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MANDELBROT FRACTALS USING FIXED-POINT TECHNIQUE OF SINE FUNCTION

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Abstract. Here, we develop escape criteria for $p_c(z) = \sin(z^n) - az + c$, $a, c \in \mathbb{C}$, $n \geq 2$, exploiting four different iterations of fixed point theory to explore various Mandelbrot sets which are different than the classical Mandelbrot set . Our concern is to utilize the lesser number of iterations that are necessary to attain the fixed point of the transcendental complex-valued sine function. Further, we investigate the effect of variables on the shape, size, color, and dynamics of fractals. Noticeably, some of the obtained fractals symbolize the Swastika (a symbol of spirituality and divinity in Indian religions), Shivling (an abstract representation of the Hindu God Shiva), flowers, spiders, butterflies, Rangoli (made mainly in the festive season in India), art on glass, and so on. Interestingly, the higher-order Mandelbrot set in Picard-orbit has a resemblance to Corona-virus.

1. Introduction

The Mandelbrot set emerged in complex dynamics and was examined initially by the French mathematicians Pierre Joseph Louis Fatou [10] and Gaston Maurice Julia [12] at the outset of the twentieth century. However, Brooks and Matelski [4] formally described and sketched it in 1978 while studying Kleinian groups and Benoit B. Mandelbrot [7] first viewed its picture on 1 March 1980 and investigated the parameter space of quadratic polynomials. On the other hand, Douady and Hubbard [8] determined the elemental characteristics of this set and named it Mandelbrot set to honor Benoit B. Mandelbrot due to his instrumental work in fractal geometry which became famous as a computer graphics promo as soon as computers turn out to be self-sufficient to display and plot the set in high resolution. Recently, a lot of research have been done to investigate the usefulness applications of fractals in, physics ([5, 3, 24]), biology ([21, 14]), cryptography ([18, 22]), electrical and electronics engineering ([13, 6]) and so on. Recently, Antal et al. [2] and Tomar et al. [23] studied more general complex-valued polynomials of higher orders of the type $z^n + az^2 - bz + c$ and $z^n + az^2 + bz + c$ respectively by using celebrated fixed point iterations equipped with s-convexity.

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In the current work, motivated by Antal et al. [1], we investigate Mandelbrot set for $\sin(z^n) - az + c, a, c \in \mathbb{C}$, $n \geq 2$, employing four different iterations, Picard, Mann, Ishikawa, Özkan [19] and Noor, for theory of fixed points. For this, we first obtain the escape criteria for the underlying function and then apply it to explore the novel variants of Mandelbrot sets. Further, we developed an algorithm and colormap for the generation of visually pleasing fractals with the help of Matlab R2015a (8.5.0). Towards the end, we compare the generated fractals in four distinct orbits to study the effect of distinct iterations on fascinating new variants for different values of involved parameters. It is worth mentioning here that 1 is an essential singularity, and the collection of escaping points is non-empty for each transcendental function (Ereneko [9]).

2. Preliminaries

The Mandelbrot set is the collection of complex numbers c so that the function $p_c(z) = z^2 + c$ does not diverge whenever iterated from point 0. Its pictures may be sketched by selecting some of the complex numbers c and then examining, if the sequence $p_c(0), p_c(p_c(0)), \ldots$ approaches to infinity. If we vary the initial value of z, considering c to be constant, we obtain the Julia set. The connected locus of a family of polynomials may also be described as the Mandelbrot set which was thought to be disconnected at first. When the analogous Julia set is connected, a point is contained in the Mandelbrot set. Mathematically, the Mandelbrot set M ([15, 7]) is described as

$$M = \{c \in \mathbb{C} : Fp_c \text{ is connected}\}.$$

Noticeably, it is the set of all complex-valued parameters so that the filled Julia set Fp_c of $p_c(z) = z^2 + c$ is connected. Equivalently

$$M = \big\{c \in \mathbb{C} : \{|p_c(z_k)| \nrightarrow \infty \text{ as } k \to \infty\big\}.$$

Definition 2.1. Let $p : \mathbb{C} \to \mathbb{C}$ be a complex-valued self-map. Then

(i) the Picard-iteration [20] is

$$z_{k+1} = p(z_k).$$

(ii) the Mann-iteration [16] is

$$z_{k+1} = (1 - \alpha)z_k + \alpha p(z_k).$$

(iii) the Ishikawa-iteration [11] is

$$\begin{aligned} z_{k+1} &= (1-\alpha)z_k + \alpha p(y_k), \\ y_k &= (1-\beta)z_k + \beta p(z_k). \end{aligned}$$

(iv) the Noor-iteration [17] is

$$\begin{aligned} z_{k+1} &= (1-\alpha)z_k + \alpha p(y_k), \\ y_k &= (1-\beta)z_k + \beta p(x_k), \\ x_k &= (1-\gamma)z_k + \gamma p(z_k). \end{aligned}$$

Here, $k = 0, 1, 2, \ldots, z_0 \in \mathbb{C}$, and $\alpha, \beta, \gamma \in (0, 1]$.

Remark 2.1. The Noor orbit (NO) reduces to:

(1) Picard orbit (PO) for $\alpha = 1, \beta = \gamma = 0$.

(2) Maan orbit (MO) for $\beta = \gamma = 0$.
(3) Ishikawa orbit (IO) for $\gamma = 0$,

We know that $|\sin(z^n)| \leq 1, \forall\ z \in \mathbb{C}$. Also for $x, y, z \in \mathbb{C}$

$$\begin{aligned}
|\sin(z^n)| &= \left|z^n - \frac{z^{3n}}{3!} + \frac{z^{5n}}{5!} - \dots\right| \\
&= |z^n|\left|1 - \frac{z^{2n}}{3!} + \frac{z^{4n}}{5!} - \dots\right|, \\
|\sin(y^n)| &= \left|y^n - \frac{y^{3n}}{3!} + \frac{y^{5n}}{5!} - \dots\right| \\
&= |y^n|\left|1 - \frac{y^{2n}}{3!} + \frac{y^{4n}}{5!} - \dots\right|,
\end{aligned}$$

and

$$\begin{aligned}
|\sin(x^n)| &= \left|x^n - \frac{x^{3n}}{3!} + \frac{x^{5n}}{5!} - \dots\right| \\
&= |x^n|\left|1 - \frac{x^{2n}}{3!} + \frac{x^{4n}}{5!} - \dots\right|,
\end{aligned}$$

Consider $x_0 = x, y_0 = y, z_0 = z, p(z) = p_c(z)$ and

(i) $\left|1 - \frac{z^{2n}}{3!} + \frac{z^{4n}}{5!} - \dots\right| \geq |\omega_1|$,
(ii) $\left|1 - \frac{y^{2n}}{3!} + \frac{y^{4n}}{5!} - \dots\right| \geq |\omega_2|$,
(iii) $\left|1 - \frac{x^{2n}}{3!} + \frac{z^{4n}}{5!} - \dots\right| \geq |\omega_3|$,

where $|\omega_1|, |\omega_2|, |\omega_3| \in (0, 1]$ apart from those values of the variables x, y, and z so that $|\omega_1| = |\omega_2| = |\omega_3| = 0$.

3. Escape Criteria

We set up escape criterion for sine function

$$p(z) = \sin(z^n) - az + c, n \geq 2, \tag{3.1}$$

where z is a complex variable whereas a and c are complex numbers.

Theorem 3.1. Suppose $|z| \geq |c| > \left(\frac{2(1+|a|)}{\alpha|\omega_1|}\right)^{\frac{1}{n-1}}$, $|z| \geq |c| > \left(\frac{2(1+|a|)}{\beta|\omega_2|}\right)^{\frac{1}{n-1}}$ and $|z| \geq |c| > \left(\frac{2(1+|a|)}{\gamma|\omega_3|}\right)^{\frac{1}{n-1}}$, where $|\omega_1|, |\omega_2|, |\omega_3| \in (0, 1]$ apart from the values of x, y, and z so that $|\omega_1| = |\omega_2| = |\omega_3| = 0$, $\alpha, \beta, \gamma \in (0, 1]$, and $p(z)$ is a complex-valued sine function (3.1). If $\{z_k\}$ is a sequence of Noor-iteration for $k \in \mathbb{N}$, then $|z_k| \to \infty$ whenever $k \to \infty$.

Proof. Let

$$|x_k| = |(1 - \gamma)z_k + \gamma p(z_k)|.$$

If $k = 0$,

$$\begin{aligned}
|x_0| &= |(1-\gamma)z_0 + \gamma p(z_0)| \\
&= |(1-\gamma)z + \gamma(\sin(z^n) - az + c)| \\
&\geq \gamma|\sin(z^n)| - \gamma|az| - \gamma|c| - |(1-\gamma)z| \\
&\geq \gamma|\sin(z^n)| - \gamma|a||z| - \gamma|z| - |z| + \gamma|z|, \ |z| \geq |c|, \\
&\geq \gamma|\omega_1||z^n| - |z||a| - |z|, \ \gamma \in (0,1], \\
&= \gamma|\omega_1||z^n| - |z|(1+|a|) \\
&= |z|(1+|a|)\Big(\frac{\gamma|\omega_1||z^{n-1}|}{1+|a|} - 1\Big).
\end{aligned}$$

Since $|\sin(z^n)| = \left|z^n - \frac{z^{3n}}{3!} + \frac{z^{5n}}{5!} - \dots\right| \geq |\omega_1||z^n|$, where z is a complex variable apart from the values of z so that $|\omega_1| = 0$, $|\omega_1| \in (0,1]$. Hence we attain

$$|x| \ \geq \ \frac{|x|}{1+|a|} = |z|\Big(\frac{\gamma|\omega_1||z^{n-1}|}{1+|a|} - 1\Big),$$

that is, $|x^n| > |z|^n\Big(\frac{\gamma|\omega_1||z^{n-1}|}{1+|a|} - 1\Big)^n \geq \gamma|\omega_1||z|^n$, since, $|z| > \Big(\frac{2(1+|a|)}{\gamma|\omega_1|}\Big)^{\frac{1}{n-1}}$.
For the next step of Noor-iteration

$$|y_k| = |(1-\beta)z_k + \beta p(x_k)|.$$

If $k = 0$,

$$\begin{aligned}
|y_0| &= |(1-\beta)z + \beta p(x)| \\
&= |(1-\beta)z + \beta(\sin(x^n) - ax + c)| \\
&\geq \beta|\sin(x^n)| - \beta|ax| - \beta|c| - |(1-\beta)z| \\
&\geq \beta|\sin(x^n)| - \beta|a||x| - \beta|z| - |z| + \beta|z|, \ |z| \geq |c|, \\
&\geq \beta|\omega_3||x^n| - \beta|x||a| - |z| \\
&\geq \beta\gamma|\omega_1||\omega_3||z^n| - \beta\gamma|\omega_1||z||a| - |z| \\
&= \beta\gamma|\omega_1||\omega_3||z^n| - |z||a| - |z|, \ \beta, \gamma, \omega_1 \in (0,1], \\
&= \beta\gamma|\omega_1||\omega_3||z^n| - |z|(1+|a|) \\
&= |z|(1+|a|)\Big(\frac{\beta\gamma|\omega_1||\omega_3||z^{n-1}|}{1+|a|} - 1\Big).
\end{aligned}$$

Since $|\sin(x^n)| = \left|x^n - \frac{x^{3n}}{3!} + \frac{x^{5n}}{5!} - \dots\right| \geq |\omega_3||x^n|$, where z is a complex variable apart from the values of z so that $|\omega_3| = 0$, $|\omega_3| \in (0,1]$. Also, $|x| \geq \gamma|w_1||z|$. Hence, we get

$$|y| \ \geq \ \frac{|y|}{1+|a|} = |z|\Big(\frac{\beta\gamma|\omega_1||\omega_3||z^{n-1}|}{1+|a|} - 1\Big),$$

that is $|y^n| > |z|^n\Big(\frac{\beta\gamma|\omega_1||\omega_3||z^{n-1}|}{1+|a|} - 1\Big)^n \geq \beta\gamma|\omega_1||\omega_3||z|^n$, since, $|y| > \Big(\frac{2(1+|a|)}{\beta|\omega_2|}\Big)^{\frac{1}{n-1}}$.
Now for the next step of Noor-iteration

$$|z_{k+1}| = |(1-\alpha)z_k + \alpha p(y_k)|.$$

Again if

$$\begin{aligned}
|z_1| &= |(1-\alpha)z + \alpha p(y)| \\
&= |(1-\alpha)z + \alpha(\sin(y^n) - ay + c)| \\
&\geq \alpha|\sin(y^n)| - \alpha|ay| - \alpha|c| - |(1-\alpha)z| \\
&\geq \alpha|\sin(y^n)| - \alpha|a||y| - \alpha|z| - |z| + \alpha|z|,\ |z| \geq |c|, \\
&\geq \alpha|\omega_2||y^n| - \alpha|y||a| - |z| \\
&\geq \alpha\beta\gamma|\omega_1||\omega_2||\omega_3||z^n| - \alpha\beta\gamma|\omega_1||\omega_3||z||a| - |z| \\
&= \alpha\beta\gamma|\omega_1||\omega_2||\omega_3||z^n| - |z||a| - |z|,\ \alpha,\beta,\gamma,\omega_1,\omega_3 \in (0,1], \\
&= \alpha\beta\gamma|\omega_1||\omega_2||\omega_3||z^n| - |z|(1+|a|) \\
&= |z|(1+|a|)\Big(\frac{\alpha\beta\gamma|\omega_1||\omega_2||\omega_3||z^{n-1}|}{1+|a|} - 1\Big).
\end{aligned}$$

Since $|\sin(y^n)| = \left|y^n - \frac{y^{3n}}{3!} + \frac{y^{5n}}{5!} - \dots\right| \geq |\omega_2||y^n|$, where z is a complex variable apart from the values of z so that $|\omega_2| = 0$, $|\omega_2| \in (0,1]$. Also, $|y| \geq \beta\gamma|\omega_1||\omega_3||z|$. Hence we attain

$$|z_1| \geq \frac{|z_1|}{1+|a|} = |z|\Big(\frac{\alpha\beta\gamma|\omega_1||\omega_2||\omega_3||z^{n-1}|}{1+|a|} - 1\Big).$$

Following the same procedure till k^{th} term, we attain

$$\begin{aligned}
|z_2| &\geq |z|\Big(\frac{\alpha\beta\gamma|\omega_1||\omega_2||\omega_3||z^{n-1}|}{1+|a|} - 1\Big)^2 \\
|z_3| &\geq |z|\Big(\frac{\alpha\beta\gamma|\omega_1||\omega_2||\omega_3||z^{n-1}|}{1+|a|} - 1\Big)^3 \\
|z_4| &\geq |z|\Big(\frac{\alpha\beta\gamma|\omega_1||\omega_2||\omega_3||z^{n-1}|}{1+|a|} - 1\Big)^4 \\
&\ \ \vdots \\
|z_k| &\geq |z|\Big(\frac{\alpha\beta\gamma|\omega_1||\omega_2||\omega_3||z^{n-1}|}{1+|a|} - 1\Big)^k.
\end{aligned}$$

Since $|z| \geq |c| > \left(\frac{2(1+|a|)}{\alpha|\omega_1|}\right)^{\frac{1}{n-1}}$, $|z| \geq |c| > \left(\frac{2(1+|a|)}{\beta|\omega_2|}\right)^{\frac{1}{n-1}}$ and $|z| \geq |c| > \left(\frac{2(1+|a|)}{\gamma|\omega_2|}\right)^{\frac{1}{n-1}}$ where $|\omega_1|, |\omega_2|, |\omega_3| \in (0,1]$, that is, $|z| \geq |c| > \left(\frac{2(1+|a|)}{\alpha\beta\gamma|\omega_1||\omega_2||\omega_3|}\right)^{\frac{1}{n-1}}$. Therefore $\frac{\alpha\beta\gamma|\omega_1||\omega_2||\omega_3||z^{n-1}|}{(1+|a|)} - 1 > 1$ and $|z_k| \to \infty$ when $k \to \infty$. □

Remark 3.1. It is interesting to observe that the conclusion of Theorem 3.1 remain unaltered even if we replace a by $-a$ in complex valued sine function "$\sin(z^n) + az + c, n \geq 2$" (see Antal et al. [1]). Hence, conclusions of all the theorems and corollary is also same for $\sin(z^n) - az + c, n \geq 2$ and consequently the escape criterion remain unaltered which are responsible for generating Mandelbrot sets (in the next section). If for some value of k, the underlying orbit $|z_k|$ lies outside the circle of radius $\max\Big\{|c|, \left(\frac{2(1+|a|)}{\alpha|\omega_1|}\right)^{\frac{1}{n-1}}, \left(\frac{2(1+|a|)}{\beta|\omega_2|}\right)^{\frac{1}{n-1}}, \left(\frac{2(1+|a|)}{\gamma|\omega_3|}\right)^{\frac{1}{n-1}}\Big\}$, then in that situation the orbit escapes. On the other hand, if $|z_k|$ lies inside the circle, then making use of the definition of a Mandelbrot set, we use algorithms to explore fractals.

4. Generation of Mandelbrot sets

Our purpose is to study the variation in the Mandelbrot set using different iterative procedures. We utilize the escape criteria developed in Section 3, to explore Mandelbrot sets in Picard, Maan, Ishikawa, and Noor orbit. In this study, when $\alpha \leq \beta \leq \gamma$, very beautiful Mandelbrot sets for the function $p(z) = \sin(z^n) - az + c$ (z is a complex variable and $a, c \in \mathbb{C}$, $n \geq 2$) are observed for Noor iterative process. Similarly, when $\alpha \leq \beta$, we obtain beautiful fractals for Ishikawa iterative process. It is fascinating to observe that as the method move from one step to three steps, more variants of amazing Mandelbrot fractals may be obtained with vibrant colors. Also a small change in the parameters $\alpha, \beta, \gamma, \omega_1, \omega_2$, and ω_3 give the major difference in the output of the respective methods. Some of the Mandelbrot sets generated in this paper are very interesting and found in surroundings. Throughout the paper, we use the standard 'jet' colormap (as shown in Figure 1).

FIGURE 1. Colormap used in the graphical examples.

Algorithm 4.1 Geometry of Mandelbrot-Set

Input: $f(z) = \sin(z^n) - az + c$, where $n \in \mathbb{N}$ and $a, c \in \mathbb{C}$; $A \subset \mathbb{C}-$ area; $K-$ a maximum number of iterations; $\alpha, \beta, \gamma \in (0,1)-$parameter of the Noor-iteration; $\omega_1, \omega_2, \omega_3 \in (0,1) - parameters\ colourmap[0..C-1]-$colour map with C colours.

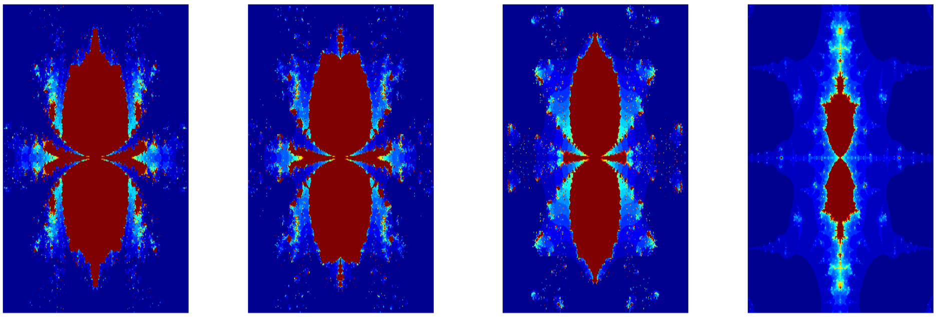
Output: Mandelbrot set for area A.

for $c \in A$ **do**
  $R_1 = \left(\frac{2(1+|a|)}{\alpha|\omega_1|}\right)^{\frac{1}{n-1}}$
  $R_2 = \left(\frac{2(1+|a|)}{\beta|\omega_2|}\right)^{\frac{1}{n-1}}$
  $R_3 = \left(\frac{2(1+|a|)}{\gamma|\omega_3|}\right)^{\frac{1}{n-1}}$
  $R = \max(|c|, R_1, R_2, R_3)$
  $n = 0$
  while $n \leq K$ **do**
    $T(z_n) = \sin(z^k) - az + c$
    $x_n = (1-\gamma)z_n + \gamma T(z_n)$
    $y_n = (1-\beta)z_n + \beta T(x_n)$
    $z_{n+1} = (1-\alpha)z_n + \alpha T(y_n)$
    if $|z_{n+1}| > R$ **then**
      break
    end if
    $n = n + 1$
  end while
  $i = (C-1)\frac{n}{K}$
  colour c with $colourmap[i]$
end for

The parameter a chosen in this experiment is a purely real number or a complex number. The fractals obtained via the Noor method are richer in their beauty as compared to Ishikawa, Maan, and Picard methods. However, some of the fractals obtained by Noor and Ishikawa do not have much difference in shape and size. The Picard method is proved to be very weak in getting Mandelbrot set with desired characteristics (shape, size, color, resolution). This leads to a topic of further research that why the Picard method fails to give a fractal as obtained by other methods in this paper.

The Figures 2 uses subsequent parameters:

	a	α	β	γ	ω_1	ω_2	ω_3	n
(a)	-1.19802032	0.17835	0.1675056	0.14323409	0.11025	0.115025	0.115025	3
(b)	-1.19802032	0.17835	0.1675056	–	0.11025	0.115025	–	3
(c)	-1.19802032	0.17835	–	–	0.11025	–	–	3
(d)	-1.19802032	–	–	–	0.11025	–	–	3

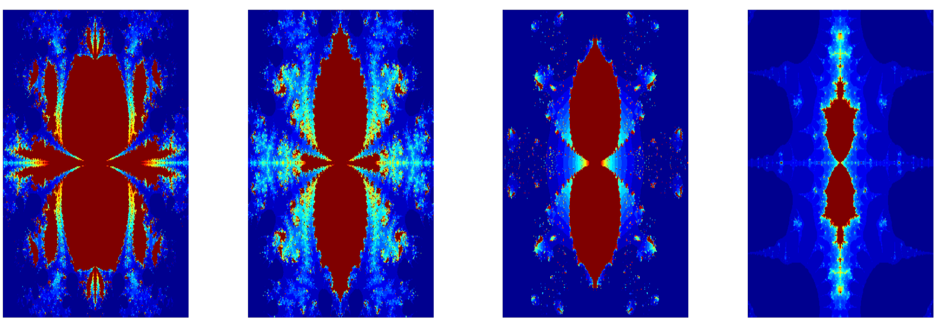


(a) (b) (c) (d)

FIGURE 2. (a) NO (b) IO (c) MO (d) PO (Cubic Mandelbrot set)

The Figures 3 uses subsequent parameters:

	a	α	β	γ	ω_1	ω_2	ω_3	n
(a)	-1.19802032	0.0077835	0.1675056	0.0814323409	0.07101025	0.079115025	0.078115025	3
(b)	-1.19802032	0.0077835	0.1675056	–	0.07101025	0.079115025	–	3
(c)	-1.19802032	0.0077835	–	–	0.07101025	–	–	3
(d)	-1.19802032	–	–	–	0.07101025	–	–	3

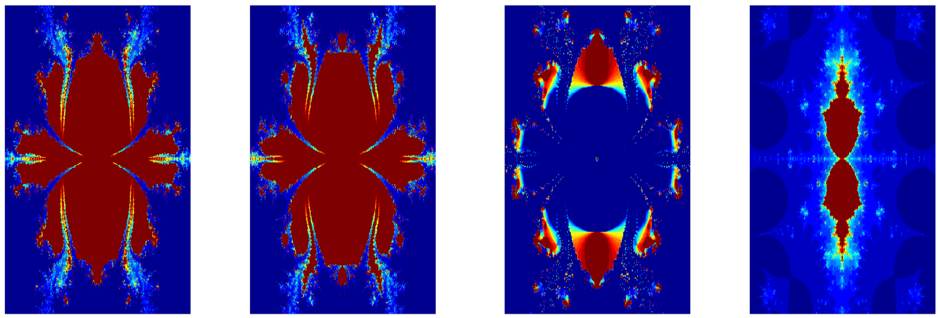


(a) (b) (c) (d)

FIGURE 3. (a) NO (b) IO (c) MO (d) PO (Cubic Mandelbrot set)

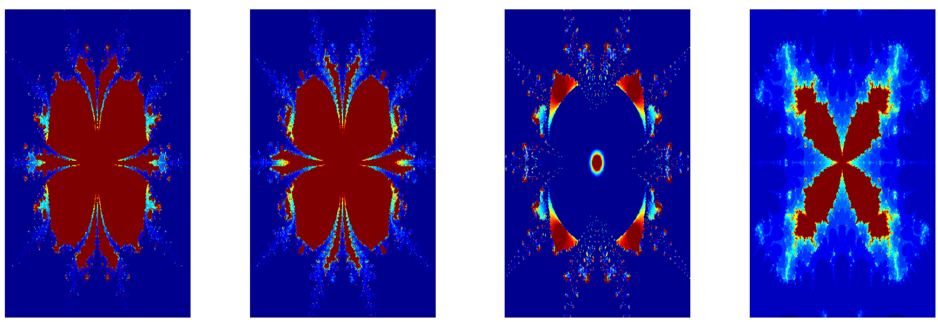
The Figures 4 - 6 uses subsequent parameters:

	a	α	β	γ	ω_1	ω_2	ω_3
(a)	-1.0229802032	0.077835	0.05675056	0.0814323409	0.000017101025	0.79115025	0.8115025
(b)	-1.0229802032	0.077835	0.05675056	–	0.000017101025	0.79115025	–
(c)	-1.0229802032	0.077835	–	–	0.000017101025	–	–
(d)	-1.0229802032	–	–	–	0.000017101025	–	–



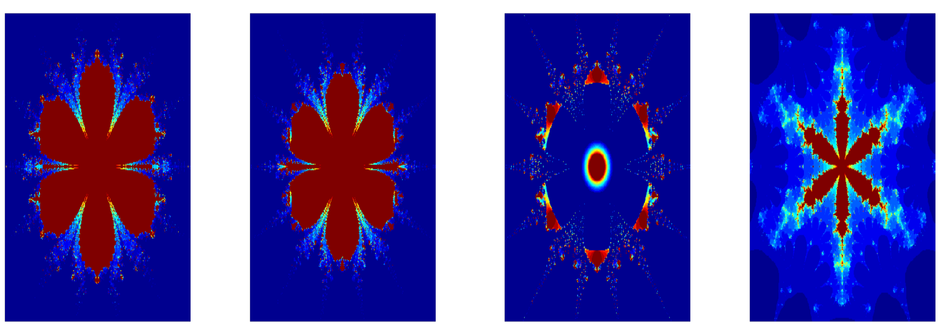
(a) (b) (c) (d)

FIGURE 4. (a) NO (b) IO (c) MO (d) PO (Cubic Mandelbrot set)



(a) (b) (c) (d)

FIGURE 5. (a) NO (b) IO (c) MO (d) PO (Quintic Mandelbrot set)

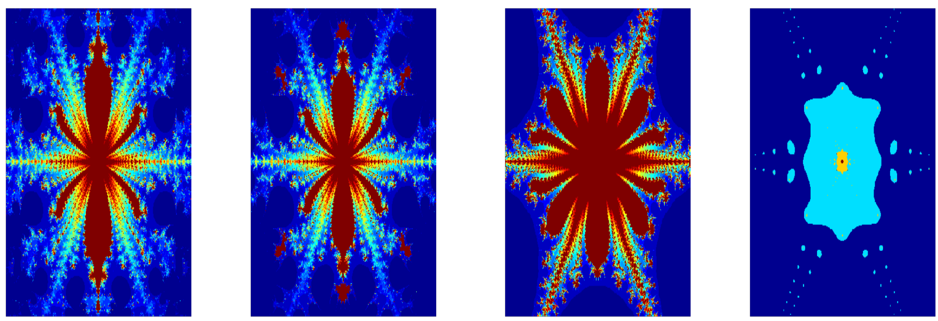


(a) (b) (c) (d)

FIGURE 6. (a) NO (b) IO (c) MO (d) PO (Septic Mandelbrot set)

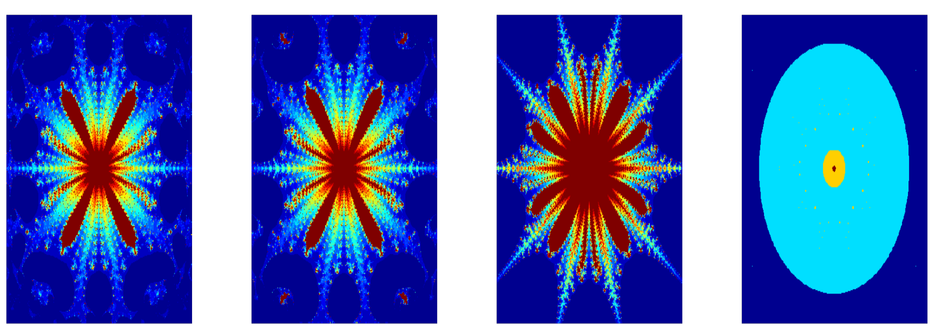
The Figures 7 - 11 uses subsequent parameters:

	a	α	β	γ	ω_1	ω_2	ω_3
(a)	-6.5	0.0097577835	0.11295675056	0.00975814323409	0.17101025	0.115025	0.8115025
(b)	-6.5	0.0097577835	0.11295675056	$-$	0.17101025	0.115025	$-$
(c)	-6.5	0.0097577835	$-$	$-$	0.17101025	$-$	$-$
(d)	-6.5	$-$	$-$	$-$	0.17101025	$-$	$-$



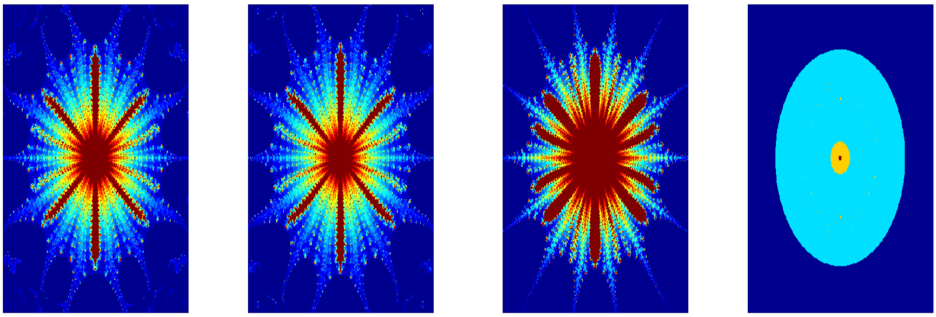
(a) (b) (c) (d)

FIGURE 7. (a) NO (b) IO (c) MO (d) PO (Cubic Mandelbrot set)



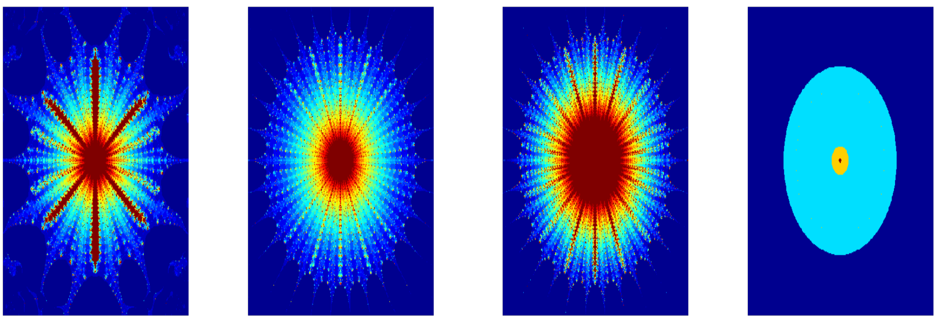
(a) (b) (c) (d)

FIGURE 8. (a) NO (b) IO (c) MO (d) PO (Quintic Mandelbrot set)



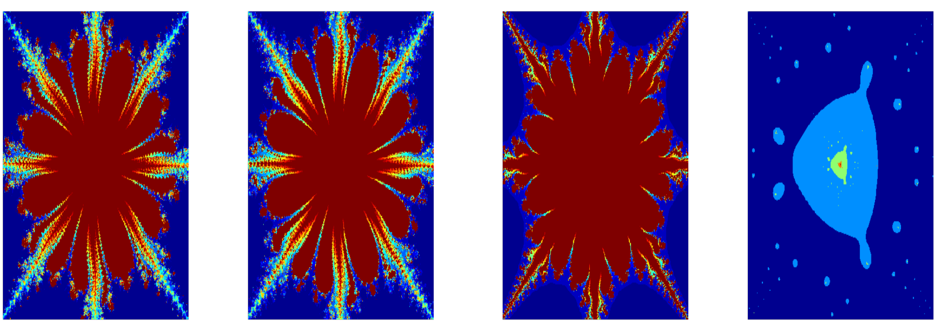
(a) (b) (c) (d)

FIGURE 9. (a) NO (b) IO (c) MO (d) PO (Septic Mandelbrot set)



(a) (b) (c) (d)

FIGURE 10. (a) NO (b) IO (c) MO (d) PO (Higher order Mandelbrot set)

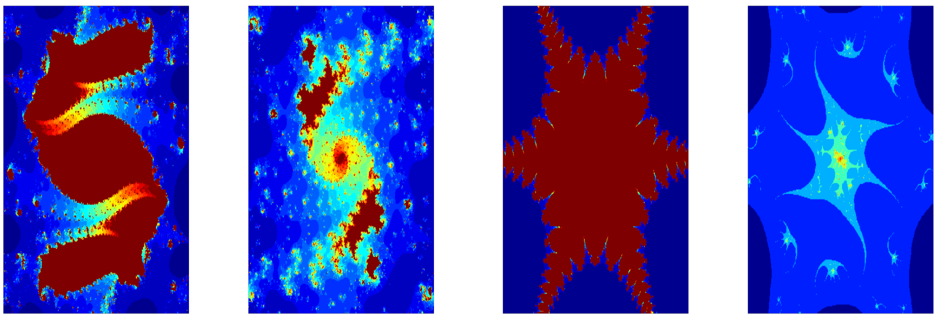


(a) (b) (c) (d)

FIGURE 11. (a) NO (b) IO (c) MO (d) PO (Quartic Mandelbrot set)

The parameters used in Figures 12 are as follows:

	a	α	β	γ	ω_1	ω_2	ω_3
(a)	$-1.594321-2.198i$	0.00097577835	0.211295675056	0.09463975814323409	0.0017101025	0.001015025	0.9115025
(b)	$-1.594321-2.198i$	0.00097577835	0.211295675056	$-$	0.0017101025	0.001015025	$-$
(c)	$-1.594321-2.198i$	0.00097577835	$-$	$-$	0.0017101025	$-$	$-$
(d)	$-1.594321-2.198i$	$-$	$-$	$-$	0.0017101025	$-$	$-$

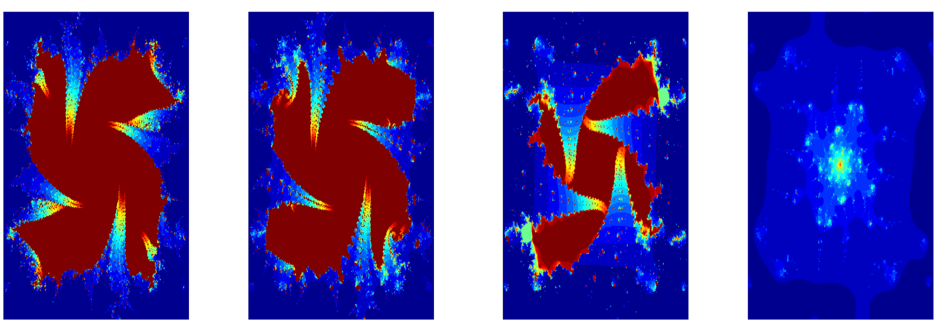


(a) (b) (c) (d)

FIGURE 12. (a) NO (b) IO (c) MO (d) PO (Cubic Mandelbrot set)

The parameters used in Figures 13 are as follows:

	a	α	β	γ	ω_1	ω_2	ω_3
(a)	$-1.0002-1.0002i$	0.27	0.0907383172	0.08509463975814323409	0.08197101025	0.982105015025	0.89115025
(b)	$-1.0002-1.0002i$	0.27	0.0907383172	$-$	0.08197101025	0.982105015025	$-$
(c)	$-1.0002-1.0002i$	0.27	$-$	$-$	0.08197101025	$-$	$-$
(d)	$-1.0002-1.0002i$	$-$	$-$	$-$	0.08197101025	$-$	$-$

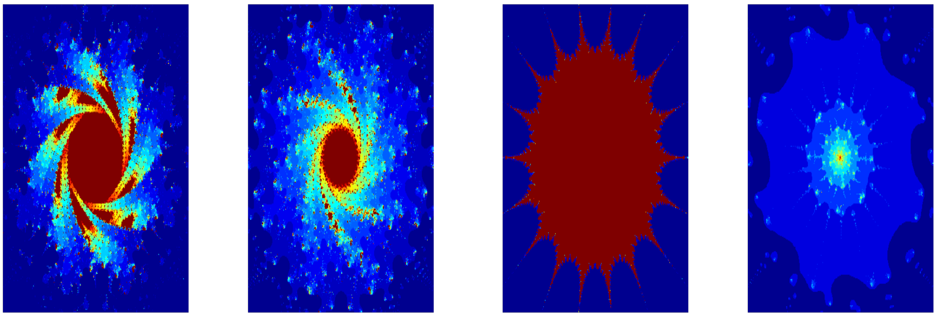


(a) (b) (c) (d)

FIGURE 13. (a) NO (b) IO (c) MO (d) PO (Quintic Mandelbrot set)

The parameters used in Figures 14 are as follows:

	a	α	β	γ	ω_1	ω_2	ω_3
(a)	$-1.1987654321 + 1.1987654321i$	0.0001	0.211295675056	0.109463975814323409	0.0017101025	0.001015025	0.99115025
(b)	$-1.1987654321 + 1.1987654321i$	0.0001	0.211295675056	–	0.0017101025	0.001015025	–
(c)	$-1.1987654321 + 1.1987654321i$	0.0001	–	–	0.0017101025	–	–
(d)	$-1.1987654321 + 1.1987654321i$	–	–	–	0.0017101025	–	–

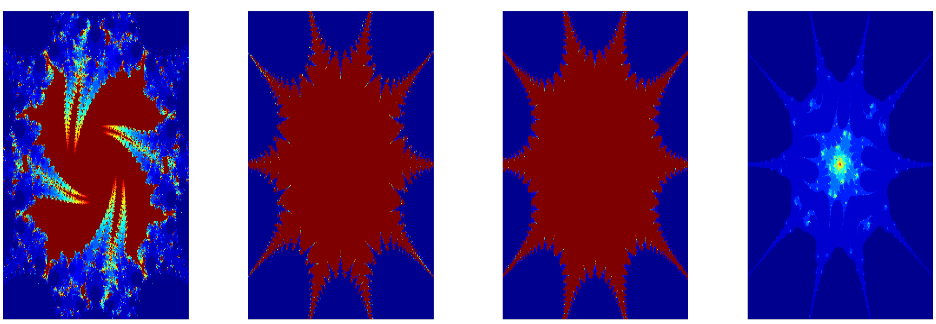


(a) (b) (c) (d)

FIGURE 14. (a) NO (b) IO (c) MO (d) PO (Nonic Mandelbrot set)

The parameters used in Figures 15 are as follows:

	a	α	β	γ	ω_1	ω_2	ω_3
(a)	$-1.00 - 1.1987654321i$	0.0001	0.0011295675056	0.109463975814323409	0.0017101025	0.001015025	0.99115025
(b)	$-1.00 - 1.1987654321i$	0.0001	0.0011295675056	0.001015025	0.0017101025	–	–
(c)	$-1.00 - 1.1987654321i$	0.0001	–	–	0.0017101025	–	–
(d)	$-1.00 - 1.1987654321i$	–	–	–	0.0017101025	–	–

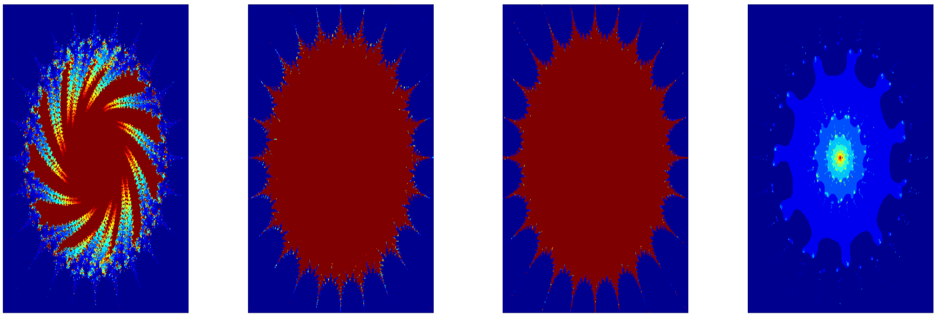


(a) (b) (c) (d)

FIGURE 15. (a) NO (b) IO (c) MO (d) PO (Quintic Mandelbrot set)

The parameters used in Figures 16 are as follows:

	a	α	β	γ	ω_1	ω_2	ω_3
(a)	$-1.200-1.1987654321i$	0.0001	0.0011295675056	0.05109463975814323409	0.0017101025	0.001015025	0.99115025
(b)	$-1.200-1.1987654321i$	0.0001	0.0011295675056	–	0.0017101025	0.001015025	–
(c)	$-1.200-1.1987654321i$	0.0001	–	–	0.0017101025	–	–
(d)	$-1.200-1.1987654321i$	–	–	–	0.0017101025	–	–

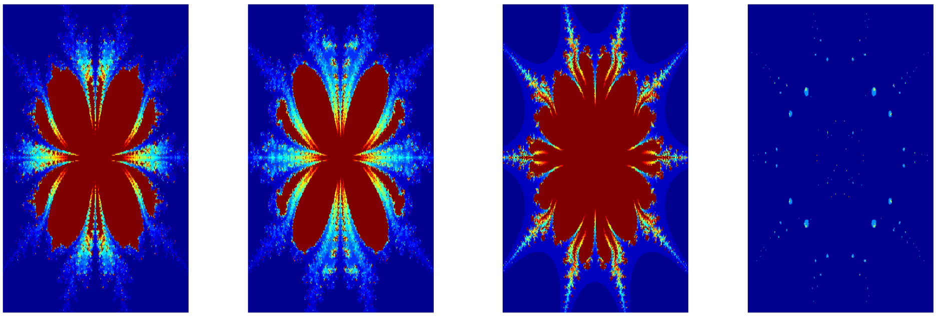


(a) (b) (c) (d)

FIGURE 16. (a) NO (b) IO (c) MO (d) PO (Higher order Mandelbrot set)

The parameters used in Figures 17 are as follows:

	a	α	β	γ	ω_1	ω_2	ω_3
(a)	-1.65	0.017	0.07383172	0.0509463975814323409	0.197101025	0.002105015025	0.09115025
(b)	-1.65	0.017	0.07383172	–	0.197101025	0.002105015025	–
(c)	-1.65	0.017	–	–	0.197101025	–	–
(d)	-1.65	–	–	–	0.197101025	–	–

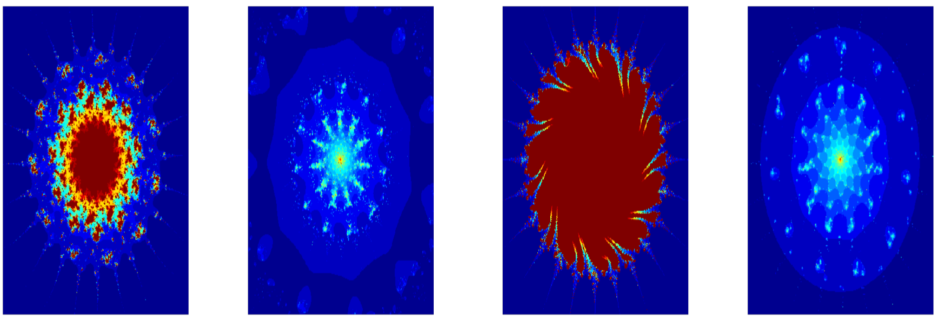


(a) (b) (c) (d)

FIGURE 17. (a) NO (b) IO (c) MO (d) PO (Quintic Mandelbrot set)

The parameters used in Figures 18 are as follows:

	a	α	β	γ	ω_1	ω_2	ω_3
(a)	$-1.0002 + 1.09i$	0.017	0.907383172	0.4509463975814323409	0.8197101025	0.982105015025	0.89115025
(b)	$-1.0002 + 1.09i$	0.017	0.907383172	–	0.8197101025	0.982105015025	–
(c)	$-1.0002 + 1.09i$	0.017	–	–	0.8197101025	–	–
(d)	$-1.0002 + 1.09i$	–	–	–	0.8197101025	–	–

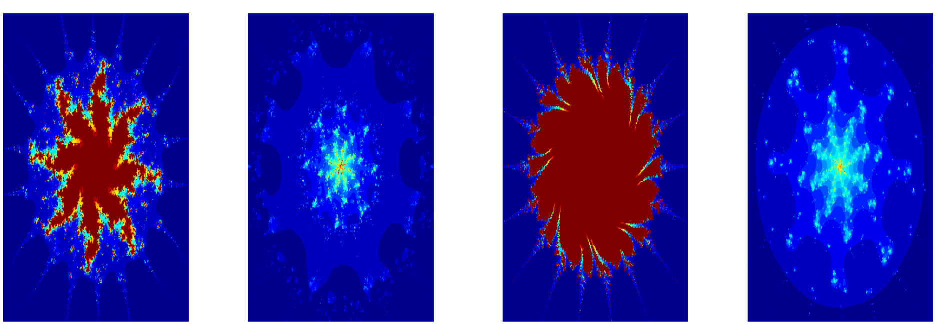


(a) (b) (c) (d)

FIGURE 18. (a) NO (b) IO (c) MO (d) PO (Higher order Mandelbrot set)

The parameters used in Figures 19 are as follows:

	a	α	β	γ	ω_1	ω_2	ω_3
(a)	$-1.0002 + 1.09i$	0.017	0.907383172	0.4509463975814323409	0.8197101025	0.982105015025	0.89115025
(b)	$-1.0002 + 1.09i$	0.017	0.907383172	–	0.8197101025	0.982105015025	–
(c)	$-1.0002 + 1.09i$	0.017	–	–	0.8197101025	–	–
(d)	$-1.0002 + 1.09i$	–	–	–	0.8197101025	–	–

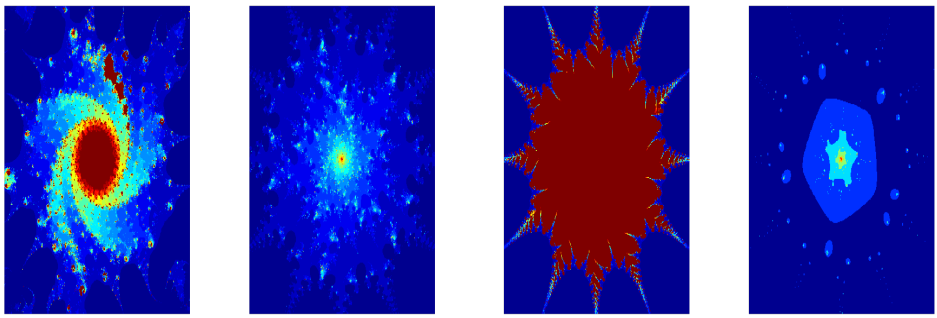


(a) (b) (c) (d)

FIGURE 19. (a) NO (b) IO (c) MO (d) PO (Nonic Mandelbrot set)

The parameters used in Figures 20 are as follows:

	a	α	β	γ	ω_1	ω_2	ω_3
(a)	$-1.92+1.92i$	0.0027	0.27383172	0.08509463975814323409	0.08197101025	0.982105015025	0.89115025
(b)	$-1.92+1.92i$	0.0027	0.27383172	–	0.08197101025	0.982105015025	–
(c)	$-1.92+1.92i$	0.0027	–	–	0.08197101025	–	–
(d)	$-1.92+1.92i$	–	–	–	0.08197101025	–	–

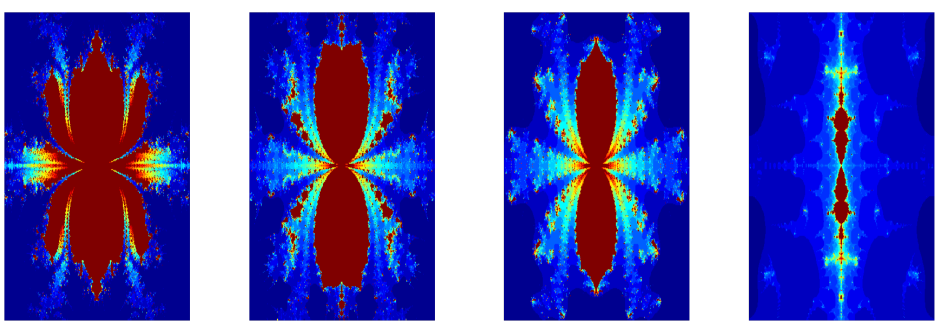


(a) (b) (c) (d)

FIGURE 20. (a) NO (b) IO (c) MO (d) PO (Sextic Mandelbrot set)

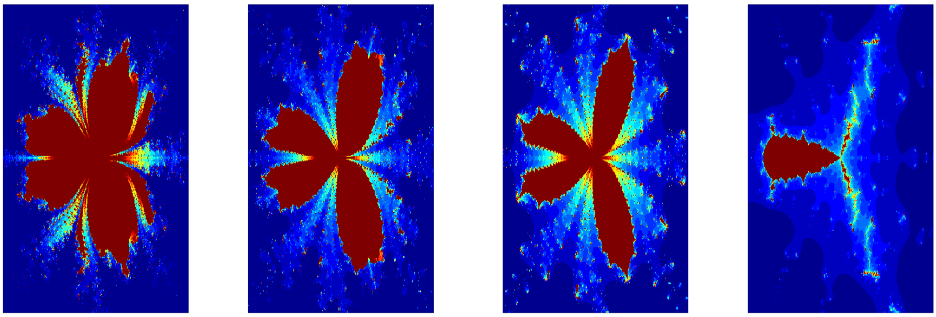
The Figures 21-25 uses subsequent parameters:

	a	α	β	γ	ω_1	ω_2	ω_3
(a)	-1.65	0.12397835	0.125675056	0.04323409	0.17101025	0.00179115025	0.009118115025
(b)	-1.65	0.12397835	0.125675056	–	0.17101025	0.00179115025	–
(c)	-1.65	0.12397835	–	–	0.17101025	–	–
(d)	-1.65	–	–	–	0.17101025	–	–



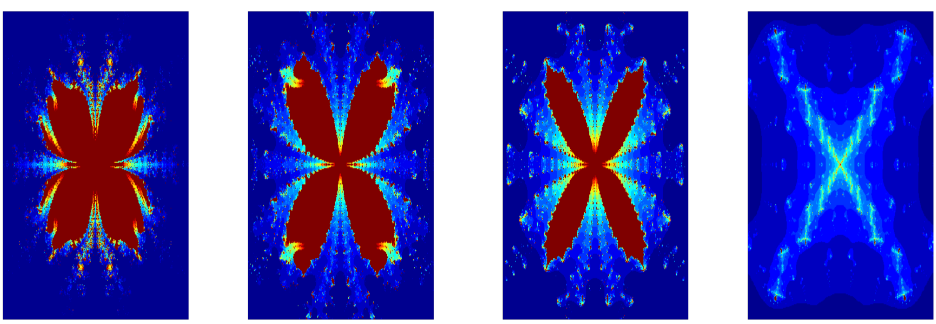
(a) (b) (c) (d)

FIGURE 21. (a) NO (b) IO (c) MO (d) PO (Quintic Mandelbrot set)



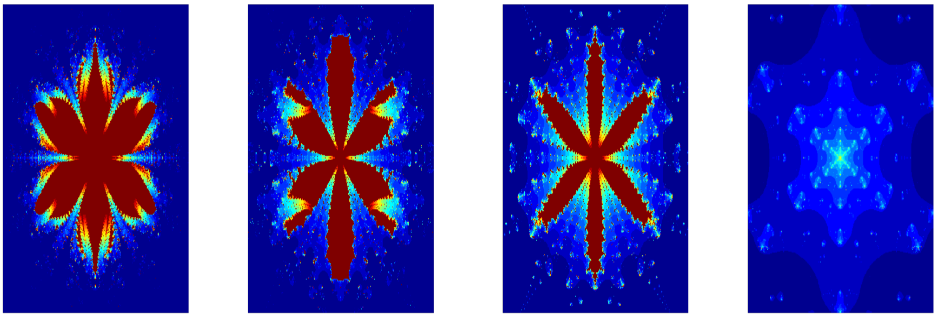
(a) (b) (c) (d)

FIGURE 22. (a) NO (b) IO (c) MO (d) PO (Quartic Mandelbrot set)



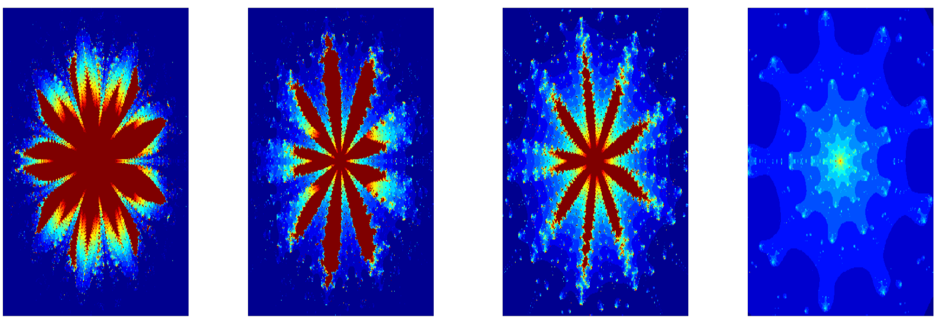
(a) (b) (c) (d)

FIGURE 23. (a) NO (b) IO (c) MO (d) PO (Quintic Mandelbrot set)



(a) (b) (c) (d)

FIGURE 24. (a) NO (b) IO (c) MO (d) PO (Septic Mandelbrot set)



(a) (b) (c) (d)

FIGURE 25. (a) NO (b) IO (c) MO (d) PO (Higher order Mandelbrot set)

Remark 4.1. Some observations which make the fractals eye-catching are

(a) All the Mandelbrot sets generated by the sine function are symmetrical about the x-axis.

(b) Mandelbrot sets have a resemblance with Spiders (see, Figures 2(a), (c), 3(b), (c), 20(c)), Starfish (see, Figure 6(c), 7 (c), 17(a)) with the large body which has both rotational as well as reflectional symmetry, well-decorated Shivling carrying a Trishul on the top (an abstract representation of Hindu God Shiva see, Figures 3(a), 4(a)-(b), 21(a)) with reflectional symmetry, a swirl or spinning wheel (see, Figure 12(b), 14(a)-(b), 16(a), 20(a)) used especially by children in Deepawali (famous Indian festival), Rangoli made during festive seasons in India (see Figure 12(c), 15(b)-(c), 17(c), 18(a), (c), 19(c), 20(c), 25(c)), Amoeboid (see, Figure 7(d)), Stars (see, Figures 11(c), 15(b)-(c)), Corona-virus (see, Figure 18(a), 25(d)), and so on.

(c) Surprisingly cubic, quintic, and septic Mandelbrot sets in Mann orbit are entirely different, and stunning having a ring in the center which is increasing in size with the increase in value of n (see Figures, 4(c), 5(c), and 6(c)).
(d) It may be seen (see, Figures 14(c), 15(c), 17(c), 19(c)) that nonic and quintic Mandelbrot sets in Mann orbit are also different having a circular shape and higher-order Mandelbrot sets are similar in shapes in Ishikawa and Mann orbits (see, 10(b)-(c), 15(b)-(c), 16(b)-(c)). Also nonic and higher-order Mandelbrot sets in Ishikawa and Picard orbit have a resemblance (Figures 18(b), (d), and 19(b), (d) are similar in shapes).
(e) Figures 2(b) and 21(b) are looking like Diya (lamp) which is symmetrical about the x-axis and the Figures 7(a)-(b), 8(a)-(b) are appear similar to butterflies. Further, Figures 9(a)-(c), 10(a)-(c) are looking like glass paintings, and Figures 17(d) and 20(b) are similar to the milky way.
(f) In Figures 13(a)-(c) and 15(a), all the fractals are similar to the Indian holy symbol Swastika. Also in Figure 12(a), an alphabet S is achieved. Figures 5(a)-(b), 6(a)-(b), 7(a)-(c), 8(a)-(c), 17(a) - (c), 21(c), 22(a)-(c), 23(a) - (c), 24(a) - 25(a) represent flowers like structures. Figures 6(a) and (b) have a resemblance to Catharanthus Roseus (Rose Periwinkle). Figure 24(a) is looking like Lotus, which is symmetrical about the x-axis.

Conclusion:

In this manuscript, we have used four, Picard, Mann, Ishikawa, and Noor iterations of fixed point theory to establish escape criteria for a sine function. We have explored new Mandelbrot sets which are dissimilar to the classical Mandelbrot set. We have observed that the size of fractals is governed by the parameters used in distinct iterations ($\alpha, \beta, \gamma \in (0, 1]$) while the symmetry and shape are governed by the parameter a, b, and c. Noticeably, some fractals are similar to the Swastika (a symbol of spirituality and divinity in Indian religions), Shivlinga, glass paintings, and Rangoli made in India, which find application in interior decoration. Some Mandelbrot sets have a resemblance to Corona-virus from which the whole world is suffering. Some represent beautiful flowers, Spiders, and butterflies found in nature.

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