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ANALYTICAL SOLUTION OF GENERALIZED DIFFUSION-LIKE EQUATION OF FRACTIONAL ORDER

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Abstract. In this paper, we consider a non-linear fractional diffusion-like equation. The existence and uniqueness of the solution of this type of equation are investigated. After that we use the homotopy perturbation method (HPM) to solve this equation by approximating a nonlinear function in a Taylor's series form and obtain an approximate solution.

1. Introduction

Nonlinear diffusion equations belong to the class of parabolic equations, which come from natural phenomena which are appearing widely in nature. Many problems that occurs in scientific phenomena like mass transport, heat conduction, conductivity etc. come with nonlinear diffusion equations. A considerable amount of work has been done on nonlinear diffusion equations [2, 3, 4, 26, 21, 1, 14, 25, 7]. Fujita considered the following nonlinear PDE

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial}{\partial x}\left(f(u)\frac{\partial u(x,t)}{\partial x}\right)$$

where u is the concentration of the diffusion at any point (x,t), $f(u)$ is the diffusion term, t is the time, and x is the space coordinate whose origin is set on the medium's surface. Fujita [2, 3, 4] studied the cases $f(u) = \frac{1}{1-\lambda u}, \frac{1}{(1-\lambda u)^2}, \frac{1}{1+2au+bu^2}$ respectively, where λ, a, b are arbitrary constants. Recently, $f(u) = u^n,\ n < 0$ and $n > 0$ are the most studied forms, called fast and slow diffusion process [26, 21, 1, 14]. Some other forms of the nonlinear diffusion equations are also considered in [26, 21, 1, 14].

In this work, we consider the following nonlinear fractional diffusion-like equation :

$$D_t^{\beta} u(x,t) = \frac{\partial}{\partial x}\left(f(u)\frac{\partial u(x,t)}{\partial x}\right), \tag{1.1}$$

with initial condition

$$u(x,0) = \omega(x). \tag{1.2}$$

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where D_t^{β} represents the Caputo fractional derivative of order β (see [17], [15]), $0 < \beta \leq 1$, $(x,t) \in B \times [0,T]$, $f(u)$ represents a class of functions that can be approximated by a polynomial and $\omega(x)$ is a known function. This kind of approach has been recently taken in [16, 22, 23].

The organization of the paper is the following

(a) Section 1: Basic definitions and preliminaries.
(b) Section 2: Existence and uniqueness of the solution of fractional nonlinear diffusion-like equations.
(c) Section 3: Derivation of the solution by Homotopy Perturbation Method.
(d) Section 4: Illustrative examples

Definition 1.1. Let $m \in \mathbb{N}, G \subset \mathbb{R}^m, [a,b] \subset \mathbb{R}$ and $f : [a,b] \times G \longrightarrow \mathbb{R}$ be a function such that $(y_1, y_2, \ldots, y_m), (\tilde{y_1}, \tilde{y_2}, \ldots, \tilde{y_m}) \in G$, then f is said to satisfy the generalized Lipschitz condition if

$$|f(x, y_1, y_2, \ldots, y_m) - f(x, \tilde{y_1}, \tilde{y_2}, \ldots, \tilde{y_m})| \leq L_1|y_1 - \tilde{y_1}| + \ldots + L_m|y_m - \tilde{y_m}|,$$

for $L_i \geq 0, i = 1, 2, \ldots, m$.

Definition 1.2. [15] Consider a mapping $T : X \longrightarrow X$ in normed space $(X, \|.\|)$. A point $x \in X$ is called a fixed point of T if $Tx = x$.

Theorem 1.1. [15] *[Banach Fixed Point Theorem] Suppose* $(X, \|.\|)$ *is nonempty Banach space. Define a map* $T : X \longrightarrow X$ *such that, for every* $x_1, x_2 \in X$, *the relation* $\|Tx_1 - Tx_2\| \leq \lambda\|x_1 - x_2\|$ *holds for* $0 \leq \lambda < 1$. *The map* T *is called a contraction mapping and* T *has a unique fixed point in* X.

The objective of this work is to study a nonlinear fractional diffusion-like equation, analyzing the conditions to guarantee the existence and uniqueness of the solutions and solve it, using a known homotopy perturbation method (HPM)(for more details, refer to the work by J.H. He in [8, 9, 10, 11, 12]), which allows us to find an approximate solution in series form.

2. Existence and Uniqueness of Generalized diffusion-like Equation of Fractional Order

Unlike the case of Ordinary Differential Equations, the problem of existence and uniqueness of the solutions for partial differential equations is much more recent (we recommend to the reader the works [18, 19, 20], for a better understanding of the subject). We consider the Banach space $C(\bar{B} \times [0,T])$ of real-valued continuous functions with the norm given by $\|u\| = \max_{(x,t)\in\bar{B}\times[0,T]} |u(x,t)|$.

Lemma 2.1. [24] *If* $w(\vec{y}, \tau)$ *and its partial derivatives are continuous on* $\bar{B} \times [0,T]$, *then* $\mathrm{D}_{y_i}^{\mu} w(\vec{y}, \tau)$, $\mathrm{D}_{y_i}^{2\mu} w(\vec{y}, \tau)$ *are bounded for* $i = 1, 2, ...n$.

The existence and uniqueness of the solution of equation (1.1) has been established by using the Banach fixed point theorem and Lemma 2.1(see [24]).

Theorem 2.1. *Let us assume that the function F defined by*

$$F(u,u',u'',f(u),f'(u)) = \frac{\partial}{\partial x}\left(f(u)\frac{\partial u(x,t)}{\partial x}\right),$$

satisfies the Lipschitz condition. That is,

$$\begin{aligned}&|F(u,u',u'',f(u),f'(u)) - F(v,v',v'',f(u),f'(u))| \le M_1|u-v|\\ &+ M_2|u'-v'| + M_3|u''-v''| + M_4|f(u)-f(u)| + M_5|f'(u)-f'(u)|.\end{aligned}$$

Further, assume that for $k_1, k_2, k_3, k_4 \ge 0$,

$$\begin{aligned}|u'-v'| &\le k_1|u-v|,\\ |u''-v''| &\le k_2|u-v|,\\ |f(u)-f(u)| &\le k_3|u-v|,\\ |f'(u)-f'(u)| &\le k_4|u-v|.\end{aligned}$$

Then, there exists a unique solution to the time fractional partial differential system (1.1)- (1.2) *if $MT^\beta < \Gamma(\beta+1)$, where $M = M_1 + M_2k_1 + M_3k_2 + M_4k_3 + M_5k_4$.*

Proof. Let us write $\Phi(u) = F(u,u',u'',f(u),f'(u))$, then using the assumptions, we have

$$\begin{aligned}|\Phi(u)-\Phi(u)| &= |F(u,u',u'',f(u),f'(u)) - F(v,v',v'',f(u),f'(u))|\\ &\le (M_1 + M_2k_1 + M_3k_2 + M_4k_3 + M_5k_4)|u-v|\\ &= M|u-v|,\end{aligned}$$

where $M = M_1 + M_2k_1 + M_3k_2 + M_4k_3 + M_5k_4$.
We can rewrite the differential system (1.1)- (1.2) as

$$D_t^\beta u(x,t) = \Phi(u(x,t)), \quad u(x,0) = w(x)$$

which is equivalent to the time fractional partial integro-differential equation

$$u(x,t) = w(x) + \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}\Phi(u(x,t))ds.$$

Now, we define the operator A on the Banach space $C(\bar{B}\times[0,T])$ as

$$Au(x,t) = w(x) + \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}\Phi(u(x,t))ds,$$

then $Au(x,t) = u(x,t)$. Now, if we show that A is a contraction, then , it follows from Banach Fixed Point Theorem that the differential system (1.1)- (1.2) has a unique solution $u(x,t)$.

Consider,

$$\begin{aligned}
&|Au(x,t) - Av(x,t)| \\
&\leq \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} |\Phi(u(x,t)) - \Phi(v(x,t))| ds \\
&\leq \frac{M}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} |u(x,t) - v(x,t)| ds \\
&\leq \frac{M}{\Gamma(\beta)} \|u-v\| \int_0^t (t-s)^{\beta-1} ds \\
&= \frac{MT^\beta}{\Gamma(\beta+1)} \|u-v\| \\
&< \|u-v\|,
\end{aligned}$$

where we have used the assumption that $MT^\beta < \Gamma(\beta+1)$. Finally, it implies that

$$\|Au - Av\| < \|u-v\|.$$

3. Derivation of the Solution by HPM

As mentioned before, we consider those functions f that can be approximated by a finite degree polynomial. For instance, the continuous function on a closed and bounded subset of real line can be approximated by a polynomial. This gives an algebraic expression for f that helps us to implement the HPM on the equation (1.1) in a convenient way. Let us assume that we can approximate the function f by a suitable nth degree polynomial P.

$$f(u) \approx P(u) = c_0 + c_1 u + c_2 u^2 + \ldots + c_n u^n, \tag{3.1}$$

where $c_0, c_1, \ldots, c_n$ are constants. Substituting (3.1) in the differential equation (1.1), we have

$$D_t^\beta u(x,t) = \frac{\partial}{\partial x}\left(P(u)\frac{\partial u(x,t)}{\partial x}\right), \tag{3.2}$$

with initial condition

$$u(x,0) = \omega(x). \tag{3.3}$$

We expand the right side of equation (3.2) and rewrite it as

$$D_t^\beta u(x,t) = P'(u)\left(\frac{\partial u}{\partial x}\right)^2 + P(u)\frac{\partial^2 u}{\partial x^2}. \tag{3.4}$$

For $p \in [0,1]$, we construct the homotopy $u(x,t;p)$ such that it is a solution of the following differential equation

$$\begin{aligned}
0 &= (1-p)\left[D_t^\beta u(x,t;p) - D_t^\beta u_0(x,t)\right] \\
&\quad + p\left[D_t^\beta u(x,t;p) - P'(u)\left(\frac{\partial u(x,t;p)}{\partial x}\right)^2 - P(u(x,t;p))\frac{\partial^2 u(x,t;p)}{\partial x^2}\right].
\end{aligned}$$

The above equation gives

$$\begin{aligned} D_t^{\beta} u &= D_t^{\beta} u_0 - p\left[D_t^{\beta} u_0 - P'(u)\left(\frac{\partial v}{\partial x}\right)^2 - P(u)\frac{\partial^2 v}{\partial x^2}\right] \\ &= D_t^{\beta} u_0 - p\left[D_t^{\beta} u_0 - \left(c_1 + 2c_2 u + \ldots + nc_n u^{n-1}\right) u_x^2 \right. \\ &\quad \left. - \left(c_0 + c_1 u + c_2 u^2 + \ldots + c_n u^n\right) u_{xx}\right]. \end{aligned} \tag{3.5}$$

Let us assume that

$$u(x,t;p) = \sum_{k=0}^{\infty} p^k u_k(x,t). \tag{3.6}$$

Substituting (3.6) in the differential equation (3.5)

$$\begin{aligned} D_t^{\beta} \sum_{k=0}^{\infty} p^k u_k &= D_t^{\beta} u_0 - p\left[D_t^{\beta} u_0 - c_1 \sum_{k=0}^{\infty} p^k \sum_{k_1+k_2=k} u_{k_{1x}} u_{k_{2x}} \right. \\ &- 2c_2 \sum_{k=0}^{\infty} p^k \sum_{k_1+k_2+k_3=k} u_{k_1} u_{k_{2x}} u_{k_{3x}} - 3c_3 \sum_{k=0}^{\infty} p^k \sum_{k_1+k_2+k_3+k_4=k} \\ & u_{k_{1x}} u_{k_{2x}} u_{k_3} u_{k_4} - \ldots - nc_n \sum_{k=0}^{\infty} p^k \sum_{k_1+k_2+k_3+\ldots+k_{n+1}=k} u_{k_{1x}} u_{k_{2x}} u_{k_3} \ldots u_{k_{n+1}} \\ &- c_0 \sum_{k=0}^{\infty} p^k u_{kxx} - c_1 \sum_{k=0}^{\infty} p^k \sum_{k_1+k_2=k} u_{k_1} u_{k_{2xx}} - c_2 \sum_{k=0}^{\infty} p^k \sum_{k_1+k_2+k_3=k} u_{k_1} u_{k_2} \\ & \left. u_{k_{3xx}} - \ldots - c_n \sum_{k=0}^{\infty} p^k \sum_{k_1+k_2+k_3+\ldots+k_{n+1}=k} u_{k_1} u_{k_2} \ldots u_{k_n} u_{k_{n+1}xx}\right]. \end{aligned}$$

Now, comparing like powers of p, we get the following set of differential equations.

$$\begin{aligned} p^0 &: D_t^{\beta} u_0 = D_t^{\beta} u_0, \\ p^1 &: D_t^{\beta} u_1 = -\left[D_t^{\beta} u_0 - c_1 u_{0x}^2 - 2c_2 u_0 u_{0x}^2 - 3c_3 u_0^2 u_{0x}^2 - \ldots \right. \\ &\quad \left. - nc_n u_0^{n-1} u_{0x}^2 - c_0 u_{0xx} - c_1 u_0 u_{0xx} - c_2 u_0^2 u_{0xx} - \ldots - c_n u_0^n u_{0xx}\ \right], \\ p^2 &: D_t^{\beta} u_2 = 2c_1 u_{0x} u_{1x} + 2c_2[2u_0 u_{0x} u_{1x} + u_{0x}^2 u_1] + 3c_3[2u_0^2 u_{0x} u_{1x} \\ &\quad + 2u_0 u_1 u_{0x}^2] + \ldots + nc_n\left[(n-1) u_1 u_0^{n-2} u_{0x}^2 + 2u_0^{n-1} u_{1x} u_{0x}\right] \\ &\quad + c_0 u_{1xx} + c_1(u_0 u_{1xx} + u_1 u_{0xx}) + c_2[u_0^2 u_{1xx} + 2u_1 u_0 u_{0xx}] \\ &\quad + \ldots + c_n\left[u_0^n u_{1xx} + n u_1 u_0^{n-1} u_{0xx}\right], \end{aligned}$$

and so on. Applying the Riemann-Liouville integral (see [17, 15]) to both sides of each equation above, we get $u_0, u_1, u_2, \ldots$. Hence the approximate solution is given by $u(x,t) = \lim_{p\to 1} v(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + \ldots$.

4. Illustrative Examples

In this section, we present some examples, which illustrate the strength and scope of the results obtained.

Example 4.1. For a general polynomial of degree two $f(u) = c_0 + c_1 u + c_2 u^2$, consider the following nonlinear diffusion-like equation

$$D_t^{\beta} u(x,t) = \frac{\partial}{\partial x}\left(f(u)\frac{\partial u(x,t)}{\partial x}\right), \quad u(x,0) = \omega(x).$$

Applying the method described in section 3, we get

$$\begin{aligned}
p^0 &: u_0(x,t) = u_0(x,t) = \omega(x),\\
p^1 &: u_1(x,t) = \mathcal{J}_t^{\beta}[c_1 u_{0x}^2 + 2c_2 u_0 u_{0x}^2 + c_0 u_{0xx} + c_1 u_0 u_{0xx}\\
&\quad + c_2 u_0^2 u_{0xx}],\\
p^2 &: u_2(x,t) = \mathcal{J}_t^{\beta}[2c_1 u_{0x} u_{1x} + 2c_2(2u_0 u_{0x} u_{1x} + u_{0x}^2 u_1) + c_0 u_{1xx}\\
&\quad + c_1(u_0 u_{1xx} + u_1 u_{0xx}) + c_2(u_0^2 u_{1xx} + 2u_1 u_0 u_{0xx})],
\end{aligned}$$

and so on. Thus, the approximate solution is given by $u(x,t) = u_0(x,t)+u_1(x,t)+u_2(x,t)+....$

In particular,
Case 1. If we take $c_0 = 1, c_1 = c_2 = 0$ in above example and $u(x,0) = \sin \pi x$, we get the following iterations.

$$\begin{aligned}
u_0(x,t) &= u(x,0) = \sin \pi x,\\
u_1(x,t) &= -\pi^2 \sin \pi x \; \frac{t^{\beta}}{\Gamma(\beta+1)},\\
u_2(x,t) &= \pi^4 \sin \pi x \; \frac{t^{2\beta}}{\Gamma(2\beta+1)}.
\end{aligned}$$

It can be observed that

$$u_n(x,t) = (-\pi^2)^n \sin \pi x \frac{t^{n\beta}}{\Gamma(n\beta+1)}.$$

Therefore,

$$\begin{aligned}
u(x,t) &= u_0(x,t) + u_1(x,t) + u_2(x,t) + ...\\
&= \sin \pi x[1 - \frac{\pi^2 t^{\beta}}{\Gamma(\beta+1)} + \frac{\pi^4 t^{2\beta}}{\Gamma(2\beta+1)} + ...].
\end{aligned}$$

For $\beta = 1$, we get $v(x,t) = \sin \pi x(e^{-\pi^2 t})$ which is the same result as obtained by [25].

Case 2 (Slow Diffusion). For $f(u) = u^2$, and $u(x,0) = \frac{x+h}{2\sqrt{c}}$, where $c > 0, h$ are arbitrary constants,

$$
\begin{aligned}
u_0(x,t) &= u_0(x,t) = \frac{x+h}{2\sqrt{c}},\\
u_1(x,t) &= (\frac{x+h}{4c\sqrt{c}})(\frac{t^\beta}{\Gamma(\beta+1)}),\\
u_2(x,t) &= (\frac{x+h}{8c^2\sqrt{c}})(\frac{3t^{2\beta}}{\Gamma(2\beta+1)}),
\end{aligned}
$$

and so on. Thus, the approximate solution is given by

$$u(x,t) = \frac{x+h}{2\sqrt{c}} + (\frac{x+h}{4c\sqrt{c}})(\frac{t^\beta}{\Gamma(\beta+1)}) + (\frac{x+h}{8c^2\sqrt{c}})(\frac{3t^{2\beta}}{\Gamma(2\beta+1)}) + ...$$

For $\beta = 1$, we have $u(x,t) = \frac{x+h}{2\sqrt{c}} + \frac{(x+h)t}{4c\sqrt{c}} + \frac{3(x+h)t^2}{16c^2\sqrt{c}} + ...$ which is the Maclaurin's formula of the exact solution (see [26]), $u(x,t) = \frac{x+h}{2\sqrt{c-t}}$.

Example 4.2. Consider the following nonlinear diffusion equation

$$D_t^\beta u = \frac{\partial}{\partial x}\left(\frac{1}{1+u^2}\frac{\partial u}{\partial x}\right), \quad u(x,0) = \tan x.$$

Since $\frac{1}{1+u^2} = 1 - u^2 + u^4 - u^6 + \ldots + (-1)^n u^{2n} + \ldots$. We consider $P(u) = 1 - u^2 + u^4 - u^6 + \ldots + (-1)^n u^{2n}$. Here $c_{2n} = (-1)^n$, and $c_{2n+1} = 0$ for $n = 0, 1, 2, \ldots$. Applying the method described in section 3,

$$
\begin{aligned}
&p^0 : u_0(x,t) = u_0(x,t) = \tan x,\\
&p^1 : u_1(x,t) = \frac{2t^\beta \tan x}{\Gamma(\beta+1)}[(1 + \tan^{4n-2} x)\\
&\quad - (1 - (-\tan^2 x)^{n-2}) - n(-1)^{n+1}\sec^2 x \tan^{2n} x],\\
&p^2 : u_2(x,t) = \big(4\sec^2 x[-\tan x + ... + n(-1)^n \tan^{2n-1} x]u_{1x}\\
&\quad + 2\sec^4 x[-1 + ... + n(2n-1)(-1)^n \tan^{2n-2} x]u_1\\
&\quad - 4\tan^2 x \sec^2 x[1 - 2\tan^2 x + ... + n(-1)^{n-1}\tan^{2n-2} x]u_1\\
&\quad +[1 - \tan^2 x + ... + (-1)^n \tan^{2n} x]u_{1xx}\big)\,\frac{t^\beta}{\Gamma(\beta+1)}
\end{aligned}
$$

Simplifying the last expression, we get the approximate solution as the sum $u(x,t) = u_0 + u_1 + u_2 + \ldots$.

Remark 4.1. In this example, as n tends to ∞, we get $u_n = 0, n = 1, 2, 3, \ldots$ and $u(x,t) = \tan x$ which is same as obtained by [21] for $\beta = 1$.

Example 4.3. Consider the following nonlinear diffusion equation

$$D_t^\beta u(x,t) = \frac{\partial}{\partial x}\left(e^u \frac{\partial u(x,t)}{\partial x}\right), \quad u(x,0) = x.$$

We consider the Maclaurin polynomial of order n generated by $f(u) = e^u$, which converges to the function e^u as n tends to infinity.

$$e^u \approx P(u) = 1 + u + \frac{u^2}{2!} + \frac{u^3}{3!} + \ldots + \frac{u^n}{n!}.$$

then, $c_n = \frac{1}{n!}$, $n = 0, 1, 2, \ldots$. Applying the HPM method described in section 3,

$$\begin{aligned}
&p^0 : u_0(x,t) = u_0(x,t) = x,\\
&p^1 : u_1(x,t) = \mathcal{J}_t^{\beta}\left[1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + ... + \frac{x^{n-1}}{(n-1)!}\right]\\
&\quad = \left[1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + ... + \frac{x^{n-1}}{(n-1)!}\right]\frac{t^{\beta}}{\Gamma(\beta+1)},\\
&p^2 : u_2(x,t) = \mathcal{J}_t^{\beta}\left[u_1\left(1 + x + \ldots + \frac{x^{n-2}}{(n-2)!}\right)\right.\\
&\quad \left. +2u_{1x}\left(1 + x + \ldots + \frac{x^{n-1}}{(n-1)!}\right) + u_{1xx}\left(1 + x + \ldots + \frac{x^n}{n!}\right)\right]\\
&\quad = \left[\left(1 + x + \frac{x^2}{2!} + \ldots + \frac{x^{n-1}}{(n-1)!}\right)\left(1 + x + \ldots + \frac{x^{n-2}}{(n-2)!}\right)\right.\\
&\quad + 2\left(1 + x + \frac{x^2}{2!} + \ldots + \frac{x^{n-2}}{(n-2)!}\right)\left(1 + x + \ldots + \frac{x^{n-1}}{(n-1)!}\right)\\
&\quad \left. + \left(1 + x + \frac{x^2}{2!} + \ldots + \frac{x^{n-3}}{(n-3)!}\right)\left(1 + x + \ldots + \frac{x^n}{n!}\right)\right]\frac{t^{2\beta}}{\Gamma(2\beta+1)},
\end{aligned}$$

and so on. Thus, the approximate solution is given by $u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + ...$

5. Conclusion

In this paper, we have introduced a nonlinear fractional diffusion-like equation which consist a nonlinear term $f(u)$. We have assumed that $f(u)$ can be approximated by a suitable finite degree polynomial. The conditions are presented which guarantee the existence and uniqueness of the solution of the nonlinear fractional differential system. The well known homotopy perturbation method has been applied to obtain an approximate analytical solution. With the help of detailed illustrative examples, the applicability of the method have been discussed. The first two examples generalises well-known examples and supports the strength of the given results. The third example has been presented for the first time, as per the author's knowledge, and illustrates that the concept can be applied to a general nonlinear fractional diffusion equation with a nonlinear term $f(u)$.

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