehat{m}-1}{2\widehat{m}}\right)\right] \tag{2.10}$$

Moreover, it is shown in [1] that $\phi_{\widehat{m}\times\widehat{m}}$ has a diagonal form.

**2.3. Derivative and integration operational matrices.** Now we introduce the operational matrices of the derivative and integration Legendre wavelets.

**Theorem 2.1.** [15]. *If $h(t)$ is a function and $W(t)$ its Legendre wavelet vector defined in 2.8, then we can write the derivative of $W(t)$ as follows*

$$\frac{dW(t)}{dt} = DW(t), \tag{2.11}$$

*where $D$ is the $2^k(M+1)$ operational matrix of derivative defined as follows*

$$D = \begin{pmatrix} \breve{A} & 0 & . & . & 0 \\ 0 & \breve{A} & 0 & . & 0 \\ . & . & . & & . \\ . & . & & . & . \\ 0 & 0 & 0 & . & \breve{A} \end{pmatrix} \tag{2.12}$$

*where* $\breve{A} = \left(\breve{A}_{i,j}\right)$, *is a* $(M+1)(M+1)$ *matrix and their elements* $\breve{A}_{i,j}$ *are given by the following relation*

$$\breve{A}_{i,j} = \begin{cases} 2^{k+1}\sqrt{(2i-1)(2j-1)} & , i = 2,\ldots,M+1,\ j = 1,\ldots,i-1 \text{ and } i+j \text{ odd} \\ 0 & , \text{otherwise} \end{cases} \tag{2.13}$$

**Corollary 2.1.** [15]. *The n-th derivative of the operational matrix can be obtained using equation* (2.11) *as follows:*

$$\frac{d^n W(t)}{dt^n} = D^n W(t), \ \text{where } D^n = \underbrace{D \times D \times ... \times D}_{n-times} \tag{2.14}$$

The integration of the vector $W(t)$, defined in (2.8), can be expressed as [14] :

$$\int_0^x W(t)dt \simeq \tilde{I}W(x), \tag{2.15}$$

where $\tilde{I}$ is the $\widehat{m} \times \widehat{m}$ operational matrix of integration for LWs.

In general, by applying $n$ times the relation (2.15), we can have $\tilde{I}^n$ and we can write

$$\underbrace{\int_0^x ... \int_0^x}_{n-times} W(t)dt...dt \simeq \tilde{I}^n W(x). \tag{2.16}$$

## 3. Convergence

**Theorem 3.1.** [7]. *If the second derivative of a continuous function* $h(t) \in L^2[0,1]$ *is bounded* $|h"(t)| \leq M$, *then the latter can be expanded as an infinite sum of Legendre wavelets and the series converges uniformly to* $h(t)$, *that is* $h(t) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} a_{nm} w_{nm}(t)$, *and coefficients* $a_{nm}$ *are bounded as*

$$|a_{nm}| \leq \frac{\sqrt{12}\widehat{M}}{(2n)^{\frac{5}{2}}(2m-3)^2} \tag{3.1}$$

**Theorem 3.2.** [9]. *Let* $A^T\ W(t)$ *be the approximation of a function h with Legendre's wavelets where* $h(t) \in L^2[0,1]$ *and its second derivative is bounded* $|h"(t)| \leq M$. *Then the error is bounded as follows:*

$$\epsilon_{\widehat{m}}(h) \leq \left(\sum_{n=1}^{\infty}\sum_{m=M}^{\infty} a_{nm}^2 + \sum_{n=2^k+1}^{\infty}\sum_{m=0}^{M-1} a_{nm}^2\right)^{\frac{1}{2}}, \tag{3.2}$$

*where*

$$\epsilon_{\widehat{m}}(h) = \left(\int_0^1 \left(h(t) - A^T\ W(t)\right)^2 dt\right)^{\frac{1}{2}}. \tag{3.3}$$

**Theorem 3.3.** [14]. *If the mixed fourth partial derivatives of a continuous function* $h(x,y) \in L^2([0,1]\times[0,1])$ *is bounded* $\left|\frac{\partial^4 h(x,y)}{\partial x^2 \partial y^2}\right| \leq \widehat{M}$, *then the Legendre wavelets expansion of the latter converges uniformly to* $h(t)$, *and coefficients* $f_{ij}$ *are bounded as*

$$|f_{ij}| \leq \frac{12\widehat{M}}{(2n_1)^{\frac{5}{2}}(2n_2)^{\frac{5}{2}}(2m_1-3)^2(2m_2-3)^2}, \tag{3.4}$$

*where* $i = M(n_1-1)+m_1+1$ *and* $j = M(n_2-1)+m_2+1$.

**Theorem 3.4.** [9]. *If* $h(x,y) \in L^2([0,1]\times[0,1])$, $\widehat{M}$ *is the bound of its second derivative and its approximation with the Legendre wavelets is* $W^T(x)\ FW(y)$. *Then the error* $\epsilon_{\widehat{m}\times\widehat{m}}(h)$ *is bounded as follows:*

$$\epsilon_{\widehat{m}\times\widehat{m}}(h) \leq \left(\sum_{i=1}^{\infty}\sum_{j=\widehat{m}+1}^{\infty} f_{ij}^2 + \sum_{i=\widehat{m}+1}^{\infty}\sum_{j=1}^{\widehat{m}} f_{ij}^2\right)^{\frac{1}{2}}, \tag{3.5}$$

*where*

$$\epsilon_{\widehat{m}\times\widehat{m}}(h) = \left(\int_0^1\int_0^1 \left(h(x,y)-W^T(x)FW(y)\right)^2 dxdy\right)^{\frac{1}{2}}. \tag{3.6}$$

## 4. The proposed method

In this section we will solve the Poisson equation with a complex solution using a calculation method based on the Legendre wavelets (LWs) and the operational matrices of integration and derivative of the LWs.

Consider the following equations

$$\Delta F(x,y) = \Gamma(x,y), \quad (x,y) \in [0,1]\times[0,1], \tag{4.1}$$

with the boundary conditions

$$\begin{cases} F(0,0) = 0 & \\ F(x,0) = h(x) & ,\ x\in[0,1] \\ F(0,y) = k(y) & ,\ y\in[0,1] \\ \dfrac{\partial F(x,1)}{\partial x} = \alpha & ,\ x\in[0,1] \\ \dfrac{\partial F(1,y)}{\partial y} = \beta & ,\ y\in[0,1], \end{cases} \tag{4.2}$$

$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ denote the Laplacian operator, $F(x,y)$ is an unknown complex function to be determined, $\Gamma(x,y)$ is a given complex function, $h(x)$ and $k(y)$ are given functions of complex variables $x,y$ which are twice continuously differentiable on the interval$[0,1]$, and $\alpha = \alpha_1 + i\alpha_2$, $\beta = \beta_1 + i\beta_2$ are two known complex constants.

**4.1. Solving the equation.** To solve this equation, we suppose

$$\begin{cases} F(x,y) = f(x,y) + ig(x,y) \\ \Gamma(x,y) = \gamma_1(x,y) + i\gamma_2(x,y) \\ h(x) = h_1(x) + ih_2(x) \\ k(y) = k_1(y) + ik_2(y). \end{cases} \tag{25}$$

So, we can rewrite the equation (4.1) and the boundary conditions (4.2) as follows:

$$\begin{cases} \Delta f = \gamma_1 \ (x,y) & (x,y) \in [0,1] \times [0,1] \\ \Delta g = \gamma_2 \ (x,y) & (x,y) \in [0,1] \times [0,1], \end{cases} \tag{4.3}$$

subject to the boundary conditions

$$\begin{cases} f(0,0) = g(0,0) = 0 & \\ f(x,0) = h_1(x), \quad g(x,0) = h_2(x) & x \in [0,1] \\ f(0,y) = k_1(y), \quad g(0,y) = k_2(y) & y \in [0,1] \\ \dfrac{\partial f(x,1)}{\partial x} = \alpha_1 & x \in [0,1] \\ \dfrac{\partial g(x,1)}{\partial x} = \alpha_2 & x \in [0,1] \\ \dfrac{\partial f(1,y)}{\partial y} = \beta_1 & y \in [0,1] \\ \dfrac{\partial g(1,y)}{\partial y} = \beta_2 & y \in [0,1]. \end{cases} \tag{4.4}$$

Now we suppose

$$\begin{cases} \dfrac{\partial^4 f}{\partial x^2 \partial y^2} = W(x)^T F \ W(y) \\ \dfrac{\partial^4 g}{\partial x^2 \partial y^2} = W(x)^T G \ W(y) \end{cases} \tag{4.5}$$

where $F = [f_{ij}]_{\widehat{m} \times \widehat{m}}$ and $G = [g_{ij}]_{\widehat{m} \times \widehat{m}}$ are the matrices we are looking for, and $W\,(.)$ is the Legendre wavelets vector that is defined in (2.7). We suppose

$$\begin{cases} f(x,0) = h_1(x) \triangleq W(x)^T H_1 \\ g(x,0) = h_2(x) \triangleq W(x)^T H_2 \\ f(0,y) = k_1(y) \triangleq K_1^T W(y) \\ g(0,y) = k_2(y) \triangleq K_2^T W(y) \end{cases} \tag{4.6}$$

By integrating (4.5) tow times with respect to $y$, we obtain

$$\begin{cases} \dfrac{\partial^2 f}{\partial x^2} = W(x)^T \left(D^2\right)^T H_1 + y \frac{\partial}{\partial y} \left(\frac{\partial^2 f}{\partial x^2}\right)_{y=0} + W(x)^T F \tilde{I}^2 W(y) \\ \dfrac{\partial^2 g}{\partial x^2} = W(x)^T \left(D^2\right)^T H_2 + y \frac{\partial}{\partial y} \left(\frac{\partial^2 g}{\partial x^2}\right)_{y=0} + W(x)^T G \tilde{I}^2 W(y). \end{cases} \tag{4.7}$$

By putting $y = 1$ in (4.7), we obtain

$$\begin{cases} \frac{\partial}{\partial y} \left(\frac{\partial^2 f}{\partial x^2}\right)_{y=0} = \frac{\partial^2 f(x,1)}{\partial x^2} - W(x)^T \left[\left(D^2\right)^T H_1 + F \ \tilde{I}^2 \ W(1)\right] \\ \frac{\partial}{\partial y} \left(\frac{\partial^2 g}{\partial x^2}\right)_{y=0} = \frac{\partial^2 g(x,1)}{\partial x^2} - W(x)^T \left[\left(D^2\right)^T H_2 + G \tilde{I}^2 \ W(1)\right], \end{cases} \tag{4.8}$$

where $H_1$ and $H_2$ are the Legendre wavelets coefficient vectors for $h_1(x)$ and $h_2(x)$ respectively, and by considering (4.4), we have $\frac{\partial^2 f(x,1)}{\partial x^2} = 0$, $\frac{\partial^2 g(x,1)}{\partial x^2} = 0$

and

$$\begin{cases} \frac{\partial}{\partial y}\left(\frac{\partial^2 f}{\partial x^2}\right)_{y=0} = W(x)^T\left[-\left(D^2\right)^T H_1 - F\tilde{I}^2 W(1)\right] \triangleq W(x)^T \wedge_1 \\ \frac{\partial}{\partial y}\left(\frac{\partial^2 g}{\partial x^2}\right)_{y=0} = W(x)^T\left[-\left(D^2\right)^T H_2 - G\tilde{I}^2 W(1)\right] \triangleq W(x)^T \wedge_2, \end{cases} \tag{4.9}$$

and by substituting (4.9) into (4.7), we have

$$\begin{cases} \frac{\partial^2 f}{\partial x^2} = W(x)^T\left(D^2\right)^T H_1 + yW(x)^T \wedge_1 + W(x)^T F\tilde{I}^2 W(y) \\ \frac{\partial^2 g}{\partial x^2} = W(x)^T\left(D^2\right)^T H_2 + yW(x)^T \wedge_2 + W(x)^T G\tilde{I}^2 W(y). \end{cases} \tag{4.10}$$

Suppose that the coefficient vectors ($LWs$) of the unit step functions and $y$ are $E$ and $Y$ respectively, then (4.10) can be written as follows:

$$\begin{cases} \frac{\partial^2 f}{\partial x^2} = W(x)^T\left[\left(D^2\right)^T H_1 E^T + \wedge_1 Y^T + F\tilde{I}^2\right] W(y) \triangleq W(x)^T A_1 W(y) \\ \frac{\partial^2 g}{\partial x^2} = W(x)^T\left[\left(D^2\right)^T H_2 E^T + \wedge_2 Y^T + G\tilde{I}^2\right] W(y) \triangleq W(x)^T A_2 W(y). \end{cases} \tag{4.11}$$

Moreover, by integrating (4.5) two times with respect to $x$, and considering (4.4), we obtain

$$\begin{cases} \frac{\partial^2 f}{\partial y^2} = k_1''(y) + x\frac{\partial}{\partial x}\left(\frac{\partial^2 f}{\partial y^2}\right)_{x=0} + W(x)^T\left(\tilde{I}^T\right)^2 FW(y) \\ \frac{\partial^2 g}{\partial y^2} = k_2''(y) + x\frac{\partial}{\partial x}\left(\frac{\partial^2 g}{\partial y^2}\right)_{x=0} + W(x)^T\left(\tilde{I}^T\right)^2 GW(y). \end{cases} \tag{4.12}$$

By putting $x = 1$, $k_1''(y) = K_1^T D^2 W(y)$ and $k_2''(y) = K_2^T D^2 W(y)$ in (4.12), we have

$$\begin{cases} \left(\frac{\partial^2 f}{\partial y^2}\right)_{x=1} = K_1^T D^2 W(y) + \frac{\partial}{\partial x}\left(\frac{\partial^2 f}{\partial y^2}\right)_{x=0} + W(1)^T\left(\tilde{I}^T\right)^2 FW(y) \\ \left(\frac{\partial^2 g}{\partial y^2}\right)_{x=1} = K_2^T D^2 W(y) + \frac{\partial}{\partial x}\left(\frac{\partial^2 g}{\partial y^2}\right)_{x=0} + W(1)^T\left(\tilde{I}^T\right)^2 G\, W(y). \end{cases} \tag{4.13}$$

By considering (4.4), $\frac{\partial^2 f(1,y)}{\partial y^2} = 0$ and $\frac{\partial^2 g(1,y)}{\partial y^2} = 0$, we have

$$\begin{cases} \frac{\partial}{\partial x}\left(\frac{\partial^2 f}{\partial y^2}\right)_{x=0} = \left[-K_1^T D^2 - W(1)^T\left(\tilde{I}^T\right)^2 F\right] W(y) \triangleq \sum_1^T W(y) \\ \frac{\partial}{\partial x}\left(\frac{\partial^2 g}{\partial y^2}\right)_{x=0} = \left[-K_2^T D^2 - W(1)^T\left(\tilde{I}^T\right)^2 G\right] W(y) \triangleq \sum_2^T W(y). \end{cases} \tag{4.14}$$

Suppose that the coefficient vectors ($LWs$) of the unit step functions and $x$ are $E$ and $X$ respectively. By considering (4.4) and by substituting (4.14) into (4.12), we can write (4.14) as follows

$$\begin{cases} \frac{\partial^2 f}{\partial y^2} = W(x)^T\left[EK_1^T D^2 + X\sum_1^T + \left(\tilde{I}^T\right)^2 F\right] W(y) \triangleq W(x)^T B_1 W(y) \\ \frac{\partial^2 g}{\partial y^2} = W(x)^T\left[EK_2^T D^2 + X\sum_2^T + \left(\tilde{I}^T\right)^2 G\right] W(y) \triangleq W(x)^T B_2 W(y). \end{cases} \tag{4.15}$$

Then, by substituting (4.11), (4.15) into (4.3), we obtain

$$\begin{cases} A_1 + B_1 - \Gamma_1 = 0 \\ A_2 + B_2 - \Gamma_2 = 0, \end{cases} \tag{4.16}$$

where $\Gamma_1$ and $\Gamma_2$ are the LWs coefficient vectors for the functions $\gamma_1(x,y)$ and $\gamma_2(x,y)$, respectively.

By solving system (4.16), we obtain the unknown matrices $F$ and $G$ and subsequently we get the functions $f(x,y)$, $g(x,y)$ by integrating (4.11) with respect to $x$ as follows, we obtain

$$\begin{cases} \frac{\partial f}{\partial x} = \left(\frac{\partial f}{\partial x}\right)_{x=0} + W(x)^T \tilde{I}^T\ A_1 W(y) \\ \frac{\partial g}{\partial x} = \left(\frac{\partial g}{\partial x}\right)_{x=0} + W(x)^T \tilde{I}^T\ A_2 W(y). \end{cases} \tag{4.17}$$

Moreover,by integrating (4.17) with respect to $x$, we obtain

$$\begin{cases} f(x,y) = f(0,y) + x\left(\frac{\partial f}{\partial x}\right)_{x=0} + W(x)^T \left(\tilde{I}^T\right)^2 A_1 W(y) \\ g(x,y) = g(0,y) + x\left(\frac{\partial g}{\partial x}\right)_{x=0} + W(x)^T \left(\tilde{I}^T\right)^2 A_2 W(y). \end{cases} \tag{4.18}$$

By considering (4.6) and by substituting into (4.18), we have

$$\begin{cases} f(x,y) = K_1^T \Psi(y) + x\left(\frac{\partial f}{\partial x}\right)_{x=0} + W(x)^T \left(\tilde{I}^T\right)^2 A_1 W(y) \\ g(x,y) = K_2^T W(y) + x\left(\frac{\partial g}{\partial x}\right)_{x=0} + W(x)^T \left(\tilde{I}^T\right)^2 A_2 W(y). \end{cases} \tag{4.19}$$

By putting $x = 1$ in (4.19), we obtain

$$\begin{cases} \left(\frac{\partial f}{\partial x}\right)_{x=0} = f(1,y) - K_1^T W(y) - W(1)^T \left(\tilde{I}^T\right)^2 A_1 W(y) \\ \left(\frac{\partial g}{\partial x}\right)_{x=0} = g(1,y) - K_2^T W(y) - W(1)^T \left(\tilde{I}^T\right)^2 A_2 W(y), \end{cases} \tag{4.20}$$

and by considering (4.4), we have

$$\begin{cases} f(1,y) = f(1,0) + \beta_1 y = h_1(1) + \beta_1 y \triangleq \omega_1^T W(y) \\ g(1,y) = g(1,0) + \beta_2 y = h_2(1) + \beta_2 y \triangleq \omega_2^T W(y). \end{cases} \tag{4.21}$$

By substituting into (4.20), we have

$$\begin{cases} \left(\frac{\partial f}{\partial x}\right)_{x=0} = \left[\omega_1^T - K_1^T - W(1)^T \left(\tilde{I}^T\right)^2 A_1\right] W(y) \triangleq \Omega_1^T W(y) \\ \left(\frac{\partial g}{\partial x}\right)_{x=0} = \left[\omega_2^T - K_2^T - W(1)^T \left(\tilde{I}^T\right)^2 A_2\right] W(y) \triangleq \Omega_2^T W(y). \end{cases} \tag{4.22}$$

By considering (4.6), (4.22) and substituting into (4.19), we have

$$\begin{cases} f(x,y) = W(x)^T \left[E K_1^T + X\Omega_1^T + \left(\tilde{I}^T\right)^2 A_1\right] W(y) \\ g(x,y) = W(x)^T \left[E K_2^T + X\Omega_2^T + \left(\tilde{I}^T\right)^2 A_2\right] W(y). \end{cases} \tag{4.23}$$

As a result, we find the approximate solution of the proposed problem using (**??**).

**4.2. Algorithm.** The algorithm of this method can be presented in the following steps:

**Input:** $M \in N, k \in N \cup \{0\}$; real constants $\alpha_1; \alpha_2; \beta_1$ and $\beta_2$; the functions $h_1(x)$, $h_2(x)$, $k_1(y)$ and $k_2(y) \in L^2[0,1]$; the functions $\gamma_1(x,y)$, $\gamma_2(x,y) \in L^2([0,1] \times [0,1])$.

**Step 1:** Define the legendre wavelets $w_{nm}(x)$ by (2.3).

**Step 2:** Prepare the legendre wavelets vector $W(x)$ from (2.8).

**Step 3:** Calculation of the legendre wavelets matrix
$\phi_{\widehat{m}\times\widehat{m}} \triangleq \left[W\left(\frac{1}{2\widehat{m}}\right), W\left(\frac{3}{2\widehat{m}}\right), \ldots, W\left(\frac{2\widehat{m}-1}{2\widehat{m}}\right)\right]$ from (2.10).

**Step 4:** Calculation of the legendre wavelets operational matrices $D^n$and $\tilde{I}^n$ using (2.14)-(2.16).

**Step 5:** Calculation of the coefficient vectors $E$, $X$, $Y$, $H_1$, $H_2$, $K_1$and $K_2$ using (2.5).

**Step 6:** Define the unknown matrices $F = [f_{ij}]_{\widehat{m}\times\widehat{m}}$ and $G = [g_{ij}]_{\widehat{m}\times\widehat{m}}$.

**Step 7:** Calculation of the vectors $\wedge_1$, $\wedge_2$ and consequently $A_1$, $A_2$ using (4.9)-(4.11).

**Step 8:** Calculation of the vectors $\sum_1$, $\sum_2$ and consequently $B_1$, $B_2$ using (4.14)-(4.15).

**Step 9:** Put $\begin{cases} A_1 + B_1 - \Gamma_1 = 0 \\ A_2 + B_2 - \Gamma_2 = 0 \end{cases}$.

**Step 10:** Find the unknown matrices $F$ and $G$ with the solution of the system of algebraic equations defined in step (2.9).

**Step 11:** Calculation of the vectors $\omega_1$, $\omega_2$ and consequently $\Omega_1$, $\Omega_2$ using (4.21)-(4.22).

**Output:** Find $f(x,y)$ and $g(x,y)$ from (4.23) and consequently $\digamma(x,y)$.

**4.2.1.** *Illustrative Test Problem.* Consider the Poisson equation on a unit square $[0,1] \times [0,1]$ which is formulated as follows:

$$\Delta\digamma(x,y) = 2x - 2 + i(2y - 2), \quad (x,y) \in [0,1] \times [0,1],$$

subject to the boundary conditions

$$\begin{cases} \digamma(0,0) = 0 & \\ \digamma(x,0) = -x - ix^2 & , \ x \in [0,1] \\ \digamma(0,y) = -y^2 - iy & , \ y \in [0,1] \\ \dfrac{\partial\digamma(x,1)}{\partial x} = 0 & , \ x \in [0,1] \\ \dfrac{\partial\digamma(1,y)}{\partial y} = 0 & , \ y \in [0,1] \end{cases}$$

So that the exact solution is given

$$\digamma(x,y) = xy^2 - x - y^2 + i\left(x^2y - x^2 - y\right) \triangleq f(x,y) + ig(x,y).$$

Boundary conditions can be written as

$$\begin{array}{ll}
f(0,0) = g(0,0) = 0 & \\
f(x,0) = h_1(x) = -x \overset{\Delta}{=} W(x)^T H_1 & , \ x \in [0,1] \\
g(x,0) = h_2(x) = -x^2 \overset{\Delta}{=} W(x)^T H_2 & , \ x \in [0,1] \\
f(0,y) = k_1(y) = -y^2 \overset{\Delta}{=} K_1^T \ W(y) & , \ y \in [0,1] \\
g(0,y) = k_2(y) = -y \overset{\Delta}{=} K_2^T \ W(y) & , \ y \in [0,1] \\
\frac{\partial f(x,1)}{\partial x} = \alpha_1 = 0 & , \ x \in [0,1] \\
\frac{\partial g(x,1)}{\partial x} = \alpha_2 = 0 & , \ x \in [0,1] \\
\frac{\partial f(1,y)}{\partial y} = \beta_1 = 0 & , \ y \in [0,1] \\
\frac{\partial g(1,y)}{\partial y} = \beta_2 = 0 & , \ y \in [0,1]
\end{array},$$

For $\widehat{m} = 3$ $(k = 1, \ M = 3)$, we have

$$D = \begin{pmatrix} 0 & 0 & 0 \\ 2\sqrt{3} & 0 & 0 \\ 0 & 2\sqrt{15} & 0 \end{pmatrix} \quad , \quad \tilde{I} = \begin{pmatrix} 1 & \frac{1}{\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{3}} & 0 & \frac{1}{\sqrt{3}\sqrt{5}} \\ 0 & -\frac{1}{\sqrt{3}\sqrt{5}} & 0 \end{pmatrix},$$

$$H_1 = K_2 = \begin{pmatrix} -\frac{1}{2} \\ -\frac{1}{2\sqrt{3}} \\ 0 \end{pmatrix} \quad , \quad H_2 = K_1 = \begin{pmatrix} -\frac{1}{3} \\ -\frac{1}{2\sqrt{3}} \\ -\frac{1}{6\sqrt{5}} \end{pmatrix},$$

$$E = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad , \quad X = Y = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2\sqrt{3}} \\ 0 \end{pmatrix},$$

$$\Gamma_1 = \Gamma_2^T = \begin{pmatrix} -1 & 0 & 0 \\ \frac{1}{\sqrt{3}} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Subsequently, we find

$$A_1 = F \begin{pmatrix} -\frac{1}{3} & 0 & \frac{1}{15}\sqrt{5} \\ \frac{1}{30}\sqrt{3} & -\frac{1}{30} & 0 \\ \frac{1}{15}\sqrt{5} & 0 & -\frac{1}{15} \end{pmatrix},$$

$$A_2 = \begin{pmatrix} -1 & \frac{1}{\sqrt{3}} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + G \begin{pmatrix} -\frac{1}{3} & 0 & \frac{1}{15}\sqrt{5} \\ \frac{1}{30}\sqrt{3} & -\frac{1}{30} & 0 \\ \frac{1}{15}\sqrt{5} & 0 & -\frac{1}{15} \end{pmatrix},$$

$$B_1 = \begin{pmatrix} -1 & 0 & 0 \\ \frac{1}{3}\sqrt{3} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} -\frac{1}{3} & \frac{1}{30}\sqrt{3} & \frac{1}{15}\sqrt{5} \\ 0 & -\frac{1}{30} & 0 \\ \frac{1}{15}\sqrt{5} & 0 & -\frac{1}{15} \end{pmatrix} F,$$

$$B_2 = \begin{pmatrix} -\frac{1}{3} & \frac{1}{30}\sqrt{3} & \frac{1}{15}\sqrt{5} \\ 0 & -\frac{1}{30} & 0 \\ \frac{1}{15}\sqrt{5} & 0 & -\frac{1}{15} \end{pmatrix} G$$

By substituting $A_1$, $A_2$, $B_1$, $B_2$, $\Gamma_1$ and $\Gamma_2$ in system (4.16)

$$\begin{cases} A_1 + B_1 - \Gamma_1 = 0 \\ A_2 + B_2 - \Gamma_2 = 0, \end{cases}$$

we have

$$F\begin{pmatrix} -\frac{1}{3} & 0 & \frac{\sqrt{5}}{15} \\ \frac{\sqrt{3}}{30} & -\frac{1}{30} & 0 \\ \frac{\sqrt{5}}{15} & 0 & -\frac{1}{15} \end{pmatrix} + \begin{pmatrix} -\frac{1}{3} & \frac{\sqrt{3}}{30} & \frac{\sqrt{5}}{15} \\ 0 & -\frac{1}{30} & 0 \\ \frac{\sqrt{5}}{15} & 0 & -\frac{1}{15} \end{pmatrix} F = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and

$$G\begin{pmatrix} -\frac{1}{3} & 0 & \frac{\sqrt{5}}{15} \\ \frac{\sqrt{3}}{30} & -\frac{1}{30} & 0 \\ \frac{\sqrt{5}}{15} & 0 & -\frac{1}{15} \end{pmatrix} + \begin{pmatrix} -\frac{1}{3} & \frac{\sqrt{3}}{30} & \frac{\sqrt{5}}{15} \\ 0 & -\frac{1}{30} & 0 \\ \frac{\sqrt{5}}{15} & 0 & -\frac{1}{15} \end{pmatrix} G = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

from where

$$F = G = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

then, we can calculate the following matrices

$$A_1 = B_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, A_2 = B_1^T = \begin{pmatrix} -1 & \frac{1}{\sqrt{3}} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{cases} f(x,y) \simeq W(x)^T \left[ EK_1^T + X\left[ \omega_1^T - K_1^T - W(1)^T \left(\tilde{I}^T\right)^2 A_1 \right] + \left(\tilde{I}^T\right)^2 A_1 \right] W(y) \\ g(x,y) \simeq W(x)^T \left[ EK_2^T + X\left[ \omega_2^T - K_2^T - W(1)^T \left(\tilde{I}^T\right)^2 A_2 \right] + \left(\tilde{I}^T\right)^2 A_2 \right] W(y). \end{cases}$$

Finally, we find the components of the approximate solution

$$\begin{cases} f(x,y) \simeq 0.999\,97xy^2 + 4.725\,5 \times 10^{-5}xy - 1.001x \\ -0.999\,98y^2 - 2.\,16\,1 \times 10^{-5}y + 2.312\,8 \times 10^{-5} \\ g(x,y) \simeq 0.999\,97x^2y - 0.999\,98x^2 + 4.\,25\,5 \times 10^{-5}xy \\ -2.416\,1 \times 10^{-5}x - 1.001y + 2.312\,8 \times 10^{-5} \end{cases}$$

In the following, we will make a comparison between the exact and approximate solution and determine the errors for a few points.

For the solution $f(x,y)$, see Figure 1 and Figure 2.

**Table 1:** The absolute errors of the proposed method for $\widehat{m} = 3$ ($k = 1,\ M = 3$) for the solution $f(x,y)$:

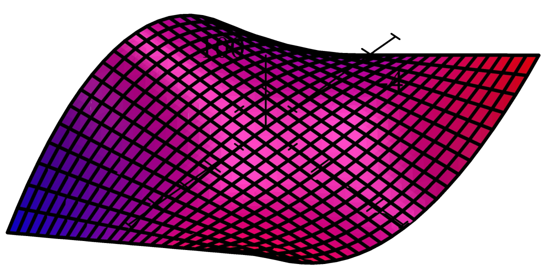
| $x_i$ | $y_i$ | | $e_i$ |
|---|---|---|---|
| 0.1 | 0.1 | | 2.13545E-05 |
| 0.2 | 0.2 | | 2.0746E-05 |
| 0.3 | 0.3 | | 2.11226E-05 |
| 0.4 | 0.4 | | 2.23044E-05 |
| 0.5 | 0.5 | | 2.41113E-05 |
| 0.6 | 0.6 | | 2.63632E-05 |
| 0.7 | 0.7 | | 2.88803E-05 |
| 0.8 | 0.8 | | 3.14824E-05 |
| 0.9 | 0.9 | | 3.39897E-05 |

For the solution $g(x, y)$, see Figure 3 and Figure 4.

**Table 2 :** The absolute errors of the proposed method for $\widehat{m} = 3$ $(k = 1,\ M = 3)$ for the solution $g(x, y)$:

| $x_i$ | $y_i$ | | $e_i$ |
|---|---|---|---|
| 0.1 | 0.1 | | 2.13545E-05 |
| 0.2 | 0.2 | | 2.0746E-05 |
| 0.3 | 0.3 | | 2.11226E-05 |
| 0.4 | 0.4 | | 2.23044E-05 |
| 0.5 | 0.5 | | 2.41113E-05 |
| 0.6 | 0.6 | | 2.63632E-05 |
| 0.7 | 0.7 | | 2.88803E-05 |
| 0.8 | 0.8 | | 3.14824E-05 |
| 0.9 | 0.9 | | 3.39897E-05 |

FIGURE 1. Approximate solution for $f(x, y)$



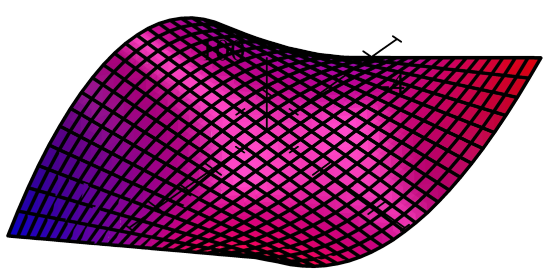
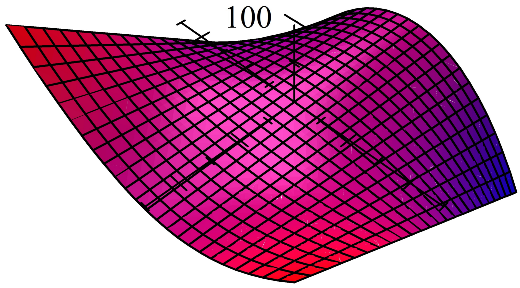


FIGURE 2. The Exact solution $f(x, y) = xy^2 - x - y^2$

FIGURE 3. Approximate solution for $g(x, y)$



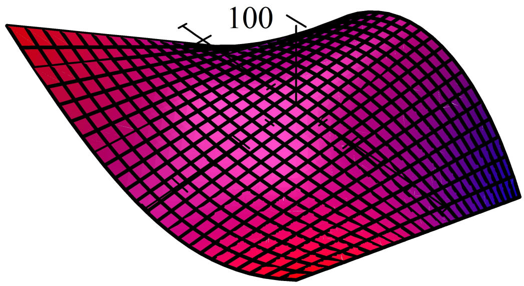




FIGURE 4. The exact solution $g(x, y) = x^2y - x^2 - y$

## 5. Conclusion

In this work, we have presented a numerical method combined with Legendre wavelets with their operational matrices of integration and derivation in order to approximate numerical solutions of the Poisson partial differential equation in $\mathbb{C}$. The performance of the method applied for the given equation is effective and has a remarkable and impressive level. The method determines the mentioned solutions in an efficient way that is quickly and accurately. The results of the illustrated examples are in agreement with the results of the proposed method which is a very powerful to find approximate solutions as well as numerical solutions.

In the future, we intend to investigate another type of nonlinear partial differential equations in a more complex domain by using the same method.

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Received: March 28, 2022; Revised: July 26, 2022; Accepted: August 21, 2022