Proceedings of the Institute of Mathematics and Mechanics,
National Academy of Sciences of Azerbaijan
Volume 00, Number 0, XXXX, Pages 000–000

NON-WEYL RESONANCE ASYMPTOTICS FOR QUANTUM GRAPH WITH THE DIRAC OPERATOR ON EDGES

ANNA G. BELOLIPETSKAIA AND IGOR Y. POPOV

Abstract. We investigate quantum graph consisting of a compact interior with a finite number of semi-infinite edges attached. The Dirac operator acts on the edges of the graph. At vertices, matching conditions of a general form are considered. For this model, the non-Weyl asymptotics of resonances (quasi-eigenvalues) is studied. The results were obtained by constructing an effective coupling matrix.

1. Introduction

Quantum graph models have been used since the 1930s, but it is only in the last few decades that quantum graphs have attracted increased interest. Currently there is a huge amount of literature on quantum graphs, a detailed description of the history of the development of the theory of quantum graphs and an extensive bibliography can be found in the works [2], [13]. In this article we study the Weyl asymptotics of the resonances of the Dirac operator on quantum graph edges with general coupling conditions.

Resonances attract great attention last time due to its physical importance (see, e.g., [1, 17, 11, 12, 14, 8, 21, 22, 20, 5, 7]). Many results have been obtained in the last 10 years on the Weyl asymptotics of the resonances of the Schrödinger operator on quantum graphs. For example, E.B.Davies and A.Pushnitski [10] proved a criterion for the non-Weyl asymptotics of resonances of the Schrödinger operator on quantum graphs with Kirchhoff coupling conditions. It was shown that the main factor in the form of asymptotics is the topological structure of the quantum graph, namely, the presence of the so-called balanced vertices. In a similar problem with general boundary conditions, the form of the asymptotics is affected by the so-called effective coupling matrix. It was first described by P.Exner and J.Lipovsky in [15]. Later E.B.Davies, P.Exner and J.Lipovsky in [9] obtained a criterion for the non-Weyl asymptotics of the resonances of the Schrödinger operator on quantum graphs with general coupling conditions through the study of the eigenvalues of the effective coupling matrix. Also in the paper [16] the influence of the magnetic field on the non-Weyl asymptotics of the resonances for the Schrödinger operator on quantum graph with general coupling conditions was studied.

2010 *Mathematics Subject Classification.* 81U20, 47N50.
Key words and phrases. quantum graph, resonance, asymptotics, Dirac operator.

Similar results for the non-Weyl asymptotics of resonances for the Dirac operator on quantum graph with general coupling conditions have not been obtained previously. This article is devoted to this topic, namely, the construction of an effective coupling matrix and the study of the asymptotic behavior of resonances for the Dirac operator on a quantum graph with general coupling conditions.

In this paper, we study quantum graphs consisting of a compact interior with a finite number of infinite edges attached. At each edge e (isomorphic to segment or to a hal-line), we consider the space $L_2(e)\otimes\mathbb{C}^2$ of 2-vector-functions $\begin{pmatrix}\psi_e^{(1)}\\ \psi_e^{(2)}\end{pmatrix}$. The Dirac operator D at edge e has the domain $W_2^1(e)\otimes\mathbb{C}^2$, $W_2^1(e)$ is the Sobolev space. It acts as follows:

$$D=-i\frac{d}{dx}\otimes\sigma_1+1\otimes\sigma_3, \tag{1.1}$$

where $\sigma_1=\begin{pmatrix}0&1\\1&0\end{pmatrix}$, $\sigma_3=\begin{pmatrix}1&0\\0&-1\end{pmatrix}$ are the Pauli matrices. One can consider the orthogonal sum of such operators over all graph edges. As for the coupling conditions at the vertices which ensure self-adjointness of the operator on the whole quantum graph, we use the general coupling conditions obtained by W.Bulla and T.Trenkler in [6]:

$$(U-iI)(\frac{1+i}{\sqrt{2}}\psi^{(1)})+i(U+I)\psi^{(2)}=0. \tag{1.2}$$

where I is the identical matrix, U is a unitary matrix which size is determined by the number of the graph edges, $\psi^{(1)}$ ($\psi^{(2)}$) are vectors, composed of $\psi_e^{(1)}$ ($\psi_e^{(2)}$).

While studying resonances, one can deal with the operator resolvent or with the scattering matrix. In this work, resonances are found as the poles of the scattering matrix. Correspondingly, the resonances are determined as roots of the following determinant (analogously to [15]):

$$F(k)=\det((U-iI)(\frac{1+i}{\sqrt{2}}c_1(k))+ik(U+I)c_2(k))=0, \tag{1.3}$$

where l_j is the length of the inner edge e_j, I_M is identity matrix of size $M\times M$ and $c_1(k), c_2(k)$ are equal to the following matrices:

$$\begin{gathered}c_1(k)=diag(c_{11}(k),...,c_{1N}(k),0\cdot I_M),\\ c_2(k)=diag(c_{21}(k),...,c_{2N}(k),i\cdot I_M),\\ c_{1j}(k)=\begin{pmatrix}0&1\\ \sin(kl_j)&\cos(kl_j)\end{pmatrix}, c_{2j}(k)=\begin{pmatrix}1&0\\ \cos(kl_j)&-\sin(kl_j)\end{pmatrix}.\end{gathered} \tag{1.4}$$

Let us introduce the function $N(R,F)$, which calculates the number of roots of the function $F(k)$, counting their multiplicities, not exceeding the parameter R:

$$N(R,F)=\{\#k : F(k)=0 \quad and \quad |k|<R\}. \tag{1.5}$$

The main result of this article will be provided for the $N(R,F)$ function.

2. Effective coupling matrix for the Dirac operator

In order to simplify the problem of studying resonances, we introduce an effective coupling matrix. This matrix is obtained by reducing the problem to a compact interior of a quantum graph.

Definition 2.1. Effective coupling matrix for the operator with coupling conditions (1.2) is such matrix $\tilde{U}$ that the resonance equation (1.3) has the form

$$\det((\tilde{U} - iI)(\frac{1+i}{\sqrt{2}}\tilde{c}_1(k)) + ik(\tilde{U} + I)\tilde{c}_2(k)) = 0, \tag{2.1}$$

where $\tilde{c}_i(k)$, $i = 1, 2$, contains only the parts of $c_i(k)$ corresponding to the internal edges.

Theorem 2.1. *The effective coupling matrix of the Dirac operator on a quantum graph with coupling conditions* (1.2) *is as follows:*

$$\tilde{U}(k) = U_1 - (\frac{1+i}{\sqrt{2}-k})U_2((\frac{1+i}{\sqrt{2}} - k)U_4 - (\frac{-1+i}{\sqrt{2}} + k)I)^{-1}U_3, \tag{2.2}$$

where I is the identity matrix $M \times M$, the matrices U_i are defined as follows:

$$U = \begin{pmatrix} U_1 & U_2 \\ U_3 & U_4 \end{pmatrix}, \tag{2.3}$$

where U_1 is the $2N \times 2N$ matrix characterizing the inner part of the graph (N is the number of internal edges), U_4 is the $M \times M$ matrix characterizing the outer part of the graph (M is the number of outer edges), matrices U_2, U_3 of size $M \times 2N, 2N \times M$, respectively, correspond to both the inner and outer parts of the graph.

Proof. Consider a quantum graph with N interior edges and M exterior edges. Due to the relation between the Dirac equation and the exactly solvable Schrödinger equation, the vectors $\psi^{(1)}, \psi^{(2)}$ can be written as follows:

$$\begin{cases} \psi^{(1)} = (f_1, ..., f_{2N}, g_1, ..., g_M)^T, \\ \psi^{(2)} = (f'_1, ..., f'_{2N}, ikg_1, ..., ikg_M)^T. \end{cases} \tag{2.4}$$

where $f_{2j-1}, f_{2j}, f'_{2j-1}, f'_{2j}$ are the values of the function and the derivative of the function at the ends of the inner edge e_j, g_j is vertex value for outer edges e_j. Substituting (2.4) into the general coupling conditions (1.2), one obtains the following equation:

$$\begin{aligned} &\tfrac{1+i}{\sqrt{2}}(U - iI)(f_1, ..., f_{2N}, g_1, ..., g_M)^T + \\ &+(U + I) \cdot diag(i \cdot I_{2N}, -k \cdot I_M) \cdot (f'_1, ..., f'_{2N}, g_1, ..., g_M)^T = 0, \end{aligned} \tag{2.5}$$

where I_M is the identity matrix of size $M \times M$, I_{2N} is the identity matrix of size $2N \times 2N$. Let's rewrite the equation 2.5 in the form:

$$V(f_1, ..., f_{2N}, f'_1, ..., f'_{2N}, g_1, ..., g_M)^T = 0. \tag{2.6}$$

It is not difficult to calculate the matrix V:

$$V = \begin{pmatrix} \frac{1+i}{\sqrt{2}}(U_1 - iI) & i(U_1 + I) & \frac{1+i}{\sqrt{2}}U_2 - kU_2 \\ \frac{1+i}{\sqrt{2}}U_3 & iU_3 & \frac{1+i}{\sqrt{2}}(U_4 - iI) - k(U_4 + i) \end{pmatrix} = \begin{pmatrix} \frac{1+i}{\sqrt{2}}(U_1 - iI) & i(U_1 + I) & (\frac{1+i}{\sqrt{2}} - k)U_2 \\ \frac{1+i}{\sqrt{2}}U_3 & iU_3 & (\frac{1+i}{\sqrt{2}} - k)U_4 - (\frac{-1+i}{\sqrt{2}} + k)I \end{pmatrix}. \tag{2.7}$$

If the matrix $(\frac{1+i}{\sqrt{2}} - k)U_4 - (\frac{-1+i}{\sqrt{2}} + k)I$ is invertible, then the vector $(g_1, ..., g_M)^T$ can be expressed from the last M lines of the equation (2.7):

$$(g_1, ..., g_M)^T = A(\frac{1+i}{\sqrt{2}}f_1 + if_1', ..., \frac{1+i}{\sqrt{2}}f_{2N} + if_{2N}')^T, \tag{2.8}$$

where

$$A = -((\frac{1+i}{\sqrt{2}} - k)U_4 - (\frac{-1+i}{\sqrt{2}} + k)I)^{-1}U_3. \tag{2.9}$$

Let us substitute the resulting expression (2.8) into the formula (2.7) and write the first $2N$ lines:

$$\begin{gathered} \frac{1+i}{\sqrt{2}}(U_1 - iI)(f_1, ..., f_{2N})^T + \frac{1+i}{\sqrt{2}}U_2A(\frac{1+i}{\sqrt{2}}(f_1, ..., f_{2N})^T + \\ i(f_1', ..., f_{2N}')^T) + i(U_1 + I)(f_1', ..., f_{2N}')^T - \\ kU_2A(\frac{1+i}{\sqrt{2}}(f_1, ..., f_{2N})^T + i(f_1', ..., f_{2N}')^T) = 0. \end{gathered} \tag{2.10}$$

Let's calculate the coefficient in front of the vector $(f_1, ..., f_{2N})^T$ in the expression (2.10):

$$\frac{1+i}{\sqrt{2}}(U_1 - iI) + iU_2A - k\frac{1+i}{\sqrt{2}}U_2A = \frac{1+i}{\sqrt{2}}(U_1 + (\frac{1+i}{\sqrt{2}} - k)U_2A - iI). \tag{2.11}$$

Similarly, one calculates the coefficient in front of the vector $(f_1', ..., f_{2N}')^T$ in the expression (2.10):

$$\frac{1+i}{\sqrt{2}} \cdot iU_2A + i(U_1 + I) - ikU_2A = i(U_1 + (\frac{1+i}{\sqrt{2}} - k)U_2A + I). \tag{2.12}$$

Using the obtained coefficients in front of vectors $(f_1, ..., f_{2N})^T$, $(f_1', ..., f_{2N}')^T$ ((2.11), (2.12)), one obtains that the effective coupling matrix has the following form:

$$\tilde{U}(k) = U_1 + (\frac{1+i}{\sqrt{2} - k})U_2A. \tag{2.13}$$

Substituting the expression for the matrix A (2.9), we get the final answer:

$$\tilde{U}(k) = U_1 - (\frac{1+i}{\sqrt{2} - k})U_2((\frac{1+i}{\sqrt{2}} - k)U_4 - (\frac{-1+i}{\sqrt{2}} + k)I)^{-1}U_3. \tag{2.14}$$

Thus, the theorem is proved. □

3. Non-Weyl resonance asymptotics for the Dirac operator

In this section we investigate the asymptotic behavior of resonances at infinity. For this purpose, it is more convenient to replace $\sin(kl_j), \cos(kl_j)$ by exponentials

e^{-ikl_j}, e^{ikl_j}. Then, functions $c_{1j}(k), c_{2j}(k)$ from formula (1.4) have the following form:

$$\begin{aligned} c_{1j}(k) &= \tfrac{1}{2}E_{1j} + \tfrac{1}{2}E_{2j} + E_{4j}, \\ c_{2j}(k) &= -i(\tfrac{1}{2}E_{1j} - \tfrac{1}{2}E_{2j} + E_{3j}), \end{aligned} \tag{3.1}$$

where the matrices $E_{1j}, E_{2j}, E_{3j}, E_{4j}$ are as follows:

$$\begin{aligned} E_{1j} &= \begin{pmatrix} 0 & 0 \\ -ie^{ikl_j} & e^{ikl_j} \end{pmatrix}, \quad E_{3j} = \begin{pmatrix} i & 0 \\ 0 & 0 \end{pmatrix}, \\ E_{2j} &= \begin{pmatrix} 0 & 0 \\ ie^{-ikl_j} & e^{ikl_j} \end{pmatrix}, \quad E_{4j} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \end{aligned} \tag{3.2}$$

Respectively, function $F(k)$ (1.3) in the new notations (3.1), (3.2) takes the following form:

$$\begin{aligned} F(k) = \det((\tfrac{1+i}{2\sqrt{2}}(U - iI) + \tfrac{1}{2}k(U + I))E_1(k) + \\ +(\tfrac{1+i}{2\sqrt{2}}(U - iI) - \tfrac{1}{2}k(U + I))E_2(k) + \\ +k(U + I)E_3(k) + \tfrac{1+i}{\sqrt{2}}(U - iI)E_4(k) - k(U + I)diag(0 \cdot I_{2N}, I_M)), \end{aligned} \tag{3.3}$$

where $E_i = diag(E_{i1}, ..., E_{iN}, 0 \cdot I_M)$. For further simplification of the obtained results, we pass to the effective coupling matrix $\tilde{U}(k)$ (2.2). Then, function $F(k)$ (3.3) takes the following form:

$$\begin{aligned} F(k) = \det((\tfrac{1+i}{2\sqrt{2}}(\tilde{U}(k) - iI) + \tfrac{1}{2}k(\tilde{U}(k) + I))E_1(k) \\ +(\tfrac{1+i}{2\sqrt{2}}(\tilde{U}(k) - iI) - \tfrac{1}{2}k(\tilde{U}(k) + I))E_2(k) \\ +k(\tilde{U}(k) + I)E_3(k) + \tfrac{1+i}{\sqrt{2}}(\tilde{U}(k) - iI)E_4(k)). \end{aligned} \tag{3.4}$$

To obtain the main result of the article, it is necessary to calculate the coefficient before $e^{\pm} = e^{\pm \sum\limits_j ikl_j}$.

Lemma 3.1. *The coefficient before* $e^{\pm}$ *in the expression* (3.4) *function* $F(k)$ *is as follows:*

$$\frac{i^N}{2^N} \cdot \det(\frac{1+i}{\sqrt{2}}(U - iI) \pm k(U + I)). \tag{3.5}$$

Proof. For convenience, we introduce auxiliary matrices $B = (b_{xy})_{2N\times 2N}, C = (c_{xy})_{2N\times 2N}$:

$$\begin{aligned} B &= \tfrac{1+i}{2\sqrt{2}}(U - iI) + \tfrac{1}{2}k(U + I), \\ C &= \tfrac{1+i}{2\sqrt{2}}(U - iI) - \tfrac{1}{2}k(U + I). \end{aligned} \tag{3.6}$$

Then, $F(k)$ can be expressed in terms of matrices B, C:

$$F(k) = \det(B \cdot E_1 + C \cdot E_2 + (B - C) \cdot E_3 + (B + C) \cdot E_4). \tag{3.7}$$

We introduce the matrix $F_0 = (f_{xy})_{2N\times 2N}$, which is obtained by removing the factor i from the columns of the matrix $F(k)$:

$$F(k) = i^N \det(F_0). \tag{3.8}$$

Then, entries f_{xy} of the matrix F_0 are presented by the following expressions:

$$\begin{cases} f_{xy} = b_{xy}e^{ikl_z} + c_{xy}e^{-ikl_z} + b_{xy-1} + c_{xy-1}, & y = 2z, z = \{1, ..., N\}, \\ f_{xy} = -b_{xy}e^{ikl_z} + c_{xy}e^{-ikl_z} + b_{xy-1} - c_{xy-1}, & y = 2z - 1, z = \{1, ..., N\}. \end{cases} \tag{3.9}$$

Note that the coefficient before e^+ of the determinant of the matrix F_0 is the same as the coefficient before e^+ of the determinant of the matrix $F^+ = (f^+_{xy})_{2N\times 2N}$, where the entries of the matrix F^+ are as follows:

$$\begin{cases} f^+_{xy} = b_{xy}e^{ikl_z} + b_{xy-1} + c_{xy-1}, & y = 2z, z = \{1, ..., N\}, \\ f^+_{xy} = -b_{xy}e^{ikl_z} + b_{xy-1} - c_{xy-1}, & y = 2z-1, z = \{1, ..., N\}. \end{cases} \tag{3.10}$$

Similarly, it is possible to introduce the matrix $F^- = (f^-_{xy})_{2N\times 2N}$ for calculation of the coefficient before e^- in the determinant of the matrix F_0:

$$\begin{cases} f^-_{xy} = c_{xy}e^{-ikl_z} + b_{xy-1} + c_{xy-1}, & y = 2z, z = \{1, ..., N\}, \\ f^-_{xy} = c_{xy}e^{-ikl_z} + b_{xy-1} - c_{xy-1}, & y = 2z-1, z = \{1, ..., N\}. \end{cases} \tag{3.11}$$

Since the determinant does not change when a linear combination of other columns is added to any column, when calculating the determinant of the matrix F^+, we add the column number $2i-1$ to the column number $2i$. The new coefficients of the determinant are as follows:

$$\begin{cases} \tilde{f}^+_{xy} = 2b_{xy-1}, & y = 2z, z = \{1, ..., N\}, \\ \tilde{f}^+_{xy} = -b_{xy}e^{ikl_z} + b_{xy-1} - c_{xy-1}, & y = 2z-1, z = \{1, ..., N\}. \end{cases} \tag{3.12}$$

Similarly, when calculating the determinant of the matrix F^-, one subtracts from the column with the number $2i$ the column with the number $2i-1$:

$$\begin{cases} \tilde{f}^-_{xy} = 2c_{xy-1}, & y = 2z, z = \{1, ..., N\}, \\ \tilde{f}^-_{xy} = c_{xy}e^{-ikl_z} + b_{xy-1} - c_{xy-1}, & y = 2z-1, z = \{1, ..., N\}. \end{cases} \tag{3.13}$$

It is easy to see that the coefficient c_0^+ before e^+ of the determinant F^+ is equal to the following expression:

$$c_0^+ = 2^N \det B. \tag{3.14}$$

Returning to the original notation, one determines the coefficient c^+ before e^+ in the expression for the function $F(k)$:

$$c^+ = \frac{i^N}{2^N}\det(\frac{1+i}{\sqrt{2}}(U - iI) + k(U + I)). \tag{3.15}$$

Analogously, one obtains the coefficient c^- before e^- for the function $F(k)$:

$$c^- = \frac{i^N}{2^N}\det(\frac{1+i}{\sqrt{2}}(U - iI) - k(U + I)). \tag{3.16}$$

This completes the proof. □

Remark 3.1. Note that the coefficients before $e^+\cdot e^{ik\sum e_j}, e^-\cdot e^{-ik\sum e_j}$, where the sum over j passes over a non-empty subset of the sets of interior edges are equal to 0. Really, formula (3.9) shows that the coefficients before $e^{\pm 2ikl_j}$ annihilates due to symmetry.

Below we will use the following theorem.

Theorem 3.1. [9] *Let* $F(k) = \sum_{r=0}^{n} k^{v_r}a_r(k)e^{ik\sigma_r}$, *where* $v_r \in \mathbb{R}$, $a_r(k)$ *are rational functions of the complex variable* k *with complex coefficients that do not vanish identically, and* $\sigma_r \in \mathbb{R}$, $\sigma_0 < \sigma_1 < ... < \sigma_n$. *Suppose also that* v_r *are chosen so that* $\lim_{k\to\infty} a_r(k) = \alpha_r$ *is finite and non-zero for all* r. *There exists a compact set* $\Omega \subset \mathbb{C}$, *real numbers* m_r *and positive* K_r, $r = 1, ..., n$ *such that the zeros of* $F(k)$

outside Ω *lie in one of* n *logarithmic strips, each one bounded between the curves* $-Imk + m_r \log|k| = \pm K_r$. *The counting function behaves in the limit* $R \to \infty$ *as*

$$N(R,F) = \frac{\sigma_n - \sigma_0}{\pi} R + O(1). \tag{3.17}$$

Now we formulate and prove a theorem that will allow us to determine the form of asymptotics (in particular, Weyl or non-Weyl) of resonances for a quantum graph, on the edges of which the Dirac operator acts with general coupling conditions.

Theorem 3.2. *Let a quantum graph be given, consisting of a compact interior, to which a finite number of edges of infinite length are attached. The Dirac operator 1.1 will act on the edges of this quantum graph. The coupling conditions at the vertices of the quantum graph will be described by the unitary matrix* U_j. *Then the asymptotics of the resonance counting function as* $R \to \infty$ *is of the form:*

$$N(R,F) = \frac{2W}{\pi} R + O(1),$$

where the W *is called the effective size of the quantum graph, it satisfies the following relation:*

$$0 \le W \le V = \sum_{j=1}^{N} l_j.$$

One should note that condition $W < V$ *is equivalent to the fact that there is a vertex where the effective coupling matrix* $\tilde{U}_j(k)$ *has an eigenvalue* $\frac{-1+i-k\sqrt{2}}{1+i+k\sqrt{2}}$ *or* $\frac{-1+i+k\sqrt{2}}{1+i-k\sqrt{2}}$ *for any* k *(the matrix* $\tilde{U}_j(k)$ *is assumed to exist).*

Proof. The condition of the theorem consists of two parts. The first part describes the general form of asymptotics, and the second part describes necessary and sufficient conditions for non-Weyl asymptotics.

To prove the first part, we will use the results of the lemma 3.1 and the remarks to it. Then using the notation from the theorem 3.2 we obtain that $-V \le \sigma_0 \le 0, 0 \le \sigma_n \le V$, where $V = \sum_{j=1}^{N} l_j$. Then $0 \le \sigma_n - \sigma_0 \le 2V$ and by the theorem 3.2 $N(R,F) = \frac{2W}{\pi} R + O(1)$, where $0 \le W \le V$, as required in the first part.

To prove the second part, it is worth noting that the non-Weyl terminology asymptotics means that the inequality $W < V$ must hold, that is, it means that at least one of the two inequalities $-V < \sigma_0$ or $\sigma_n < V$. That is, the coefficient before e^+, e^- must be equal to 0. In the lemma 3.1 it was obtained that the coefficient before $e^\pm$ is equal to $\frac{i^N}{2^N} \cdot \det(\frac{1+i}{\sqrt{2}}(U - iI) \pm k(U+I))$, which will be zero if and only if, when $\frac{-1+i-k\sqrt{2}}{1+i+k\sqrt{2}}$ or $\frac{-1+i+k\sqrt{2}}{1+i-k\sqrt{2}}$ will be the eigenvalue of the effective matching matrix $\tilde{U(k)}$ for any k. Thus, the theorem is proved. ∎

Thus, in order to determine the form of the asymptotic behavior of the resonances, it is sufficient to study the eigenvalues of the effective coupling matrix.

Acknowledgements. This research was supported by Russian Science Foundation (grant number 23-21-00096 (https://rscf.ru/en/project/23-21-00096/)).

References

[1] A. Aslanyan, L. Parnovski, D. Vassiliev, Complex resonances in acoustic waveguides, *Q. J. Mech. Appl. Math.* **53** (2000), no. 3, 429-447.

[2] G. Berkolaiko, P. Kuchment, *Introduction to quantum graphs*, American Mathematical Soc., Providence, 2013.

[3] I.V. Blinova, I.Y. Popov, Quantum graph with the Dirac operator and resonance states completeness. *Operator Theory: Adv. Appl.* **268** (2018), 111-124

[4] A.A. Boitsev, I.Y. Popov, A model of an electron in a quantum graph interacting with a two-level system, *Nanosystems: Phys., Chem., Math.* **10** (2019), no. 2, 131–140.

[5] Ph. Briet, A model of sheared nanoribbons, *Nanosystems: Phys. Chem. Math.* **13** (2022), no 1, 12-16.

[6] W. Bulla, T. Trenkler, The free Dirac operator on compact and noncompact graphs, *J. Math. Phys.* **31** (1990), 1157-1163.

[7] G. Cardone, T. Durante, Asymptotic analysis of a planar waveguide perturbed by a non periodic perforation, *Nanosystems: Phys. Chem. Math.* **13** (2022), no 1, 5-11.

[8] T. Christiansen, Some upper bounds on the number of resonances for manifolds with infinite cylindrical ends, *Ann. Henri Poincare* **3** (2002), 895-920.

[9] E.B. Davies, P. Exner, J. Lipovsky, Non-Weyl asymptotics for quantum graphs with general coupling conditions, *J. Phys. A: Math. Theor.* **43** (2010), 474013.

[10] E.B. Davies, A. Pushnitski, Non-Weyl resonance asymptotics for quantum graphs, *Analysis and PDE* **4** (2011), 729-756.

[11] P. Duclos, P. Exner, B. Meller, Open quantum dots: Resonances from perturbed symmetry and bound states in strong magnetic fields, *Rep. Math. Phys.* **47** (2001), no. 2, 253-267.

[12] J. Edward, On the resonances of the Laplacian on waveguides, *J. Math. Anal. Appl.* **272** (2002), 89-116.

[13] P. Exner, et al. (ed.). *Analysis on Graphs and Its Applications.* Papers from the program held in Cambridge, January 8 – June 29, 2007. Proceedings of Symposia in Pure Mathematics, 77. American Mathematical Society, 2008, 705 pp.

[14] P. Exner, H. Kovarik, *Quantum waveguides*, Springer, Berlin, 2015.

[15] P. Exner, J. Lipovsky, Resonances from perturbations of quantum graphs with rationally related edges, *J. Phys. A: Math. Theor.* **43** (2010), 105301.

[16] P. Exner, J. Lipovsky, Non-Weyl resonance asymptotics for quantum graphs in a magnetic field, *Phys. Lett. A* **375** (2011), 805-807.

[17] P. Exner, V. Lotoreichik, M. Tater, On resonances and bound states of Smilansky Hamiltonian, *Nanosystems: Phys. Chem. Math.* **7** (2016), 789-802.

[18] D.A. Gerasimov, I.Y. Popov, Completeness of resonance states for quantum graph with two semi-infinite edges, *Compl. Var. Elliptic Eq.* **63** (2018), 996-1010.

[19] I.Y. Popov, I.V. Blinova, Quantum graph in a magnetic field and resonance states completeness, *Operator Theory: Adv. Appl.* **276** (2020) 540-553.

[20] T.H. Rasulov, B.I. Bahronov, Existence of the eigenvalues of a tensor sum of the Friedrichs models with rank 2 perturbation, *Nanosystems: Phys. Chem. Math.* **14** (2023), no.2, 151-157.

[21] J. Sjöstrand, M. Zworski, Complex scaling and the distribution of scattering poles, *J.Amer. Math. Soc.* **4** (1991), 729-769.

[22] A.M. Vorobiev, I.Y. Popov, E.S. Trifanova, Resonance asymptotics for a pair quantum waveguides with common semitransparent perforated wall, *Nanosystems: Phys. Chem. Math.* **11** (2020), no. 6, 619-627.

Anna G. Belolipetskaia
ITMO University, St. Petersburg, Russia
E-mail address: annabel1502@mail.ru

Igor Y. Popov
ITMO University, St. Petersburg, Russia
E-mail address: popov1955@gmail.com

Received: June 26, 2023; Revised: August 5, 2023; Accepted: August 10, 2023