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SCATTERING PROBLEM OF THE PERTURBED AIRY EQUATION ON A HALF-AXIS

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Abstract. The perturbed Airy equation $-y'' - xy + q(x)y = \lambda y$ on the half-axis $0 \le x < +\infty$ with the $y'(0) = hy(0)$ boundary condition is considered. By means of transformation operators the direct and inverse scattering problems are studied. We obtain the main integral equation of the inverse problem. An effective algorithm for reconstruction of perturbed potential is indicated.

1. Introduction

Consider the differential equation

$$-y'' - xy + q(x)y = \lambda y,\ 0 \le x < +\infty. \tag{1.1}$$

Equation (1.1) is a perturbation of the Airy equation, which has numerous applications. In [2], the scattering problem for equation (1.1) with the Dirichlet boundary condition $y\,(0) = 0$ was studied. However, in [2] there are errors associated with the solution of the inverse problem (see, for example, Lemma 5.1). Therefore, the results of [2] cannot be considered satisfactory and the study of the boundary value problem of scattering theory for (1.1) is very topical.

For equation (1.1), we set the following boundary condition:

$$y'\,(0) = hy\,(0)\,, \tag{1.2}$$

where h is an arbitrary real number (including the value $h = \infty$). We will assume that the function $q\,(x)$ is real and satisfies the condition

$$\int_0^\infty \left(1 + x^4\right) e^{2x^{\frac{3}{2}}} \left|q\,(x)\right| dx < \infty. \tag{1.3}$$

In the present work, by means of method of transformation operator, we study the scattering problem for problem (1.1)-(1.2) in the class of potentials (1.3). The inverse scattering problem (1.1)-(1.2) consists of the following parts:

a) Reconstruction. Give an algorithm for recovering the potential from the scattering data.

b) Uniqueness. Prove that the scattering data uniquely determines the potential.

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c) Characterization. Give conditions for some data to be the scattering data of some potential.

This work is devoted to the first item. We obtain a Marchenko-type integral equation for problem (1.1)–(1.2). An effective algorithm for the reconstruction of the potential $q(x)$ is indicated.

We note that the inverse scattering problem in the case without an additional linear potential has been studied in detail by many authors (see [9], [10], [12] and the references therein). In the presence of an additional linear potential, the continuous spectrum of the unperturbed equation is associated with eigenfunctions that in contrast to an exponential function do not have a multiplicative property, and it was therefore necessary to modify several classical arguments in [9], [12].

Inverse problems for the Schrödinger equation with increasing potentials were studied in [3]–[8]. Various problems of the spectral analysis for equation (1.1) and for its unperturbed case were studied in [11], [14].

2. Preliminaries and the scattering problem for the unperturbed equation

We consider the unperturbed equation

$$-y^{''}-xy=\lambda y. \tag{2.1}$$

It is well known (see, e.g., [1]) that the Airy functions $Ai\left(-x-\lambda\right)$ and $Bi\left(-x-\lambda\right)$ are linearly independent solutions of equation (2.1). Below, we need some (see [1]) asymptotic equalities related to the Airy functions:

$$\begin{aligned}&Ai\left(z\right)\sim\pi^{-\frac{1}{2}}z^{-\frac{1}{4}}e^{-\zeta}\left[1+O\left(\zeta^{-1}\right)\right],\\&Ai'\left(z\right)\sim-\pi^{-\frac{1}{2}}z^{\frac{1}{4}}e^{-\zeta}\left[1+O\left(\zeta^{-1}\right)\right],\ \left|\arg z\right|<\pi,\ z\to\infty,\end{aligned}\tag{2.2}$$

$$\begin{aligned}&Ai\left(-z\right)\sim\pi^{-\frac{1}{2}}z^{-\frac{1}{4}}\sin\left(\zeta+\tfrac{\pi}{4}\right)\left[1+O\left(\zeta^{-2}\right)\right]-\\&-\pi^{-\frac{1}{2}}z^{-\frac{1}{4}}\cos\left(\zeta+\tfrac{\pi}{4}\right)O\left(\zeta^{-1}\right),\quad\left|\arg z\right|<\tfrac{2\pi}{3},z\to\infty,\\&Ai'\left(-z\right)\sim-\pi^{-\frac{1}{2}}z^{\frac{1}{4}}\cos\left(\zeta+\tfrac{\pi}{4}\right)\left[1+O\left(\zeta^{-2}\right)\right]-\\&-\pi^{-\frac{1}{2}}z^{\frac{1}{4}}\cos\left(\zeta+\tfrac{\pi}{4}\right)O\left(\zeta^{-1}\right),\quad\left|\arg z\right|<\tfrac{2\pi}{3},z\to\infty,\end{aligned}\tag{2.3}$$

$$\begin{aligned}&Bi\left(z\right)\sim\pi^{-\frac{1}{2}}z^{-\frac{1}{4}}e^{-\zeta}\left[1+O\left(\zeta^{-1}\right)\right],\ \left|\arg z\right|<\tfrac{\pi}{3},\ z\to\infty,\\&Bi'\left(z\right)\sim\pi^{-\frac{1}{2}}z^{-\frac{1}{4}}e^{-\zeta}\left[1+O\left(\zeta^{-1}\right)\right],\ \left|\arg z\right|<\tfrac{\pi}{3},\ z\to\infty.\end{aligned}\tag{2.4}$$

$$\begin{aligned}&Bi\left(-z\right)\sim\pi^{-\frac{1}{2}}z^{-\frac{1}{4}}\cos\left(\zeta+\tfrac{\pi}{4}\right)\left[1+O\left(\zeta^{-2}\right)\right]+\\&+\pi^{-\frac{1}{2}}z^{-\frac{1}{4}}\sin\left(\zeta+\tfrac{\pi}{4}\right)O\left(\zeta^{-1}\right),\quad\left|\arg z\right|<\tfrac{2\pi}{3},\ z\to\infty.\\&Bi'\left(-z\right)\sim\pi^{-\frac{1}{2}}z^{\frac{1}{4}}\sin\left(\zeta+\tfrac{\pi}{4}\right)\left[1+O\left(\zeta^{-2}\right)\right]-\\&-\pi^{-\frac{1}{2}}z^{\frac{1}{4}}\cos\left(\zeta+\tfrac{\pi}{4}\right)O\left(\zeta^{-1}\right),\quad\left|\arg z\right|<\tfrac{2\pi}{3},\ z\to\infty.\end{aligned}\tag{2.5}$$

where $\zeta = \frac{2}{3}z^{\frac{3}{2}}$.

We also introduce the special solutions

$$f_0(x,\lambda) = \pi^{\frac{1}{2}}\left[\mathrm{A}i(-x-\lambda) - i\mathrm{B}i(-x-\lambda)\right] \tag{2.6}$$

of the unperturbed equation (2.1), which satisfies the relation (see [13]) $f_0(x,\lambda) \in L_2(0,\infty)$ for $Im\lambda > 0$. Obviously, for real values of λ another solution to equation (2.1) is $\overline{f_0(x,\lambda)} = \pi^{\frac{1}{2}}\left[\mathrm{A}i(-x-\lambda) + i\mathrm{B}i(-x-\lambda)\right]$. Moreover, solutions $f_0(x,\lambda)$ and $\overline{f_0(x,\lambda)}$ are linearly independent, because their Wronskian is equal to $-2i$ (see [13]). Denote by $\varphi_0(x,\lambda)$the solution of equation (1.1) satisfying the conditions $\varphi_0(0,\lambda) = 1$, $\varphi_0'(0,\lambda) = h$. Since for real values of λ the solution $\varphi_0(x,\lambda)$takes real values, we have

$$\frac{2i\varphi_0(x,\lambda)}{f_0'(0,\lambda) - hf_0(0,\lambda)} = \overline{f_0(x,\lambda)} - S_0(\lambda)\,f_0(x,\lambda),$$

where

$$S_0(\lambda) = \frac{\overline{f_0'(0,\lambda) - hf_0(0,\lambda)}}{f_0'(0,\lambda) - hf_0(0,\lambda)} =$$

$$= \frac{Ai'(-\lambda) + iBi'(-\lambda) - h\left[Ai(-\lambda) + iBi(-\lambda)\right]}{Ai'(-\lambda) - iBi'(-\lambda) - h\left[Ai(-\lambda) - iBi(-\lambda)\right]}. \tag{2.7}$$

In the case $h = \infty$, the last equality takes the form

$$S_0(\lambda) = \frac{Ai(-\lambda) + iBi(-\lambda)}{Ai(-\lambda) - iBi(-\lambda)}.$$

The function $S_0(\lambda)$, defined by formula (2.7) is called the scattering function for the boundary value problem (2.1), (1.2), and the function $u_0(x,\lambda) = \frac{2i\varphi_0(x,\lambda)}{f_0'(0,\lambda) - hf_0(0,\lambda)}$ is called the solution of the scattering problem. Note that the function $S_0(\lambda)$ is continuous on the real axis and satisfies the identity $|S_0(\lambda)| = 1$.

3. Scattering problem for a perturbed equation

Let us pass to the description of the scattering problem for the boundary value problem (1.1), (1.2). Differential equation (1.1), together with the boundary condition (1.2), defines in space $L_2(0,+\infty)$ a self-adjoint operator L, which can be obtained by closure of symmetric operator defined by equation (1.1) and the boundary condition (1.2) on twice continuously differentiable compactly supported functions. It is known that [14] the spectrum of the operator L is continuous and fills the entire real axis, i.e., real values of the energy λ are associated with the continuous spectrum of problem (1.1), (1.2). The eigenfunctions of the continuous spectrum are determined by asymptotic condition:

$$f(x,\lambda) = f_0(x,\lambda) + o(1),\, x \to \infty.$$

Let $\sigma_i(x) = \int\limits_x^\infty t^i e^{2t^{\frac{3}{2}}}|q(t)|\,dt$, $i = 0,1$. It follows from [7] under condition (1.3), that the solution $f(x,\lambda)$ satisfies triangular representations, which demonstrate the scattering effect,

$$f(x,\lambda) = f_0(x,\lambda) + \int_x^\infty K(x,t)f_0(t,\lambda)dt. \tag{3.1}$$

The kernel $K(x,t)$ is real and satisfies the relations

$$|K(x,t)| \le \frac{1}{2}\sigma_0(\frac{x+t}{2})e^{\sigma_1(x)}, \tag{3.2}$$

$$K(x,x) = \frac{1}{2}\int_x^\infty q(t)\,dt. \tag{3.3}$$

Moreover, the kernel $K(x,t)$ is a continuously differentiable function and for any fixed x, $K(x,y)$ belongs to $L_1(0,\infty)$. From the formula (3.1) we obtain

$$f'(0,\lambda) = f_0'(0,\lambda) - K(0,0)\,f_0(0,\lambda) + \int_0^\infty K_x'(0,t)f_0(t,\lambda)dt \tag{3.4}$$

We introduce a solution $\varphi(x,\lambda)$ to equation (1.1) that satisfies the conditions $\varphi(0,\lambda) = 1$, $\varphi'(0,\lambda) = h$. Let us study the connection between solutions $\varphi(x,\lambda)$ and $f(x,\lambda)$. Because the function $q(x)$ is real, for real values of λ the function $\overline{f(x,\lambda)}$ is also a solution of equation (1.1). Moreover, the solutions $f(x,\lambda)$ and $\overline{f(x,\lambda)}$ are linearly independent, because their Wronskian is equal to $-2i$ (see [7]). Because $\varphi(x,\lambda)$ is real, we have

$$\frac{2i\varphi(x,\lambda)}{f'(0,\lambda) - hf(0,\lambda)} = \overline{f(x,\lambda)} - S(\lambda)\,f(x,\lambda)\,,\ Im\lambda = 0. \tag{3.5}$$

and the coefficient $S(\lambda)$ is given by

$$S(\lambda) = \frac{\overline{f'(0,\lambda) - hf(0,\lambda)}}{f'(0,\lambda) - hf(0,\lambda)}. \tag{3.6}$$

In the case $h = \infty$, the last equality takes the form

$$S(\lambda) = \frac{\overline{f(0,\lambda)}}{f(0,\lambda)}. \tag{3.7}$$

It follows from (3.4)-(3.6), (3.7) that the function $S(\lambda)$ is continuous on the real axis and $|S(\lambda)| = 1$.

Consider the difference $S_0(\lambda) - S(\lambda)$. Using then relations (2.2)-(2.6), (3.6) (3.7), we find that

$$S_0(\lambda) - S(\lambda) = O\left(\lambda^{-\frac{1}{2}}\right),\ \lambda \to \pm\infty. \tag{3.8}$$

Let us deduce the formula for the expansion in eigenfunctions of the continuous spectrum for problem (1.1), (1.2). We consider the equation

$$-y'' + xy + q(x) - \lambda y = h(x)\,,$$

where $h(x)$ is an arbitrary function in $L_2(0,\infty)$ for which we must obtain an expansion. Following Titchmarsh's arguments [14], we further deduce that the functions $\varphi(x,\lambda)$ and $\psi(x,\lambda)$, defined up to a factor in the general theory [14], respectively coincide with $\varphi(x,\lambda)$ and $f(x,\lambda)$. Therefore, the corresponding Green's function $G(x,y,\lambda)$ has the form

$$G(x,y,\lambda) = \begin{cases} \frac{f(x,\lambda)\varphi(y,\lambda)}{f'(0,\lambda)-hf(0,\lambda)}, y \le x, \\ \\ \frac{\varphi(x,\lambda)f(y,\lambda)}{f'(0,\lambda)-hf(0,\lambda)}, y > x \end{cases} \tag{3.9}$$

Moreover, we have the expansion formula [14]

$$h(x) = -\frac{1}{\pi}\int_{-\infty}^{+\infty} Im\Phi(x,\lambda)\,d\lambda = \frac{1}{\pi}\int_{-\infty}^{+\infty} Re\{i\Phi(x,\lambda)\}\,d\lambda. \tag{3.10}$$

where

$$\Phi(x,\lambda) = \int_0^{\infty} G(x,y,\lambda)\,h(y)\,dy.$$

We note that

$$\begin{aligned} &Re\left\{\left[\overline{f(y,\lambda)} - S(\lambda) f(y,\lambda)\right] f(x,\lambda)\right\} = \\ &= Re\left\{\left[\overline{f(x,\lambda)} - S(\lambda) f(x,\lambda)\right] f(y,\lambda) + \left[\overline{f(y,\lambda)} f(x,\lambda) - \overline{f(x,\lambda)} f(y,\lambda)\right]\right\} = \\ &= Re\left\{\left[\overline{f(x,\lambda)} - S(\lambda) f(x,\lambda)\right] f(y,\lambda)\right\}\end{aligned}$$

The last equality together with relations (3.5), (3.9), (3.10) leads us to the expansion formula

$$h(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(y)\,Re\left\{\left[\overline{f(x,\lambda)} - S(\lambda) f(x,\lambda)\right] f(y,\lambda)\right\} dy d\lambda$$

or

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} Re\left\{\left[\overline{f(x,\lambda)} - S(\lambda) f(x,\lambda)\right] f(y,\lambda)\right\} d\lambda = \delta(x-y), \tag{3.11}$$

where δ is the Dirac delta function. In the case when $q(x) = 0$, the last formula takes the form

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} Re\left\{\left[\overline{f_0(x,\lambda)} - S_0(\lambda) f_0(x,\lambda)\right] f_0(y,\lambda)\right\} d\lambda = \delta(x-y). \tag{3.12}$$

Let us turn to the study of the inverse scattering problem. The inverse scattering problem for problem (1.1), (1.2) consists in reconstructing the potential $q(x)$ from the scattering function $S(\lambda)$. An important role in the solution of the inverse problem is played by the Marchenko-type integral equation.

Consider a function

$$F(x,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} Re\{[S_0(\lambda) - S(\lambda)]\, f_0(x,\lambda)\, f_0(y,\lambda)\}\,d\lambda. \tag{3.13}$$

It follows from relations (2.2)-(2.6) that the improper integral (3.13) converges and for any fixed x, $F(x,y)$ belongs to $L_2(0,\infty)$. Moreover, the function $F(x,y)$ is obviously symmetrical in its arguments.

Theorem 3.1. *For each fixed $x \geq 0$, the function $K(x,y)$ appearing in representation* (3.1) *satisfies the integral equation*

$$F(x,y) + K(x,y) + \int_x^{+\infty} K(x,t)\,F(t,y)\,dt = 0, y > x. \tag{3.14}$$

Proof. It follows from the well-known properties of the transformation operators (see,e.g., [12]) and representations (3.1) that

$$f_0(y,\lambda) = f(y,\lambda) + \int_y^{\infty} K_0(y,t) f(t,\lambda) dt,$$

where the kernel $K_0(y,t)$ satisfies a relation similar to (3.2). The last formula with regard to (3.11) then implies

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} Re\left\{\left[\overline{f(x,\lambda)}-S(\lambda)f(x,\lambda)\right]f_0(y,\lambda)\right\}d\lambda=$$

$$=\delta(x-y)+\int_y^{\infty}K_0(y,t)\left(\frac{1}{2\pi}\int_{-\infty}^{\infty}Re\left\{\left[\overline{f(x,\lambda)}-S(\lambda)f(x,\lambda)\right]f(t,\lambda)\right\}d\lambda\right)dt=$$

$$=\delta(x-y)+\int_y^{\infty}K_0(y,t)\,\delta(x-t)\,dt=\delta(x-y)+K_0(y,x)=\delta(x-y),$$

since $K_0(y,x)=0$ for $x<y$. On the other hand, using (3.1) and (3.12), we obtain

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} Re\left\{\left[\overline{f(x,\lambda)}-S(\lambda)f(x,\lambda)\right]f_0(y,\lambda)\right\}d\lambda=$$

$$=\frac{1}{2\pi}\int_{-\infty}^{\infty} Re\left\{\left[\overline{f_0(x,\lambda)}-S_0(\lambda)f_0(x,\lambda)\right]f_0(y,\lambda)\right\}d\lambda+$$

$$+\int_x^{\infty}K(x,t)\left\{\frac{1}{2\pi}\int_{-\infty}^{\infty}Re\left\{\left[\overline{f_0(t,\lambda)}-S_0(\lambda)f_0(t,\lambda)\right]f_0(y,\lambda)\right\}d\lambda\right\}dt-$$

$$-\frac{1}{2\pi}\int_{-\infty}^{\infty}Re\left\{\left[S(\lambda)-S_0(\lambda)\right]f_0(x,\lambda)f_0(y,\lambda)\right\}d\lambda-$$

$$-\int_x^{\infty}K(x,t)\left\{\frac{1}{2\pi}\int_{-\infty}^{\infty}Re\left\{\left[S(\lambda)-S_0(\lambda)\right]f_0(t,\lambda)f_0(y,\lambda)\right\}d\lambda\right\}dt=$$

$$=\delta(x-y)+\int_x^{\infty}K(x,t)\,\delta(t-y)\,dt-$$

$$-\frac{1}{2\pi}\int_{-\infty}^{\infty}Re\left\{\left[S(\lambda)-S_0(\lambda)\right]f_0(x,\lambda)f_0(y,\lambda)\right\}d\lambda-$$

$$-\int_x^{\infty}K(x,t)\left\{\frac{1}{2\pi}\int_{-\infty}^{\infty}Re\left\{\left[S(\lambda)-S_0(\lambda)\right]f_0(t,\lambda)f_0(y,\lambda)\right\}d\lambda\right\}dt=$$

$$=\delta(x-y)+K(x,y)+F(x,y)+\int_x^{\infty}K(x,t)\,F(t,y)\,dt.$$

Comparing the last two equalities, we obtain the integral equation (3.14). □

Equation (3.14) is called the main equation. Under certain conditions imposed on the function $F(x,y)$, the main equation is uniquely solvable in the space $L_p(x,\infty), p=1,2$. In conclusion, we note that the solution of the inverse problem can be constructed by the following algorithm.

Algorithm. Let the scattering function $S(\lambda)$ be given. Then:
1. Calculate the function $F(x,y)$ by (3.13).
2. Find $K(x,y)$ by solving the main equation (3.14).
3. Construct $q(x)$ by (3.3), i.e. $q(x)=-2\frac{dK(x,x)}{dx}$.

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