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JUSTIFICATION OF COLLOCATION METHOD FOR ONE CLASS OF SYSTEMS OF CURVILINEAR INTEGRAL EQUATIONS

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Abstract. This work studies the approximate solution of the system of integral equations in the problems of shielding electromagnetic fields for cylindrical bodies.

1. Introduction and problem statement

Consider a cylindrical body Ω_1 in a homogeneous isotropic space, with a cross section D_1, bounded by a thin screen S_Λ, with a thickness Λ and a generatrix directed along the z axis. To simplify the system of integral equations, the screen S_Λ is replaced with the perfectly thin surface S, on which special boundary conditions are imposed (see [10]). Denote by Ω_2 the domain beyond the surface S, and by D_2 the corresponding section by the plane $z = const$. The domain Ω_j $\left(j = \overline{1,2}\right)$ is characterized by the electromagnetic parameters $\gamma_j = 0$, ε_j, μ_j, and the material of the shield is characterized by the parameters γ, ε, μ. Let the closed Lyapunov curve Γ be a contour of the orthogonal section of the surface S, and $\vec{n}(y)$ be an outer unit normal at the point $y \in \Gamma$. It was shown in [4] that if the electromagnetic field propagates orthogonally with respect to the cylinder, then the boundary value problem of shielding is reduced to the following boundary value problem:

$$\begin{gathered} \Delta u_1 + k_1^2 u_1 = 0 \text{ in } D_1, \quad \Delta u_2' + k_2^2 u_2' = 0 \text{ in } D_2, \\ u_1|_\Gamma = \left(\alpha_1 \frac{\partial u_1}{\partial \vec{n}} + \alpha_2 \frac{\partial u_2}{\partial \vec{n}} \right)\Bigg|_\Gamma, \\ u_2|_\Gamma = \left(\beta_1 \frac{\partial u_1}{\partial \vec{n}} + \beta_2 \frac{\partial u_2}{\partial \vec{n}} \right)\Bigg|_\Gamma, \quad u_2 = u_0 + u_2', \end{gathered} \tag{1.1}$$

where Δ is a Laplace operator, $k_j = \omega\sqrt{\varepsilon_j \mu_j}$ $\left(j = \overline{1,2}\right)$,

$$\alpha_1 = \frac{1}{2\omega\mu_1}\left(\frac{1}{\Pi} - N\right), \quad \alpha_2 = -\frac{1}{2\omega\mu_2}\left(\frac{1}{\Pi} + N\right),$$

$$\beta_1 = \frac{1}{2\omega\mu_1}\left(\frac{1}{\Pi} + N\right), \quad \beta_2 = -\frac{1}{2\omega\mu_2}\left(\frac{1}{\Pi} - N\right),$$

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$$N=\frac{1}{2}\omega\,\mu\,\Lambda,\quad \Pi=\frac{1}{2}\omega\left(\varepsilon+i\frac{\gamma}{\omega}\right)\Lambda,$$

ω is a circular field frequency, and u_0 is a potential defining a given source field.

Let $G_j(x,y)=\frac{\pi i}{2}H_0^{(1)}(k_j|x-y|)$ be a fundamental solution of the Helmholtz equation in D_j $\left(j=\overline{1,\,2}\right)$, $H_0^{(1)}$ be a zero degree Hankel function of the first kind defined by the formula $H_0^{(1)}(w)=J_0(w)+i\,N_0(w)$,

$$J_0(w)=\sum_{m=0}^{\infty}\frac{(-1)^m}{(m!)^2}\left(\frac{w}{2}\right)^{2m}$$

be a Bessel function of zero degree,

$$N_0(w)=\frac{2}{\pi}\left(\ln\frac{w}{2}+C\right)J_0(w)+\sum_{m=1}^{\infty}\left(\sum_{l=1}^{m}\frac{1}{l}\right)\frac{(-1)^{m+1}}{(m!)^2}\left(\frac{w}{2}\right)^{2m}$$

be a Neumann function of zero degree, and $C=0.57721...$ be an Euler's constant. It was proved in [4] that the simple-layer potentials

$$u_1(x)=\int_{\Gamma}G_1(x,y)\,\varphi_1(y)\,dl_y,\quad x\in D_1,$$

and

$$u_2'(x)=\int_{\Gamma}G_2(x,y)\,\varphi_2(y)\,dl_y,\quad x\in D_2,$$

are the solutions of the boundary value problem (1.1), if the functions φ_1 and φ_2 satisfy the system of integral equations

$$\pi\varphi_1(x)+\int_{\Gamma}\left[Q_{11}(x,y)\;\varphi_1(y)+Q_{12}(x,y)\;\varphi_2(y)\right]dl_y=\theta_2u_0(x),$$

$$\pi\varphi_2(x)+\int_{\Gamma}\left[Q_{21}(x,y)\;\varphi_1(y)+Q_{22}(x,y)\;\varphi_2(y)\right]dl_y=\delta_2u_0(x)-\frac{\partial u_0(x)}{\partial\vec{n}(x)},$$

which we rewrite as

$$\begin{aligned}\varphi_1(x)+(B_{11}\varphi_1)(x)+(B_{12}\varphi_2)(x)&=\frac{1}{\pi}\theta_2u_0(x),\\ \varphi_2(x)+(B_{21}\varphi_1)(x)+(B_{22}\varphi_2)(x)&=\frac{1}{\pi}\left(\delta_2u_0(x)-\frac{\partial u_0(x)}{\partial\vec{n}(x)}\right),\end{aligned}\tag{1.2}$$

where

$$(B_{jm}f)(x)=\frac{1}{\pi}\int_{\Gamma}Q_{jm}(x,y)\;f(y)\;dl_y,\quad x\in\Gamma,\quad j,m=\overline{1,2},$$

$$Q_{11}(x,y)=\frac{\partial G_1(x,y)}{\partial\vec{n}(x)}-\theta_1G_1(x,y),\quad Q_{12}(x,y)=-\theta_2G_2(x,y),$$

$$Q_{21}(x,y)=\delta_1G_1(x,y),\quad Q_{22}(x,y)=\delta_2G_2(x,y)-\frac{\partial G_2(x,y)}{\partial\vec{n}(x)},$$

$$\theta_1=-\frac{1}{2}\omega\mu_1\left(\frac{1}{N}-\Pi\right),\quad \theta_2=\frac{1}{2}\omega\mu_1\left(\frac{1}{N}+\Pi\right),$$

$$\delta_1=-\frac{1}{2}\omega\mu_2\left(\frac{1}{N}+\Pi\right),\quad \delta_2=\frac{1}{2}\omega\mu_2\left(\frac{1}{N}-\Pi\right).$$

It is known that, in general, it is impossible to find an exact solution to the system of integral equations (1.2). Therefore, you have to study the approximate solution of the system of integral equations (1.2) with theoretical justification. Note that in [2, 3, 5, 6, 13, 14], the approximate solution of some classes of systems of integral equations has been studied. But the approximate solution of the system of integral equations (1.2) has not yet been studied. In [8], a quadrature formula for simple-layer and double-layer logarithmic potentials has been constructed, while in [11] a quadrature formula for simple-layer and double-layer potentials has been presented. However, in [11], the asymptotic formula for the zero degree Hankel functions of the first kind has been used to construct the quadrature formulas, which does not allow to find the convergence rate of these quadrature formulas. This work is just dedicated to the justification of collocation method for the system of integral equations (1.2).

2. Justification of collocation method for the system of equations (1.2)

Assume that the curve $\Gamma \subset R^2$ is defined by the parametric equation $x(t) = (x_1(t), x_2(t))$, $t \in [a,b]$. Let's divide the interval $[a,b]$ into $n > 2M_0(b-a)/d$ equal parts: $t_p = a + \frac{(b-a)\,p}{n}$, $p = \overline{0,n}$, where

$$M_0 = \max_{t \in [a,b]} \sqrt{(x_1'(t))^2 + (x_2'(t))^2} < +\infty$$

(see [12, p. 560]) and d is a standard radius (see [16, p. 400]). As control points, we consider $x(\tau_p)$, $p = \overline{1,n}$, where $\tau_p = a + \frac{(b-a)\,(2p-1)}{2n}$. Then the curve Γ is divided into elementary parts: $\Gamma = \bigcup\limits_{p=1}^{n} \Gamma_p$, where $\Gamma_p = \{\, x(t) : \; t_{p-1} \le t \le t_p\}$.

It is known ([8]) that

(1) $\forall p \in \{1,2,...,n\}$: $r_p(n) \sim R_p(n)$, where

$$r_p(n) = \min\{|x(\tau_p) - x(t_{p-1})|\,,\, |x(t_p) - x(\tau_p)|\},$$

$$R_p(n) = \max\{|x(\tau_p) - x(t_{p-1})|\,,\, |x(t_p) - x(\tau_p)|\},$$

and $a(n) \sim b(n)$ means that

$$C_1 \le \frac{a(n)}{b(n)} \le C_2,$$

where C_1 and C_2 are positive constants independent of n;

(2) $\forall p \in \{1,2,...,n\} : R_p(n) \le d/2$;

(3) $\forall p, j \in \{1,2,...,n\} : r_j(n) \sim r_p(n)$;

(4) $r(n) \sim R(n) \sim \frac{1}{n}$, where $R(n) = \max\limits_{p=\overline{1,n}} R_p(n)$, $r(n) = \min\limits_{p=\overline{1,n}} r_p(n)$.

In the sequel, we will call this kind of division a division of the curve Γ into "regular" elementary parts.

By $C(\Gamma)$ we denote the space of all continuous functions on Γ with the norm $\|\varphi\|_\infty = \max\limits_{x \in \Gamma} |\varphi(x)|$, and for the function $\varphi \in C(\Gamma)$ we introduce the modulus of continuity of the form

$$\omega\left(\varphi,h\right)=\max_{\substack{|x-y|\le h\\ x,\,y\in\Gamma}}\left|\varphi(x)-\varphi(y)\right|,\quad h>0.$$

Let

$$Q_{11}^{n}\left(x,y\right)=\frac{\partial G_{1}^{n}\left(x,y\right)}{\partial\vec{n}\left(x\right)}-\theta_{1}G_{1}^{n}\left(x,y\right),\quad Q_{12}^{n}\left(x,y\right)=-\theta_{2}G_{2}^{n}\left(x,y\right),$$

$$Q_{21}^{n}\left(x,y\right)=\delta_{1}G_{1}^{n}\left(x,y\right),\quad Q_{22}^{n}\left(x,y\right)=\delta_{2}G_{2}^{n}\left(x,y\right)-\frac{\partial G_{2}^{n}\left(x,y\right)}{\partial\vec{n}\left(x\right)},$$

where

$$G_{j}^{n}\left(x,\,y\right)=\frac{\pi i}{2}H_{0,n}^{(1)}\left(k_{j}\left|x-y\right|\right),\quad x,y\in\Gamma,\quad x\neq y,\quad j=\overline{1,2},$$

$$H_{0,n}^{(1)}\left(w\right)=J_{0,n}\left(w\right)+i\,N_{0,n}\left(w\right),\quad J_{0,n}\left(w\right)=\sum_{m=0}^{n}\frac{\left(-1\right)^{m}}{\left(m!\right)^{2}}\left(\frac{w}{2}\right)^{2m}$$

and

$$N_{0,n}\left(w\right)=\frac{2}{\pi}\left(\ln\frac{w}{2}+C\right)J_{0,n}\left(w\right)+\sum_{m=1}^{n}\left(\sum_{l=1}^{m}\frac{1}{l}\right)\frac{\left(-1\right)^{m+1}}{\left(m!\right)^{2}}\left(\frac{w}{2}\right)^{2m}.$$

It is not difficult to calculate that

$$\frac{\partial G_{j}^{n}\left(x,y\right)}{\partial\vec{n}\left(x\right)}=\frac{\pi i}{2}\left(\frac{\partial J_{0,n}\left(k_{j}\left|x-y\right|\right)}{\partial\vec{n}\left(x\right)}+i\,\frac{\partial N_{0,n}\left(k_{j}\left|x-y\right|\right)}{\partial\vec{n}\left(x\right)}\right),\quad j=\overline{1,2},$$

where

$$\frac{\partial J_{0,n}\left(k_{j}\left|x-y\right|\right)}{\partial\vec{n}\left(x\right)}=\left(x-y,\vec{n}\left(x\right)\right)\sum_{m=1}^{n}\frac{\left(-1\right)^{m}k_{j}^{2m}\left|x-y\right|^{2m-2}}{2^{2m-1}\left(m-1\right)!\,m!},\quad j=\overline{1,2},$$

and

$$\frac{\partial N_{0,n}\left(k_{j}\left|x-y\right|\right)}{\partial\vec{n}\left(x\right)}=\frac{2}{\pi}\left(\ln\frac{k_{j}\left|x-y\right|}{2}+C\right)\frac{\partial J_{0,n}\left(k_{j}\left|x-y\right|\right)}{\partial\vec{n}\left(x\right)}+$$

$$+\frac{2\left(x-y,\vec{n}\left(x\right)\right)}{\pi\left|x-y\right|^{2}}J_{0,n}\left(k_{j}\left|x-y\right|\right)+$$

$$+\left(x-y,\vec{n}\left(x\right)\right)\sum_{m=1}^{n}\left(\sum_{l=1}^{m}\frac{1}{l}\right)\frac{\left(-1\right)^{m+1}k_{j}^{2m}\left|x-y\right|^{2m-2}}{2^{2m-1}\left(m-1\right)!\,m!},\quad j=\overline{1,2}.$$

Then, using the quadrature formula for simple-layer and double-layer potentials obtained in [7], it is easy to prove the validity of the following theorem.

Theorem 2.1. *Let $\Gamma\subset R^{2}$ be a simple closed Lyapunov curve with index $0<\alpha\le 1$ and $\varphi_{1},\varphi_{2}\in C\left(\Gamma\right)$. Then the expressions*

$$\left(B_{km}^{n}\varphi_{m}\right)\left(x\left(\tau_{p}\right)\right)=$$

$$=\frac{b-a}{\pi n}\sum_{\substack{j=1\\ j\neq p}}^{n}Q_{km}^{n}\left(x\left(\tau_{p}\right),\,x\left(\tau_{j}\right)\right)\sqrt{\left(x_{1}'\left(\tau_{j}\right)\right)^{2}+\left(x_{2}'\left(\tau_{j}\right)\right)^{2}}\,\varphi_{m}\left(x\left(\tau_{j}\right)\right),\quad k,m=\overline{1,2},$$

are the quadrature formulas for the integrals $(B_{km}\varphi_m)(x)$, $x \in \Gamma$, *respectively, at the control points* $x(\tau_p)$, $p = \overline{1, n}$, *with*

$$\max_{p=\overline{1,n}} \left|(B_{km}\varphi_m)(x(\tau_p)) - (B^n_{km}\varphi_m)(x(\tau_p))\right| \le$$
$$\le M^1\left(\omega(\varphi_m,\, 1/n) + \|\varphi_m\|_\infty \frac{\ln n}{n^\alpha}\right), k, m = \overline{1,2}.$$

Let C^{2n} be the space of $2n$–dimensional vectors $z^{2n} = \left(z_1^{2n}, z_2^{2n}, \ldots, z_{2n}^{2n}\right)^{\mathrm{T}}$, $z_l^{2n} \in C$, $l = \overline{1, 2n}$, with the norm $\left\|z^{2n}\right\| = \max\limits_{l=\overline{1,2n}} \left|z_l^{2n}\right|$, where "$a^{\mathrm{T}}$" means the transposition of the vector a. Let's consider the $2n$ −dimensional matrix $B^{2n} = (b_{pj})_{p,j=1}^{2n}$ with the elements

$$b_{pj} = \frac{|sgn\,(p-j)|\,(b-a)}{\pi n} Q^n_{11}(x(\tau_p),\, x(\tau_j))\sqrt{(x_1'(\tau_j))^2 + (x_2'(\tau_j))^2}$$

for $p = \overline{1, n}$ and $j = \overline{1, n}$;

$$b_{pj} = \frac{|sgn\,(p-j+n)|\,(b-a)}{\pi n} Q^n_{12}(x(\tau_p),\, x(\tau_{j-n}))\sqrt{(x_1'(\tau_{j-n}))^2 + (x_2'(\tau_{j-n}))^2}$$

for $p = \overline{1, n}$ and $j = \overline{n+1, 2n}$;

$$b_{pj} = \frac{|sgn\,(p-j-n)|\,(b-a)}{\pi n} Q^n_{21}(x(\tau_{p-n}),\, x(\tau_j))\sqrt{(x_1'(\tau_j))^2 + (x_2'(\tau_j))^2}$$

for $p = \overline{n+1, 2n}$ and $j = \overline{1, n}$;

$$b_{pj} = \frac{|sgn\,(p-j)|\,(b-a)}{\pi n} Q^n_{22}(x(\tau_{p-n}),\, x(\tau_{j-n}))\sqrt{(x_1'(\tau_{j-n}))^2 + (x_2'(\tau_{j-n}))^2}$$

for $p = \overline{n+1, 2n}$ and $j = \overline{n+1, 2n}$.

If we denote by z_p^{2n}, $p = \overline{1, n}$, the approximate values of $\varphi_1(x(\tau_p))$, and by z_{p+n}^{2n}, $p = \overline{1, n}$, the approximate values of $\varphi_2(x(\tau_p))$, then, using the quadrature formulas constructed for the integrals $(B_{jm}f)(x)$, $x \in \Gamma$, $j, m = \overline{1, 2}$, we can replace the system of integral equations (1.2) by the system of algebraic equations with respect to $z^{2n} \in C^{2n}$, written as

$$\begin{cases} z_p^{2n} + \sum\limits_{j=1}^{2n} b_{pj} z_j^{2n} = \frac{1}{\pi}\theta_2 u_0(x(\tau_p)), \quad p = \overline{1, n}, \\ z_p^{2n} + \sum\limits_{j=1}^{2n} b_{pj} z_j^{2n} = \frac{1}{\pi}\left(\delta_2 u_0(x(\tau_p)) - \dfrac{\partial u_0(x(\tau_p))}{\partial \vec{n}(x(\tau_p))}\right), \quad p = \overline{n+1, 2n}. \end{cases} \tag{2.1}$$

Now let's state the main result of this work.

Theorem 2.2. *Let the function* u_0 *be continuous on the curve* Γ. *Then, for non eigenfrequencies, the systems of equations* (1.2) *and* (2.1) *have unique solutions* $\rho_* = (\varphi_1^*,\, \varphi_2^*)^{\mathrm{T}} \in C(\Gamma) \times C(\Gamma)$ *and* $w^{2n} \in C^{2n}$ $(n \ge n_0)$, *respectively, with*

$$\max_{p=\overline{1,n}} \left| w_p^{2n} - \varphi_1^*(x(\tau_p))\right| \le M\left(\omega(u_0,\, 1/n) + \frac{\ln n}{n^\alpha}\right),$$
$$\max_{p=\overline{1,n}} \left| w_{p+n}^{2n} - \varphi_2^*(x(\tau_p))\right| \le M\left(\omega(u_0,\, 1/n) + \frac{\ln n}{n^\alpha}\right).$$

[1]Hereinafter M denotes a positive constant which can be different in different inequalities.

Proof. To justify the method of collocation, we will use Vainikko's convergence theorem for linear operator equations (see [15]). For this aim, let's first write the equations (1.2) and (2.1) in the operator form.

Let's consider the second order matrix operator

$$B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix},$$

defined in the space $C(\Gamma) \times C(\Gamma)$. Then the system of integral equations (1.2) can be rewritten as follows:

$$(I + B)\,\rho = \chi, \tag{2.2}$$

where I is a unit operator in $C(\Gamma) \times C(\Gamma)$, $\rho = (\varphi, \psi)^{\mathrm{T}}$ and

$$\chi(x) = \frac{1}{\pi}\left(\theta_2 u_0(x),\ \delta_2 u_0(x) - \frac{\partial u_0(x)}{\partial \vec{n}(x)}\right)^{\mathrm{T}}.$$

Note that $C(\Gamma) \times C(\Gamma)$ is a Banach space with the norm

$$\|\,\rho\,\|_1 = \max\{\|\varphi_1\|_\infty, \|\varphi_2\|_\infty\}.$$

Obviously, the system of algebraic equations (2.1) can be rewritten as follows:

$$\left(I^{2n} + B^{2n}\right) z^{2n} = \chi^{2n}, \tag{2.3}$$

where I^{2n} is a unit matrix of order $2n$, $\chi^{2n} = p^{2n}\chi$, and $p^{2n} : C(\Gamma) \times C(\Gamma) \to C^{2n}$ is a linear bounded operator defined by the formula

$$p^{2n}\rho =$$

$$= (\varphi_1(x(\tau_1)), \varphi_1(x(\tau_2)), ..., \varphi_1(x(\tau_n)), \varphi_2(x(\tau_1)), \varphi_2(x(\tau_2)), ..., \varphi_2(x(\tau_n)))^{\mathrm{T}}.$$

Now let's verify the fulfilment of the conditions of Theorem 4.2 from [15], using the notations and definitions of the same work. It was proved in [4] that the system of integral equations (1.2) for non eigenfrequencies $k_2 \neq k_2^{(s)} = \omega_s\sqrt{\varepsilon_2\mu_2}$ $(s = 1, 2, ...)$ has the only solution in the space $C(\Gamma) \times C(\Gamma)$ for any continuous right-hand side, where $k_2^{(s)}$ are the eigenvalues of the interior Dirichlet boundary value problem

$$\Delta u_2 + k_2^2 u_2 = 0 \text{ in } R^2 \backslash \bar{D},$$
$$u_2|_\Gamma = 0.$$

Consequently, $Ker\ (I + B) = \{0\}$. Besides, the operators $I^{2n} + B^{2n}$ are Fredholm operators of index zero. Taking into account the way the curve Γ has been divided into "regular" elementary parts, we obtain the following equality for every $\rho \in C(\Gamma) \times C(\Gamma)$:

$$\lim_{n\to\infty} \left\|p^{2n}\rho\right\| = \lim_{n\to\infty} \max\left\{\max_{l=\overline{1,n}} |\varphi_1(x(\tau_l))|\ ,\ \max_{l=\overline{1,n}} |\varphi_2(x(\tau_l))|\right\} =$$

$$= \max\left\{\max_{x\in\Gamma} |\varphi_1(x)|\ ,\ \max_{x\in\Gamma} |\varphi_2(x)|\right\} = \|\rho\|_1.$$

Consequently, the system of operators $P = \left\{p^{2n}\right\}$ is a connecting system for the spaces $C(\Gamma) \times C(\Gamma)$ and C^{2n}. Then $\chi^{2n} \xrightarrow{P} \chi$, and, by Definition 2.1 of [15], it follows from Theorem 2.1 that $I^{2n} + B^{2n} \xrightarrow{PP} I + B$. Due to Definition 3.2 of [15], $I^{2n} \to I$ stably. Then, by Proposition 3.5 and Definition 3.3 of [15], it remains to verify the compactness condition, which, in view of Proposition 1.1 of [15], is equivalent to the following condition: $\forall \left\{z^{2n}\right\}$, $z^{2n} \in C^{2n}$, $\left\|z^{2n}\right\| \leq M$ there exists a relatively compact sequence $\left\{B_{2n} z^{2n}\right\} \subset C\left(\Gamma\right) \times C\left(\Gamma\right)$ such that

$$\left\|B^{2n}\, z^{2n} - p^{2n}\left(\ B_{2n} z^{2n}\right)\right\| \to 0 \ \text{ as } \ n \to \infty.$$

As $\left\{B_{2n} z^{2n}\right\}$ we consider the sequence

$$\left(B_{2n} z^{2n}\right)\,(x) = \left(\sum_{j=1}^{2n} b_j^{(1)}\,(x)\ z_j^{2n},\ \ \sum_{j=1}^{2n} b_j^{(2)}\,(x)\ z_j^{2n}\right)^{\mathrm{T}},$$

where

$$b_j^{(1)}\,(x) = \frac{1}{\pi}\int\limits_{\Gamma_j} Q_{11}^n\,(x,y)\,dl_y \text{ and } b_j^{(2)}\,(x) = \frac{1}{\pi}\int\limits_{\Gamma_j} Q_{21}^n\,(x,y)\,dl_y,\ \text{ if } j = \overline{1,\,n},$$

$$b_j^{(1)}\,(x) = \frac{1}{\pi}\int\limits_{\Gamma_{j-n}} Q_{12}^n\,(x,y)\,dl_y \text{ and } b_j^{(2)}\,(x) = \frac{1}{\pi}\int\limits_{\Gamma_{j-n}} Q_{22}^n\,(x,y)\,dl_y,\ \text{if } j = \overline{n+1,\,2n}.$$

It is clear that

$$\left|J_0\left(k_1\left|x-y\right|\right)\right| \leq \sum_{m=0}^{\infty} \frac{\left(\left|k_1\right|\ diamL\right)^{2m}}{4^m\left(m!\right)^2} \leq M, \forall x, y \in \Gamma, \tag{2.4}$$

and

$$\left|\sum_{m=1}^{\infty}\left(\sum_{l=1}^{m}\frac{1}{l}\right)\frac{\left(-1\right)^{m+1}}{\left(m!\right)^2}\left(\frac{k_1\left|x-y\right|}{2}\right)^{2m}\right| \leq$$

$$\leq \sum_{m=1}^{\infty}\left(\sum_{l=1}^{m}\frac{1}{l}\right)\frac{\left(\left|k_1\right|\ diamL\right)^{2m}}{4^m\left(m!\right)^2} \leq M, \forall x, y \in \Gamma, \tag{2.5}$$

Besides, taking into account the inequality

$$\left|\left(x-y, \vec{n}\left(x\right)\right)\right| \leq M\left|x-y\right|^{1+\alpha} \tag{2.6}$$

(see [16, p. 403]), it is not difficult to show that for arbitrary points $x, y \in \Gamma, x \neq y$, and for arbitrary positive integer n the following estimates are true:

$$\left|Q_{km}^n\left(x,y\right)\right| \leq M\left(\left|\ln\left|x-y\right|\right| + \frac{1}{\left|x-y\right|^{1-\alpha}}\right), k, m = \overline{1,2}.$$

Hence it follows that

$$\left|\sum_{j=1}^{2n} b_j^{(m)}\,(x)\ z_j^{2n}\right| \leq M\ \left\|z^{2n}\right\|, \forall\, x \in \Gamma, \forall\, m = 1, 2,$$

i.e.

$$\left|\left(B_{2n} z^{2n}\right)\,(x)\right| \leq M\left\|z^{2n}\right\|, \forall\, x \in \Gamma.$$

Then, due to the condition $\left\|z^{2n}\right\| \leq M$, we obtain the uniform boundedness of the sequence $\left\{B_{2n} z^{2n}\right\}$.

Now let's consider two arbitrary points x', $x'' \in \Gamma$ such that $|x' - x''| = h < d/2$. Then, proceeding as in [1], we can show that

$$\left| \sum_{j=1}^{2n} b_j^{(1)}\left(x'\right)\, z_j^{2n} - \sum_{j=1}^{2n} b_j^{(1)}\left(x''\right)\, z_j^{2n} \right| \leq M \left\| z^{2n} \right\| \; h \left| \ln h \right|, \quad \forall\, x',\, x'' \in \Gamma,$$

and

$$\left| \sum_{j=1}^{2n} b_j^{(2)}\left(x'\right)\, z_j^{2n} - \sum_{j=1}^{2n} b_j^{(2)}\left(x''\right)\, z_j^{2n} \right| \leq M \left\| z^{2n} \right\| \; h \left| \ln h \right|, \quad \forall\, x',\, x'' \in \Gamma.$$

Consequently,

$$\left| \left(B_{2n} z^{2n}\right)\,\left(x'\right) - \left(B_{2n} z^{2n}\right)\,\left(x''\right) \right| \leq M \left\| z^{2n} \right\| \left| x' - x'' \right| \left| \ln \left| x' - x'' \right| \right|, \quad \forall x', x'' \in \Gamma,$$

and, therefore, $\left\{ B_{2n} z^{2n} \right\} \subset C\left(\Gamma\right) \times C\left(\Gamma\right)$. This immediately implies the equicontinuity of the sequence $\left\{ B_{2n} z^{2n} \right\}$. Then from the Arzela theorem we obtain the relative compactness of the sequence $\left\{ B_{2n} z^{2n} \right\}$. Besides, proceeding as in [7], we get

$$\left\| B^{2n}\, z^{2n} - p^{2n} \left(\; B_{2n} z^{2n} \right) \right\| \to 0 \quad \text{as} \quad n \to \infty.$$

Now, applying Theorem 4.2 from [15], we see that the equations (2.2) and (2.3) have unique solutions $\rho_* = \left(\varphi_1^*, \varphi_2^*\right)^{\mathrm{T}} \in C\left(\Gamma\right) \times C\left(\Gamma\right)$ and $w^{2n} \in C^{2N}$ $(n \geq n_0)$, respectively, with

$$c_1\, \delta_n \leq \left\| \, w^{2n} - p^{2n} \rho_* \right\| \leq \; c_2\, \delta_n,$$

where

$$c_1 \; = 1/ \sup_{n \geq n_0} \; \left\| \, I^{2n} + B^{2n} \right\| > 0, \; c_2 \; = \sup_{n \geq n_0} \; \left\| \, \left(I^{2n} + B^{2n} \right)^{-1} \right\| < +\infty,$$

$$\delta_n = \left\| \left(I^{2n} + B^{2n} \right)\; \left(p^{2n} \rho_* \right) - \chi^{2n} \right\|.$$

Taking into account the equality

$$\chi^{2n} = p^{2n} \chi = p^{2n} \rho_* + p^{2n} \left(B \rho_* \right)$$

and the error estimates for the quadrature formulas constructed for the integrals $\left(B_{jm} f\right)(x)$, $x \in \Gamma$, $j, m = 1, 2$, we have

$$\delta_n = \left\| B^{2n} \left(p^{2n} \rho_* \right) - p^{2n} \left(B \rho_* \right) \right\| \leq M \; \left(\left\| \, \rho_* \right\|_1 \frac{\ln n}{n^{\alpha}} + \omega \; \left(\rho_*, \; 1/n \right) \right).$$

The modulus of continuity of the vector function ρ_* here is defined by the formula

$$\omega \; \left(\rho_*, \; h \right) = \max_{\substack{|x-y| \leq h \\ x,\, y \in \Gamma}} \sqrt{\left(\varphi_1^*\left(x\right) - \varphi_1^*\left(y\right) \right)^2 + \left(\varphi_2^*\left(x\right) - \varphi_2^*\left(y\right) \right)^2}, \; h > 0.$$

It is clear from the inequalities (2.4), (2) and (2.6) that for arbitrary points $x, y \in \Gamma, x \neq y$, the relation

$$|Q_{km}(x,y)| \leq M \left(|\ln|x-y|| + \frac{1}{|x-y|^{1-\alpha}} \right), \quad k,m=\overline{1,2},$$

holds. Then, proceeding as in [9], we can show that

$$\left|(B_{jm}f)\left(x'\right)-(B_{jm}f)\left(x''\right)\right| \leq$$

$$M\left\|f\right\|_{\infty}\left|x'-x''\right|\left|\ln\left|x'-x''\right|\right|, \quad \forall x',\ x''\in\Gamma, \quad \forall j,m=1,2.$$

Consequently,

$$\left|(B\rho_*)\left(x'\right)-(B\rho_*)\left(x''\right)\right| \leq M\left\|\rho_*\right\|_1\left|x'-x''\right|\left|\ln\left|x'-x''\right|\right|, \quad \forall x',x''\in\Gamma,$$

i.e.

$$\omega\ (B\rho_*,\ 1/n) \leq M\left\|\rho_*\right\|_1\frac{\ln n}{n}.$$

Then, taking into account the inequalities

$$\omega\left(\rho_*,\ 1/n\right)=\omega\left(\chi-B\rho_*,\ 1/n\right)\leq$$

$$\leq\omega\left(\chi,\ 1/n\right)+\omega\left(B\rho_*,\ 1/n\right)\leq\omega\left(u_0,\ 1/n\right)+M\left\|\rho_*\right\|_1\frac{\ln n}{n}$$

and

$$\left\|\rho_*\right\|_1\leq\left\|(I+B)^{-1}\right\|\ \left\|\chi\right\|_1,$$

we obtain

$$\delta_n\leq M\left(\omega\ (u_0,\ 1/n)+\frac{\ln n}{n^{\alpha}}\right).$$

The theorem is proved. □

From Theorem 2.2 we obtain the following corollary.

Corollary 2.1. *Let the function* u_0 *be continuous on the curve* Γ, $w^{2n}=\left(w_1^{2n},w_2^{2n},...,w_{2n}^{2n}\right)$ *be a solution to the system of algebraic equations (2.1),* $x_1\in D_1$ *and* $x_2\in D_2$*. Then the sequence*

$$u_1^n(x_1)=\frac{b-a}{n}\sum_{j=1}^{n}G_1^n(x_1,x(\tau_j))\sqrt{(x_1'(\tau_j))^2+(x_2'(\tau_j))^2}w_j^{2n}$$

converges to the value $u_1(x_1)$ *and the sequence*

$$u_2^n(x_2)=\frac{b-a}{n}\sum_{j=1}^{n}G_2^n(x_2,x(\tau_j))\sqrt{(x_1'(\tau_j))^2+(x_2'(\tau_j))^2}w_{n+j}^{2n}$$

converges to the value $u_2'(x_2)$*, and the following inequalities hold:*

$$|u_1(x_1)-u_1^n(x_1)|\leq M\left(\omega(u_0,1/n)+\frac{\ln n}{n^{\alpha}}\right),$$

$$\left|u_2'(x_2)-u_2^n(x_2)\right|\leq M\left(\omega(u_0,1/n)+\frac{\ln n}{n^{\alpha}}\right).$$

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