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VARIABLE MUCKENHOUPT WEIGHTS WITH APPLICATIONS TO OPERATORS

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Abstract. Variable Muckenhoupt weights are considered in variable exponent Lebesgue spaces. Boundedness of averaging operator is proved. Almost all weighted norm inequalities are obtained using this transference result. Potential type approximate identities are considered. Translations of Steklov operator are proved to be bounded in these spaces.

1. Introduction

Classical Muckenhoupt's A_p class of weights have many important properties for investigations related to weighted norm inequalities corresponding to several operators. For example, among others, A_p weights are precisely the weights for which many singular integral and maximal operators are bounded in the appropriate weighted space. Properties of Muckenhoupt's weights can be found in books [19, 36], and paper [32]. Although Muckenhoupt's class A_p comprehensively studied by many mathematicians working on weight theory of operators, there is still need to wider class of weights for which characterization problems of (weighted) operators related to variable exponent function spaces. To overcome such type problems there has been appeared some variable Muckenhoupt's class $A_{p(\cdot)}$. As in the classical L_p case of A_p, there are two type of definition of weighted variable exponent Lebesgue spaces: (i) weights as multiplier and (ii) weights as a measure. Corresponding to these two approaches definition of variable Muckenhoupt's class $A_{p(\cdot)}$ changes suitably. Papers [13, 14] consider the case (i) and paper [16] considers approach (ii). Also, interestingly, there has been occurred (see [33]) a mixed type of (i) and (ii). All these three types of approaches have their own positive and negative circumstances. Variable Muckenhoupt's class $A_{p(\cdot)}$ of above used in many applications to solve several contemporary problems of weighted operator theory.

In this work, we consider approach (ii) and corresponding class of $A_{p(\cdot)}$. Main properties, inequalities or equivalences for weights in $A_{p(\cdot)}$ has been proved by Diening Hästö ([16]) in the case $1 < essinf_{x\in(-\infty,\infty)}p(x)$ and $esssup_{x\in(-\infty,\infty)}p(x) < \infty$. In this study we will consider the case $essinf_{x\in[-\pi,\pi]}p(x) \geq 1$ and $esssup_{x\in[-\pi,\pi]}p(x) < \infty$. Boundedness of averaging operator is proved to gain

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a transference result. Almost all weighted norm inequalities are obtained using this transference result. Potential type approximate identities are considered. Translations of Steklov operator are proved to be bounded in these spaces. K-functional is a good apparatus for measuring smoothness properties of functions given in these function classes.

2. Preliminary definitions

2.0.1. ***Local log-Hölder continuity condition.*** For measurable set $B \subseteq \mathrm{T} := [-\pi,\pi]$, let $\mathcal{P}(B)$ be the class of Lebesgue measurable functions $p = p(x) : B \to [1,\infty)$ such that $1 \leq p_B^- \leq p_B^+ < \infty$ where $p^-(B) := essinf_{x\in B} p(x)$ and $p^+(B) := esssup_{x\in B} p(x)$. We set $\mathcal{P} := \mathcal{P}(\mathrm{T})$, $p^- := p^-(\mathrm{T})$ and $p^+ := p^+(\mathrm{T})$.

Variable exponent $p(\cdot)$ is said to be satisfy *Local log-Hölder continuity condition* ([15]) on $B \subseteq \mathrm{T}$ if there exists a constant $c_{\log}(p) > 0$, dependent only on p, such that

$$|p(x) - p(y)| \leq \frac{c_{\log}(p)}{\ln(e + 1/|x-y|)} \text{ for all } x, y \in B. \tag{2.1}$$

We define $\mathcal{P}^{\log}(B) := \{p \in \mathcal{P}(B) : 1/p \text{ satisfy } (2.1)\}$ and set $\mathcal{P}^{\log} := \mathcal{P}^{\log}(\mathrm{T})$.

2.0.2. ***Weighted variable exponent Lebesgue space.*** Suppose that $B \subseteq \mathrm{T}$ is a measurable set and $\omega : B \to [0,\infty]$ is a 2π periodic weight function, i.e., ω is Lebesgue measurable and almost everywhere (a.e.) positive function on B. We define $\omega(B) := \int_B \omega(x)\,dx$. Let $B(x,r)$ is a ball in $\mathbb{R} := (-\infty,\infty)$ with center x and radius $r > 0$.

Definition 2.1. We define weighted variable exponent Lebesgue spaces ([16, 9]) as a collection $L_{2\pi,\omega}^{p(\cdot)}(B)$ of 2π periodic measurable functions $f : B \to \mathbb{R}$ satisfying

$$\|f\|_{B,p(\cdot),\omega} := \inf\left\{\alpha > 0 : \rho_{B,p(\cdot),\omega}\left(\frac{f}{\alpha}\right) \leq 1\right\} < \infty \tag{2.2}$$

where ω is a weight function on B, $p \in \mathcal{P}$, measurable $B \subseteq \mathrm{T}$ and

$$\rho_{B,p(\cdot),\omega}(f) := \int_B |f(x)|^{p(x)}\,\omega(x)\,dx.$$

We set $L_{2\pi,\omega}^{p(\cdot)} := L_{2\pi,\omega}^{p(\cdot)}(\mathrm{T})$, $\rho_{p(\cdot),\omega}(f) := \rho_{\mathrm{T},p(\cdot),\omega}(f)$, $\|f\|_{p(\cdot),\omega} := \|f\|_{\mathrm{T},p(\cdot),\omega}$. $L_{2\pi,\omega}^{p(\cdot)}$ is a Banach space ([15, Theorem 3.2.7],[35]) with norm (2.2) when ω is a weight function on T and $p \in \mathcal{P}$. We denote $L_{2\pi,\omega}^{p(\cdot)}(B) = L^p(B)$ when $\omega \equiv 1$, $p(x) = p$ is a constant and measurable $B \subseteq \mathrm{T}$.

Definition 2.2. Let $p \in \mathcal{P}$, and ω be a weight on T. We define $p'(x) := p(x)/(p(x)-1)$ for $p(x) > 1$ and $p'(x) := \infty$ for $p(x) = 1$.

2.1. Variable Muckenhoupt weight. Given a measurable set $A \subset \mathbb{R}$, symbol $|A|$ will represent the Lebesgue measure of A. For a $p \in \mathcal{P}$, the class of weights ω satisfying the variable exponent Muckenhoupt condition (see Diening Hästö [16])

$$[\omega]_{A_{p(\cdot)}} := \sup_{B \subseteq \mathrm{T}} \frac{\|\omega\|_{L^1(B)}}{|B|^{p_B}} \left\|\frac{1}{\omega}\right\|_{B,(p'(\cdot)/p(\cdot))} < \infty \tag{2.3}$$

for any $B \subseteq$T, will be denoted by $A_{p(\cdot)} := A_{p(\cdot)}(\mathrm{T})$ where

$$(p_B)^{-1} := \frac{1}{|B|}\int\limits_B \frac{dx}{p(x)}.$$

When $p(x) = p$ =constant, class $A_{p(\cdot)}$ in (2.3) turns into the classical Muckenhoupt class A_p ([32]).

Note that different formulations and definitions of variable exponent Muckenhoupt weights are also known. See for example the papers [14], [13]) and [33] (local variable exponent Muckenhoupt class).

Formulation (2.3) is different from definitions given in [13, 14, 33] and the class $A_{p(\cdot)}$ is increasing in $p(\cdot)$. Namely,

$$A_1 \subset A_{p^-} \subset A_{p(\cdot)} \subset A_{p^+} \subset A_\infty \tag{2.4}$$

where $1 < p^- \leq p(\cdot) \leq p^+ < \infty$, $A_\infty := \cup_{p\geq 1} A_p$ and

$$\begin{aligned} \omega &\in A_1 \Leftrightarrow [\omega]_{A_1} := \sup_{B\subseteq \mathrm{T}} \frac{\omega(B)}{|B|} ess \sup_{x\in B} \frac{1}{\omega(x)} < \infty, \\ \omega &\in A_p \Leftrightarrow [\omega]_{A_p} := \sup_{B\subseteq \mathrm{T}} \frac{\omega(B)}{|B|^p}\left(\int_B (\omega(x))^{-\frac{1}{p-1}}\right)^{p-1} < \infty. \end{aligned}$$

We define dual weight as $\omega'(\cdot) := (\omega(\cdot))^{1-p'(\cdot)}$.

2.2. Averaging operator and transference result.

Definition 2.3. ([20, p.96]) Let $\mathbb{N} := \{1, 2, 3, \cdots\}$ be natural numbers and $\mathbb{N}_0 := \mathbb{N}\cup\{0\}$.

(a) A family Q of measurable sets $E \subset \mathbb{R}$ is called locally N-finite ($N \in \mathbb{N}$) if

$$\sum_{E\in Q} \chi_E(x) \leq N$$

almost everywhere in $\mathbb{R}$ where χ_U is the characteristic function of the set U.

(b) A family Q of open bounded sets $U \subset \mathbb{R}$ is locally 1-finite if and only if the sets $U \in Q$ are pairwise disjoint.

(c) Let $U \subset$T be a measurable set and

$$A_U f := \frac{1}{|U|}\int\limits_{U\cap \mathrm{T}} |f(t)|\, dt.$$

(d) For a family Q of open sets $U \subset$T we define averaging operator by

$$T_Q : L^1_{loc}(T) \to L^0(T),$$

$$T_Q f(x) := \sum_{U\in Q} \chi_{U\cap \mathrm{T}}(x) A_U f, \quad x \in \mathrm{T},$$

where $L^0(\mathrm{T})$ is the set of measurable functions on T.

Definition 2.4. (i) Throughout this paper symbol $\mathfrak{A} \lessapprox \mathcal{B}$ will mean that there exists a constant C depending only on unimportant parameters in question such that inequality $\mathfrak{A} \leq \mathrm{C}\mathcal{B}$ holds.

(ii) We will use symbol C for generic constants that does not depend on main parameters and changes with placements. By $\mathbb{S}_i$ ($i = 0, 1, \cdots, 13$) we denote specific constants that defined particularly in context. These constants $\mathbb{S}_i$ $(i=0,1,\cdots,13)$ are depend on main parameters of the problem. On the other hand there are some other specific constants c_{inc}, $c_{\log}(p)$, $[\omega]_{A_{p(\cdot)}}$, $\mathfrak{D}$, $\mathfrak{E}$, $\mathfrak{S}$ each of them defined in context particularly. We will give explicit constants in the proofs but these constants are not best constants.

Theorem 2.1. *If $p\in\mathcal{P}^{\log}$ and $\omega\in A_{p(\cdot)}$, then there exist positive constants depend only on p,ω, such that*

(i) $L_{2\pi,\omega}^{p(\cdot)}(B)\subset L^1(B)$ and

$$\|f\|_{B,1,1}\precsim\|f\|_{B,p(\cdot),\omega}$$

for any subset $B\subseteq$T with $(1/4)<|B|\leq 2$;

(ii) $L_{2\pi,\omega}^{p(\cdot)}\subset L^1(\mathrm{T})$ and

$$\|f\|_{1,1}\precsim\|f\|_{p(\cdot),\omega}.$$

When $\omega\equiv1$ Theorem 2.1 is well known. See for example [34, Theorem 2.1] of Sharapudinov. When $p(\cdot)$=p=constant and $\omega\in A_p$ Theorem 2.1 was established in Ky [30], Israfilov [21, Lemma 2], Kokilashvili Yildirir [28, In Proposition 3.3], and Berkson and Gillespie [10, Remark 2.12 p. 934]. In the case $p^->1$ see also Aydın [7, Proposition 2.1].

Theorem 2.2. *Suppose that $p\in\mathcal{P}^{\log}$, $\omega\in A_{p(\cdot)}$, and $f\in L_{2\pi,\omega}^{p(\cdot)}$. If Q is 1-finite family of open bounded subsets of $\mathbb{R}$ having Lebesgue measure 1, then, the averaging operator T_Q is uniformly bounded in $L_{2\pi,\omega}^{p(\cdot)}$, namely,*

$$\|T_Qf\|_{p(\cdot),\omega}\precsim\|f\|_{p(\cdot),\omega}$$

holds with a positive constant depending only on p,ω.

In case of $\omega\equiv1$ Theorem 2.2 is obtained by Diening Harjulehto Hästö Růžička [15] and by Harjulehto Hästö [20].

Lemma 2.1. *([8, p.352])If $p\in\mathcal{P}^{\log}$, $\omega\in A_{p(\cdot)}$, then, dual of $L_{2\pi,\omega}^{p(\cdot)}$ is $L_{2\pi,\omega'}^{p'(\cdot)}$.*

Definition 2.5. Let $p\in\mathcal{P}^{\log}$, $\omega\in A_{p(\cdot)}$, and $f\in L_{2\pi,\omega}^{p(\cdot)}$. For an $F\in L_{2\pi,\omega'}^{p'(\cdot)}\cap C^\infty$, $\|F\|_{p'(\cdot),\omega'}\leq1$, we define auxiliary function

$$\mathcal{U}_{f,F}(u):=\int_{\mathrm{T}}f(x+u)\,|F(x)|\,dx,\quad u\in\mathrm{T}. \tag{2.5}$$

Auxiliary function $\mathcal{U}_{f,F}(\cdot)$ of (2.5) will frequently be used in the rest of the work and we set $\mathbb{S}_0:=\|F\|_\infty$ for further references.

Theorem 2.3. *Let $p\in\mathcal{P}^{\log}$, $\omega\in A_{p(\cdot)}$, $F\in L_{2\pi,\omega'}^{p'(\cdot)}\cap C^\infty$, $\|F\|_{p'(\cdot),\omega'}\leq1$, and $f\in L_{2\pi,\omega}^{p(\cdot)}$. Then, the function $\mathcal{U}_{f,F}(u)$, $u\in$T, defined in (2.5) is a uniformly continuous function on* T.

Theorem 2.4. *Let $p\in\mathcal{P}^{\log}$, $\omega\in A_{p(\cdot)}$ and $f,g\in L_{2\pi,\omega}^{p(\cdot)}$. If*

$$\|\mathcal{U}_{f,F}\|_{C(\mathrm{T})}\precsim\|\mathcal{U}_{g,F}\|_{C(\mathrm{T})},$$

with an absolute positive constant, then, weighted norm inequality

$$\|f\|_{p(\cdot),\omega}\precsim\|g\|_{p(\cdot),\omega}.$$

holds with a positive constant depend only on p, ω.

Theorem 2.4 is new also for $\omega \equiv 1$ and/or for $p(\cdot)=p=$constant and $\omega \in A_p$.

Definition 2.6. (a) Let $f \in L^{p(\cdot)}_{2\pi,\omega}$, $\lambda > 0$, $\tau \in \mathbb{R}$, $x \in$T and

$$\mathrm{S}_{\lambda,\tau}\, f(x) := \lambda \int_{x+\tau-1/(2\lambda)}^{x+\tau+1/(2\lambda)} f(t)\, dt,$$

(b) If $0 < h < \infty$ we define

$$T_h f(x) := \frac{1}{h} \int_x^{x+h} f(t)\, dt. \tag{2.6}$$

As a corollary of Theorem 2.4 we get the following result.

Theorem 2.5. *Suppose that* $p \in \mathcal{P}^{\log}$, $\omega \in A_{p(\cdot)}$, $f \in L^{p(\cdot)}_{2\pi,\omega}$, $\lambda > 0$, *and* $u \in$T. *Then, (a) the family of operators*

$$\{\mathrm{S}_{\lambda,u}\}_{\lambda>0, u\in \mathrm{T}}$$

is uniformly bounded (in λ *and* u*) in* $L^{p(\cdot)}_{2\pi,\omega}$*, namely,*

$$\|\mathrm{S}_{\lambda,u}\ f\|_{p(\cdot),\omega} \lessapprox \|f\|_{p(\cdot),\omega}$$

holds with a positive constant depend only on p, ω.

(b) If $0 < h < \infty$*, then, we get*

$$\|T_h f\|_{p(\cdot),\omega} \lessapprox \|f\|_{p(\cdot),\omega}\,.$$

Theorem 2.5(a) for $\lambda \geq 1$ and $|u| \leq \pi/\lambda^{\rho}$ $(\rho > 0)$ was obtained by Sharapudinov [34, Lemma 3.1] and for $\lambda \geq 1$ and $u \in \mathbb{R}$ was obtained by Łenski Szal [31, Lemma 2]. Also see papers [3, 5, 6, 17, 22, 23, 24, 25, 26, 27, 39] for similar expressions.

2.3. Approximate identities.

Definition 2.7. Let f and g be two real-valued $2\pi-$periodic measurable functions on T. We define the convolution $f * g$ of f and g by setting $(f * g)(x) = \int_{\mathrm{T}} f(y) g(x-y) dy$ for $x \in$T for which the integral exists in T.

Definition 2.8. ([11])Let B be a measurable set $B \subseteq$T, $\phi \in L^1(B)$ and $\int_B \phi(t)\, dt = 1$. For each $t > 0$ we define $\phi_t(x) = \frac{1}{t}\phi\left(\frac{x}{t}\right)$. Such a sequence $\{\phi_t\}$ will be called approximate identity. A function

$$\tilde{\phi}(x) = \sup_{|y| \geq |x|} |\phi(y)|$$

will be called radial majorant of ϕ. If $\tilde{\phi} \in L^1(B)$, then, sequence $\{\phi_t\}$ will be called potential-type approximate identity.

Using the same proof of of Corollary 4.6.6 of [15, p.130] we can obtain the following theorem.

Theorem 2.6. *(Corollary 4.6.6 of* [15, p.130]*) Suppose* $p \in \mathcal{P}^{\log}$*,* $\omega \in A_{p(\cdot)}$*,* $f \in L_{2\pi,\omega}^{p(\cdot)}$*, and* ϕ *is a potential-type approximate identity with radial majorant* $\tilde{\phi} \in L^1$*. Then, for any* $t > 0$*,*

$$\|f * \phi_t\|_{p(\cdot),\omega} \precapprox \left\|\tilde{\phi}\right\|_1 \|f\|_{p(\cdot),\omega}$$

and

$$\lim_{t \to 0} \|f * \phi_t - f\|_{p(\cdot)} = 0$$

hold with a positive constant depend only on p, ω*.*

Definition 2.9. (i) Let X be a Banach space on T. For $r \in \mathbb{N}$, we denote by W_X^r collection of functions $f \in X$ such that $f^{\,(r-1)}$ is absolutely continuous (AC) and $f^{\,(r)} \in X$. We define $\|f\|_{W_X^r} := \|f\|_X + \left\|f^{\,(r)}\right\|_X$.

(ii) In particular: (a) Let $C(\mathrm{T})$ be the collection of functions $f : \mathrm{T} \to \mathbb{R}$ such that f is continuous function on T. In the case $X = C(\mathrm{T})$ we set $W_X^r := W_{C(\mathrm{T})}^r$.
(b) In the case $p \in \mathcal{P}$, $\omega \in A_{p(\cdot)}$ and $X = L_{2\pi,\omega}^{p(\cdot)}$ we set $W_X^r := W_{p(\cdot),\omega}^r$. W_X^r becomes a Banach space with norm $\|f\|_{W_X^r}$ for $X = C(\mathrm{T})$ and for $X = L_{2\pi,\omega}^{p(\cdot)}$ where $p \in \mathcal{P}$, $\omega \in A_{p(\cdot)}$.

(c) For $r \in \mathbb{N}$, we define C^r consisting of every member $f \in C(\mathrm{T})$ such that the derivative $f^{(k)}$ exists and is continuous on T for $k = 1, ..., r$. We set $C^\infty := \{f \in C^r \text{ for any } r \in \mathbb{N}\}$.

Theorem 2.7. *Let* $p \in \mathcal{P}$*,* $\omega \in A_{p(\cdot)}$*, and* $F \in C^\infty$ *be as in (2.5).*

(a) If $r \in \mathbb{N}$ *and* $f \in W_{p(\cdot),\omega}^r$*, then,* $\mathcal{U}_{f,F} \in C^r$ *and* $(\mathcal{U}_{f,F})^{(r)} = \left(\mathcal{U}_{f^{(r)},F}\right)$.

(b) $\mathcal{U}_{g*h,F} = (\mathcal{U}_{g,F}) * h$ *for* $g, h \in L_{2\pi,1}^1$.

Definition 2.10. Let $0 < h < \infty$, $r \in \mathbb{N}$, $p \in \mathcal{P}^{\log}$, $\omega \in A_{p(\cdot)}$, $f \in L_{2\pi,\omega}^{p(\cdot)}$, $\delta \geq 0$.

(a) We define difference operator

$$\Delta_h^r f(\cdot) := (T_h - I)^r f(\cdot)$$

where $T_h f$ is from (2.6), and I is the identity operator.

(b) We define *modulus of smoothness of order* r as

$$\Omega_r(f,\delta)_{p(\cdot),\omega} := \|(I - T_\delta)^r f\|_{p(\cdot),\omega}; \quad \Omega_0(f,\delta)_{p(\cdot),\omega} := \|f\|_{p(\cdot),\omega}; \quad \delta > 0,$$
$$\Omega_r(f,0)_{p(\cdot),\omega} := 0 := \Omega_0(f,0)_{p(\cdot),\omega}.$$

Note that, there is another formulation of the modulus of smoothness concept. See [37].

The following proposition is immediate from Theorem 2.5.

Proposition 2.1. *If* $p \in \mathcal{P}^{\log}$*,* $\omega \in A_{p(\cdot)}$*,* $r \in \mathbb{N}$ *and* $f \in L_{2\pi,\omega}^{p(\cdot)}$*, then,*

(i) *there exists a positive constant depend only on* p, ω *such that*

$$\|(I - T_\delta)^r f\|_{p(\cdot),\omega} \precapprox \|f\|_{p(\cdot),\omega};$$

(ii) $\Omega_r(\cdot,\delta)_{p(\cdot),\omega}$ *is non-negative and non-decreasing function of* $\delta \geq 0$*;*

(iii) $\Omega_r(f_1 + f_2, \cdot)_{p(\cdot),\omega} \leq \Omega_r(f_1, \cdot)_{p(\cdot),\omega} + \Omega_r(f_2, \cdot)_{p(\cdot),\omega}$.

3. Auxiliary results

Here we will collect some auxiliary definions and results required for proofs.

Definition 3.1. We denote by $S(\mathrm{T})$ the collection of simple functions on T. We set $S_0(\mathrm{T}) := \{f \in S(\mathrm{T}) : f \text{ has a compact support in } \mathrm{T}\}$.

From Corollary 3.2.14 of [15, p.79], Remark 3.11 of [16, p.14] and proof of Lemma 6.7 of [16, p.23] we have the following proposition.

Proposition 3.1. *(Corollary 3.2.14 of* [15, p.79]*) Let* $p \in \mathcal{P}^{\log}$, $\omega \in A_{p(\cdot)}$. *Then*

$$\frac{1}{2}\|f\|_{p(\cdot),\omega} \leq \sup_{g \in L^{p'(\cdot)}_{2\pi,\omega'} : \|g\|_{p'(\cdot),\omega'} \leq 1} \int_{\mathrm{T}} |f(x)|\,|g(x)|\,dx \leq 2\|f\|_{p(\cdot),\omega} \tag{3.1}$$

holds for all $f \in L^{p(\cdot)}_{2\pi,\omega}$. *Furthermore, the supremum in (3.1) is unchanged if we replace the condition* $g \in L^{p'(\cdot)}_{2\pi,\omega'}$ *by* $g \in S(\mathrm{T})$ *or* $g \in S_0(\mathrm{T})$.

Using Theorem 2.2, Corollary 4.6.6 of [15, p.130] and Theorem 2.6 we have the following proposition.

Proposition 3.2. *Let* $p \in \mathcal{P}^{\log}$, $\omega \in A_{p(\cdot)}$. *Then*

$$\frac{1}{12\mathbb{S}_5}\|f\|_{p(\cdot),\omega} \leq \sup_{g \in L^{p'(\cdot)}_{2\pi,\omega'} \cap C^\infty : \|g\|_{p'(\cdot),\omega'} \leq 1} \int_{\mathrm{T}} |f(x)|\,|g(x)|\,dx \leq 2\|f\|_{p(\cdot),\omega}$$

holds for all $f \in L^{p(\cdot)}_{2\pi,\omega}$.

Convolution of Definition 2.7 exists for every $x \in \mathrm{T}$ and measurable function. Furthermore, in the classical Lebesgue spaces, there holds

$$\|f * g\|_{p,1} \leq \|f\|_{p,1}\|g\|_{1,1}.$$

If f is continuous (respectively absolutely continuous ($\equiv$AC)) then $f * g$ is continuous (respectively AC).

Lemma 3.1. *(*[34]*)Let B be measurable set $B \subseteq \mathrm{T}$ and ω be a weight function on B. For $p \in \mathcal{P}$, $1 \leq p(x) \leq q(x) \leq q_B^+ < \infty$ there holds*

$$\|f\|_{B,p(\cdot),\omega} \leq (\omega(B) + 1)\|f\|_{B,q(\cdot),\omega} \tag{3.2}$$

when the left hand side of (3.2) is finite.

Lemma 3.2. *(*[8, p.352, Theorem 3]*)For measurable set $B \subseteq \mathrm{T}$, $p \in \mathcal{P}^{\log}(B)$ and a weight function ω on B, the following Hölder's inequality*

$$\|fg\|_{B,1,1} \leq 2\|f\|_{B,p(\cdot),\omega}\|g\|_{B,p'(\cdot),\omega'}$$

holds for $f \in L^{p(\cdot)}_{2\pi,\omega}(B)$ *and* $g \in L^{p'(\cdot)}_{2\pi,\omega'}(B)$ *and* $1 = \frac{1}{p(\cdot)} + \frac{1}{p'(\cdot)}$.

From Lemma 3.1 of [16] we get the following theorem.

Theorem 3.1. *If* $p, q \in \mathcal{P}^{\log}$, $q^-, p^- > 1$, $q \leq p$ *and* $\omega \in A_{q(\cdot)}$, *then*

$$[\omega]_{A_{p(\cdot)}} \leq c_{inc}[\omega]_{A_{q(\cdot)}}$$

holds with

$$c_{inc} := \begin{cases} 16e^{9\left(c_{\log}p^{-1}+c_{\log}q^{-1}\right)} & ;\ p, q \text{ nonconstant,} \\ 16e^{9c_{\log}p^{-1}} & ;\ \text{only } q \text{ constant.} \end{cases}$$

Using the same proof of Proposition 4.33 of [12, p.152] we get the following result.

Theorem 3.2. *Let $1 \leq p < \infty$, ω be a weight on* T *and $f \in L^p_{2\pi,\omega}$. In this case,*

$$\int_{U\cap \mathrm{T}} \left[A_U(|f(x)|)\right]^p \omega(x)dx \leq [\omega]_{A_p} \int_{U\cap \mathrm{T}} |f(x)|^p \omega(x)dx \qquad \forall U \subset \mathrm{T}$$

if and only if $\omega \in A_p$.

The following result follows directly from (2.3).

Lemma 3.3. *If $p \in \mathcal{P}^{\log}$ and $\omega \in A_{p(\cdot)}$, then $\omega \in L^1(\mathrm{T})$.*

Lemma 3.4. *(*[29, Corollary of Theorem 4.1]*)Suppose that $p \in \mathcal{P}^{\log}$, $\omega \in A_{p(\cdot)}$, $f \in L^{p(\cdot)}_{2\pi,\omega}$. In this case the set C^∞ of infinitely continuously differentiable functions on* T, *is a dense subset of $L^{p(\cdot)}_{2\pi,\omega}$.*

Using the same proof of Lemma 3.4 of [2, p.15] we have the following theorem.

Theorem 3.3. *Let $p \in \mathcal{P}^{\log}$ and $\omega \in A_{p(\cdot)}$. If $m \in \mathbb{N}$ satisfies $m > 2^{p^+}[\omega]_{A_{p^+}}$, then*

$$\int_{\mathrm{T}} \frac{\omega(x)dx}{(e+|x|)^m} \leq \omega(B(0,1))\, 2^{2p^+} [\omega]^2_{A_{p^+}} \left(\frac{2^{p^+}[\omega]_{A_{p^+}}}{2^m - 2^{p^+}[\omega]_{A_{p^+}}} + 1 \right).$$

Theorem 3.4. *If $p \in \mathcal{P}^{\log}$, $\omega \in A_{p(\cdot)}$ and $f \in L^{p(\cdot)}_{2\pi,\omega}$, then,*

$$\lim_{\delta\to 0^+} \Omega_r(f,\delta)_{p(\cdot),\omega} = 0.$$

From Theorem 7.1 of [18, p:37,135] we have the following weak type inequality for Hardy Littlewood's maximal operator M.

Theorem 3.5. *(*[18, p:37,135]*)If $\omega \in A_1$, $\lambda > 0$ and $f \in L^1_{2\pi,\omega}$, then,there exists an absolute constant $\mathfrak{D}$ (corresponding to the one dimensional case) such that*

$$\|\lambda \chi_{\Upsilon_\lambda}\|_{1,\omega} \leq \mathfrak{D}\, \|f\|_{1,\omega}$$

where $\Upsilon_\lambda := \{x \in \mathrm{T} : Mf(x) > \lambda\}$.

The following order relation for the first modulus of smoothness is important.

Lemma 3.5. *Let $0 < h \leq \delta < \infty$, $p \in \mathcal{P}^{\log}$, $\omega \in A_{p(\cdot)}$ and $f \in L^{p(\cdot)}_{2\pi,\omega}$. Then*

$$\|(I-T_h)f\|_{p(\cdot),\omega} \precapprox \|(I-T_\delta)f\|_{p(\cdot),\omega}$$

holds.

4. Proof of results

Proof of Theorem 2.1. (i) Let $B \subset \mathbb{T}$ be a subset and $(1/4) < |B| \leq 2$. If $p_B^- > 1$, then $p_B > 1$ and

$$\int_B |f(x)|\,dx \leq 2\max\left\{1, \left(\rho_{B,p'(\cdot),\omega}\left(\omega^{-1}\right)\right)^{1/p^-}\right\}\|f\|_{B,p(\cdot),\omega}$$

$$= 2\max\left\{1, \max\left\{1, \frac{[\omega]_{A_{p(\cdot)}}|B|^{p_B}}{\omega(B)}\right\}^{p^+/p^-}\right\}\|f\|_{B,p(\cdot),\omega}$$

$$\leq 2\max\left\{1, \frac{[\omega]_{A_{p(\cdot)}}}{\omega(B(0,1))}(1+\pi)^{p+}\right\}^{p^+/p^-}\|f\|_{B,p(\cdot),\omega} \leq \mathbb{S}_1\|f\|_{B,p(\cdot),\omega}$$

holds with

$$\mathbb{S}_1 := \frac{[\omega]_{A_{p(\cdot)}}2(1+\pi)^{(p^+)^2}}{(\omega(B(0,1)))^{p^+/p^-}}.$$

We consider the case $p_B^- = 1$ and $p^+ > 1$. In this case we can decompose B as

$$B := (\cup_j G_j) \cup (\cup_\iota b_\iota) \cup (\cup_i N_i)$$

where G_j, b_ι are subintervals of B, and N_i are singletons with the measure $meas(N_i)$ of N_i is equal to zero and

$$G_j := \{x \in B : p(x) > 1\},\; b_\iota := \{x \in B : p(x) = 1\},$$

$$|G_j| \leq 2, \quad |b_\iota| \leq 2.$$

After then we can proceed as

$$\int_B |f(x)|\,dx = \sum\nolimits_j \int_{G_j}|f(x)|\,dx + \sum\nolimits_\iota \int_{b_\iota}|f(x)|\,dx := \breve{I} + \ddot{I}.$$

We estimate $\ddot{I}$ first. As $\left.\frac{p'(\cdot)}{p(\cdot)}\right|_{b_\iota} = \infty$ we get

$$\ddot{I} = \sum_\iota \int_{b_\iota}|f(x)|\,dx = \sum\nolimits_\iota \int_{b_\iota}|f(x)|\,\omega(x)\,\omega(x)^{-1}\,dx$$

$$\leq \sum\nolimits_\iota \|f\|_{b_\iota,p(\cdot),\omega}\left\|\omega^{-1}\right\|_{b_\iota,\infty} = \sum\nolimits_\iota \frac{\omega(b_\iota)}{|b_\iota|^{p_{b_\iota}}}\left\|\frac{1}{\omega}\right\|_{b_\iota,\frac{p'(\cdot)}{p(\cdot)}}\frac{|b_\iota|^{p_{b_\iota}}}{\omega(b_\iota)}\|f\|_{b_\iota,p(\cdot),\omega}$$

$$\begin{aligned}
&\leq \frac{2(1+\pi)^{(p^+)^2}[\omega]_{A_{p(\cdot)}}}{(\omega(B(0,1)))^{p^+/p^-}}\sum\nolimits_\iota \|f\|_{b_\iota,p(\cdot),\omega} \leq \frac{2(1+\pi)^{(p^+)^2}[\omega]_{A_{p(\cdot)}}}{(\omega(B(0,1)))^{p^+/p^-}}\|f\|_{B,p(\cdot),\omega} \\
&= \mathbb{S}_1\|f\|_{B,p(\cdot),\omega}\,.
\end{aligned}$$

For the second expression $\breve{I}$ we find

$$\breve{I} = \sum\nolimits_j \int_{G_j}|f(x)|\,dx \leq \frac{[\omega]_{A_{p(\cdot)}}2(1+\pi)^{(p^+)^2}}{(\omega(B(0,1)))^{p^+/p^-}}\sum_j \|f\|_{G_j,p(\cdot),\omega}$$

$$\leq \mathbb{S}_1\|f\|_{B,p(\cdot),\omega}\,.$$

If $p_B^- = 1$ and $p^+ = 1$, then $p \equiv 1$. Now result

$$\int_B |f(x)|\,dx \le \frac{[\omega]_{A_1}|B|}{\omega(B)}\|f\|_{B,1,\omega}$$

is known from Remark 2.11 of [10, p.934].

Since

$$\frac{|B|}{\omega(B)}=\frac{|B|^{p_B}}{\omega(B)} \le \frac{(4\mathfrak{D})^{p^+}(1+\pi)^{p^+}}{\min\left\{(\omega(B(0,1)))^{p^-/p^+_{B(0,1)}},(\omega(B(0,1)))^{p^+/p^-_{B(0,1)}}\right\}}:=\mathbb{S}_2$$

we get

$$\int_B |f(x)|\,dx \le [\omega]_{A_1}\mathbb{S}_2\|f\|_{B,1,\omega}.$$

As a result, setting $\mathbb{S}_3 := \mathbb{S}_1 \vee \left([\omega]_{A_1}\mathbb{S}_2\right)$ we have

$$\int_B |f(x)|\,dx \le \mathbb{S}_3\|f\|_{B,1,\omega}$$

with $a \vee b := \max\{a,b\}$.

(ii) In this case we can decompose $\mathrm{T}:=\cup_j G_j$ where $(1/4) < |G_j| \le 2$. Then,

$$\int_{\mathrm{T}} |f(x)|\,dx = \sum\nolimits_j \int_{G_j}|f(x)|\,dx \le \sum\nolimits_j \mathbb{S}_3\|f\|_{G_j,p(\cdot),\omega}$$
$$= \mathbb{S}_3\sum\nolimits_j \|f\|_{G_j,p(\cdot),\omega} = \mathbb{S}_3\|f\|_{p(\cdot),\omega}.$$

□

Proof of Theorem 2.2. Let us consider $f \in L^{p(\cdot)}_{2\pi,\omega}$ with $\|f\|_{p(\cdot),\omega} \le 1$ and $p^+ > 1$. Suppose that $Q := \{U : U \text{ open and } |U| = 1\}$ be a 1-finite family. We define constant $\mathbb{S}_4$ as

$$\mathbb{S}_4 := \mathfrak{C}\left(2(1+\mathbb{S}_3)(1+\pi)\right)^{p^+}(1+\omega(B(0,1)))\left(\left(p^+\right)'[\omega]_{A_{p^+}}\right)^{\frac{1}{p^+-1}}$$

where absolute constant $\mathfrak{C} > 1$ is come from p^+-Buckley's univariate estimate of Hardy Littlewood maximal function. Then using Corollary 2.2.2 of [20, p.20] and Theorem 2.1 we obtain

$$\rho_{p(\cdot),\omega}\left(\frac{1}{\mathbb{S}_4}T_Q f\right) = \frac{1}{\mathbb{S}_4}\int_{\mathrm{T}}\left|\sum_{U\in Q}\chi_{U\cap\mathrm{T}}(x)\,A_U(f)\right|^{p(x)}\omega(x)dx$$
$$\le \frac{1}{\mathbb{S}_4}\sum_{U\in Q}\chi_{U\cap\mathrm{T}}(x)\int_{U\cap\mathrm{T}}|A_U(f)|^{p(x)}\omega(x)dx$$
$$\le \frac{1}{\mathbb{S}_4}\sum_{U\in Q}\chi_U(x)\int_{U\cap\mathrm{T}}\left(\mathbb{S}_3\|f\|_{p(\cdot),\omega}\right)^{p(x)}\omega(x)dx$$
$$\le \frac{1}{\mathbb{S}_4}\sum_{U\in Q}\chi_U(x)\int_{U\cap\mathrm{T}}\left((1+\mathbb{S}_3)^{p^+}\|f\|_{p(\cdot),\omega}\right)\omega(x)dx$$
$$\le \frac{(1+\mathbb{S}_3)^{p^+}}{\mathbb{S}_4}\sum_{U\in Q}\chi_U(x)\,\omega(U\cap\mathrm{T}) \le \frac{(1+\mathbb{S}_3)^{p^+}}{\mathbb{S}_4}\sum_{U\in Q}\chi_U(x)\,\omega(U)$$

$$\leq \frac{(1+\mathbb{S}_3)^{p^+} 2^{p^+} \mathfrak{E}}{\mathbb{S}_4} (1+\pi)^{p^+} (1+\omega(B(0,1))) \left((p^+)' [\omega]_{A_{p^+}}\right)^{\frac{1}{p^+-1}} = 1$$

and hence

$$\|T_Q f\|_{p(\cdot),\omega} \leq \mathbb{S}_4. \tag{4.1}$$

General case

$$\|T_Q f\|_{p(\cdot),\omega} \leq \mathbb{S}_4 \|f\|_{p(\cdot),\omega} \quad \text{for } f \in L^{p(\cdot)}_{2\pi,\omega}$$

can be obtained easily from (4.1).

If $p^+ = 1$, then, the following result

$$\|T_Q f\|_{1,\omega} \leq [\omega]_{A_1} \|f\|_{1,\omega}$$

follows from Theorem 3.2:

$$\int_{\mathrm{T}} \left| \sum_{U\in Q} \chi_{U\cap \mathrm{T}}(x) A_U(f) \right| \omega(x)dx \leq \sum_{U\in Q} \chi_{U\cap \mathrm{T}}(x) \int_{U\cap \mathrm{T}} |A_U(f)| \omega(x)dx$$

$$\leq [\omega]_{A_1} \sum_{U\in Q} \chi_U(x) \int_{U\cap \mathrm{T}} |f(x)| \omega(x)dx \leq [\omega]_{A_1} \|f\|_{1,\omega}.$$

If we set $\mathbb{S}_5 := \mathbb{S}_4 \vee [\omega]_{A_1}$ and combine results obtained above, then,

$$\|T_Q f\|_{p(\cdot),\omega} \leq \mathbb{S}_5 \|f\|_{p(\cdot),\omega}.$$

□

Proof of Theorem 2.3. (a) Since C^∞ is a dense subset of $L^{p(\cdot)}_{2\pi,\omega}$ (see Theorem 3.4), we consider functions $f \in C^\infty$. For any $\varepsilon > 0$, there exists $\delta := \delta(\varepsilon) > 0$ so that $|f(x+u_1) - f(x+u_2)| < \varepsilon$ for any $u_1, u_2 \in$T with $|u_1 - u_2| < \delta$. Then, for F of Definition 2.5, there holds inequality

$$|\mathcal{U}_{f,F}(u_1) - \mathcal{U}_{f,F}(u_2)| \leq \int_{\mathrm{T}} |f(x+u_1) - f(x+u_2)| |F(x)| dx$$

$$\leq \max_{x,u_1,u_2\in \mathrm{T}} |f(x+u_1) - f(x+u_2)| \|F\|_1 \leq \frac{\varepsilon}{\mathbb{S}_3} \mathbb{S}_3 \|F\|_{p'(\cdot),\omega'} \leq \varepsilon$$

for any $u_1, u_2 \in$T with $|u_1 - u_2| < \delta$. Thus conclusion of Theorem 2.3 follows. For the general case $f \in L^{p(\cdot)}_{2\pi,\omega}$ there exists an $g \in C^\infty$ so that

$$\|f - g\|_{p(\cdot),\omega} < \frac{\xi}{4\mathbb{S}_3\mathbb{S}_0}$$

for any $\xi > 0$. Therefore

$$|\mathcal{U}_{f,F}(u_1) - \mathcal{U}_{f,F}(u_2)| = |\mathcal{U}_{f,F}(u_1) - \mathcal{U}_{g,F}(u_1)| + |\mathcal{U}_{g,F}(u_1) - \mathcal{U}_{g,F}(u_2)| +$$

$$\begin{aligned} + |\mathcal{U}_{g,F}(u_2) - \mathcal{U}_{f,F}(u_2)| &= |\mathcal{U}_{f-g,F}(u_1)| + \frac{\xi}{2} + |\mathcal{U}_{g-f,F}(u_2)| \\ &\leq 2\mathbb{S}_3\mathbb{S}_0 \|f-g\|_{p(\cdot),\omega} + \frac{\xi}{2} < \xi. \end{aligned}$$

As a result $\mathcal{U}_{f,F}$ is uniformly continuous on T. □

Proof of Theorem 2.4. Let $0 \leq f, g \in L^{p(\cdot)}_{2\pi,\omega}$. If $\|g\|_{p(\cdot),\omega} = 0$, then, the result is obvious. So we may assume that $\|g\|_{p(\cdot),\omega} > 0$. In this case, there exists an absolute constant C such that

$$\|\mathcal{U}_{f,F}\|_{C(\mathrm{T})} \leq C \|\mathcal{U}_{g,F}\|_{C(\mathrm{T})} = C \max_{u\in\mathrm{T}} \left| \int_{\mathrm{T}} g(x+u) |F(x)| \, dx \right|$$
$$= C\mathbb{S}_0 \|g\|_1 \leq \mathbb{S}_3 C \mathbb{S}_0 \|g\|_{p(\cdot),\omega} .$$

On the other hand, for any $\varepsilon > 0$ and appropriately chosen $F_\varepsilon \in L^{p'(\cdot)}_{2\pi,\omega'}$ with

$$\int_{\mathrm{T}} f(x) |F_\varepsilon(x)| \, dx \geq \frac{1}{12\mathbb{S}_5} \|f\|_{p(\cdot),\omega} - \varepsilon, \qquad \|F_\varepsilon\|_{p'(\cdot),\omega'} \leq 1,$$

one can find

$$\|\mathcal{U}_{f,F}\|_{C(\mathrm{T})} \geq |\mathcal{U}_{f,F}(0)| \geq \int_{\mathrm{T}} f(x) |F(x)| \, dx \geq \frac{1}{12\mathbb{S}_5} \|f\|_{p(\cdot),\omega} - \varepsilon$$

In the last inequality we take as $\varepsilon \to 0+$ and obtain

$$\|\mathcal{U}_{f,F}\|_{C(\mathrm{T})} \geq \frac{1}{12\mathbb{S}_5} \|f\|_{p(\cdot),\omega} .$$

Combining these inequalities we get

$$\|f\|_{p(\cdot),\omega} \leq 12\mathbb{S}_5 \|\mathcal{U}_{f,F}\|_{C(\mathrm{T})} \leq 12\mathbb{S}_5 C \|\mathcal{U}_{g,F}\|_{C(\mathrm{T})}$$
$$\leq 12\mathbb{S}_3\mathbb{S}_5 C\mathbb{S}_0 \|g\|_{p(\cdot),\omega} .$$

For general $f, g \in L^{p(\cdot)}_{2\pi,\omega}$ we obtain

$$\|f\|_{p(\cdot),\omega} \leq 24\mathbb{S}_3\mathbb{S}_5 C\mathbb{S}_0 \|g\|_{p(\cdot),\omega} .$$

□

Proof of Theorem 2.5. Proof of (a): Since

$$\mathcal{U}_{\mathrm{S}_{\lambda,\tau} f,F} = \mathrm{S}_{\lambda,\tau} \mathcal{U}_{f,F}$$

we find

$$\left\|\mathcal{U}_{\mathrm{S}_{\lambda,\tau} f,F}\right\|_{C(\mathrm{T})} = \|\mathrm{S}_{\lambda,\tau}(\mathcal{U}_{f,F})\|_{C(\mathrm{T})} \leq \|\mathcal{U}_{f,F}\|_{C(\mathrm{T})} .$$

Now from Theorem 2.4 we get

$$\|\mathrm{S}_{\lambda,\tau} f\|_{p(\cdot),\omega} \leq 24\mathbb{S}_3\mathbb{S}_5\mathbb{S}_0 \|f\|_{p(\cdot),\omega} .$$

Proof of (b) is the same with (a). □

Proof of Theorem 2.7. (a) follows from Theorem 2.3 and (2.5).

(b) For any $u \in$T,

$$\mathcal{U}_{g*h,F} = \int_{\mathrm{T}} (g*h)(x+u) F(x) \, dx = \int_{\mathrm{T}} \int_{\mathrm{T}} g(x+u-t) h(t) dt F(x) \, dx$$

$$= \int_{\mathrm{T}} \int_{\mathrm{T}} g(x+u-t) F(x) \, dx h(t) dt = \int_{\mathrm{T}} F_{g,F}(u-t) h(t) dt = (\mathcal{U}_{g,F}) * h.$$

□

Proof of Lemma 3.5. Let $0 < h \leq \delta < \infty$, $p \in \mathcal{P}^{\log}$, $\omega \in A_{p(\cdot)}$ and $f \in L^{p(\cdot)}_{2\pi,\omega}$. Then, using Proposition 2.4 of [4]

$$\left\|\mathcal{U}_{(I-T_h)f,F}\right\|_{C(\mathrm{T})} = \left\|(I-T_h)\mathcal{U}_{f,F}\right\|_{C(\mathrm{T})} \leq 72 \left\|(I-T_\delta)\mathcal{U}_{f,F}\right\|_{C(\mathrm{T})}$$
$$\leq 72 \left\|\mathcal{U}_{(I-T_\delta)f,F}\right\|_{C(\mathrm{T})}.$$

From Theorem 2.4 we get

$$\left\|(I-T_h) f\right\|_{p(\cdot),\omega} \leq 1728\mathbb{S}_3\mathbb{S}_5\mathbb{S}_0 \left\|(I-T_\delta) f\right\|_{p(\cdot),\omega}.$$

□

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