Proceedings of the Institute of Mathematics and Mechanics,
National Academy of Sciences of Azerbaijan
Volume 00, Number 0, XXXX, Pages 000–000

TRIANGULAR REPRESENTATION OF THE JOST-TYPE SOLUTION TO THE PERTURBED MODIFIED MATHIEU EQUATION

AGIL KH. KHANMAMEDOV AND GUNAY F. RZAYEVA

Abstract. The perturbed modified Mathieu equation $-y'' + (chx)\,y + q\,(x)\,y = \lambda y$ on the half-axis $0 \le x < \infty$ is considered. By means of transformation operator an integral representation of the Jost-type solution is found. An estimate is obtained with respect to the kernel of the transformation operator. A connection is established between the kernel of the integral representation and the perturbation potential.

1. Introduction

The Mathieu differential equation often arises in solving scientific and engineering problems (see [1], [5], [6], [8], [13], [15]). The most interesting example is the problem of oscillations of an elliptical membrane (see [5], [6]). The Mathieu equations also arise when studying the propagation of electromagnetic waves in an elliptical cylinder, when considering surface waves of a liquid in a vessel having the shape of an elliptical cylinder, and when solving a number of other issues (see [13], [15]).

Consider the following perturbed modified Mathieu equation

$$-y'' + (chx)\,y + q\,(x)\,y = \lambda y,\ \ 0 < x < \infty, \quad \lambda \in C, \tag{1.1}$$

where the real potential $q\,(x)$satisfies the conditions

$$q\,(x) \in C^{(1)}\,[0, \infty]\,, \quad \int_0^\infty e^x\,|q\,(x)|\,dx < \infty. \tag{1.2}$$

Note that equation (1.1) is a one-dimensional Schrödinger equation with an additional exponentially growing potential. The last equation is of particular interest from a physical point of view (see [2]).

It is known [13] that equation (1.1) with $q\,(x) = 0$ has a special solution $f_0\,(x, \lambda)$, which can be represented as

$$f_0\,(x, \lambda) = \left(\sqrt{2}e^{\frac{x}{2}}\right)^{-1} e^{-\sqrt{2}e^{\frac{x}{2}}} \sum_{r=0}^{\infty} (-1)^r\, c_r \left(\sqrt{2}e^{\frac{x}{2}}\right)^{-r}, \tag{1.3}$$

2010 *Mathematics Subject Classification.* 34A55, 34L40.

Key words and phrases. perturbed equation, Mathieu equation, Schrödinger equation, transformation operator, Jost-type solution, Riemann function.

where

$$c_0 = 1,\ c_1 = \tfrac{4\lambda+1^2}{8},\ c_2 = \tfrac{(4\lambda+1^2)(4\lambda+3^2)}{8^2 2!},$$

$$c_3 = \tfrac{(4\lambda+1^2)(4\lambda+3^2)(4\lambda+5^2)}{8^3 3!} - \tfrac{4}{3!}, \tag{1.4}$$

$$c_4 = \tfrac{(4\lambda+1^2)(4\lambda+3^2)(4\lambda+5^2)(4\lambda+7^2)}{8^4 4!} - \tfrac{2}{4!}(4\lambda+13), ...$$

Moreover, for each x the solution $f_0(x,\lambda)$ is an entire function with respect to λ. Note that, by virtue of (1.3), (1.4), for each fixed λ the function $f_0(x,\lambda)$ belongs to the space $L_2(0,+\infty)$. Of particular interest is the Jost-type solution of the perturbed equation (1.1), i.e., solution $f(x,\lambda)$ that satisfies the condition $f(x,\lambda) = f_0(x,\lambda)[1+o(1)]$, $x \to \infty$.

In the present paper, by means of the transformation operator, we find a representation of a special Jost-type solution $f(x,\lambda)$ of equation (1.1). When obtaining an integral equation for the kernel of the representation, it turned out to be natural to use the Riemann function method. The results of this paper can be used to study various spectral problems for equation (1.1).

Note that a transformation operator with a condition at infinity is constructed in [10], [11] for the one-dimensional Schrödinger equation with a rapidly decreasing potential. At the same time, for the Schrödinger equations with unbounded potentials, the construction of transformation operators encounters significant difficulties in comparison with a rapidly decreasing potential. In this direction, we note the works [4], [7], [12], [14], in which were studied the transformation operators for the Schrödinger equation with an additional potential of the form cx^α, $\alpha = 1, 2$.

Note that the usual Mathieu equations are special cases of the Hill equation. For perturbations of the last equation, the transformation operators are studied in detail(see [9] and references therein)

Let

$$\sigma_0(x) = \int_x^\infty |q(t)|\,dt,\ \ \sigma_1(x) = \int_x^\infty \sigma_0(t)\,dt\,. \tag{1.5}$$

The main result of the present paper is as follows.

Theorem 1.1. *If the potential $q(x)$ satisfies condition (1.2), then for all values of λ, the equation (1.1) has solution $f(x,\lambda)$ representable as*

$$f(x,\lambda) = f_0(x,\lambda) + \int_x^{+\infty} K(x,t)\ f_0(t,\lambda)\,dt, \tag{1.6}$$

where the kernel $K(x,t)$ is a continuously differentiable function and satisfies the following conditions:

$$|K(x,t)| \le \frac{1}{2}\sigma_0\left(\frac{x+t}{2}\right)e^{\sigma_1(x)}, \tag{1.7}$$

$$K(x,x) = \frac{1}{2}\int_x^{+\infty} q\ (t)\,dt. \tag{1.8}$$

2. Proof of the Theorem

Substituting the representation (1.6) into Eq. (1.1), we can obtain for the kernel $K(x,t)$ the following problem:

$$\frac{\partial K(x,t)}{\partial x^2} - \frac{\partial K(x,t)}{\partial t^2} - (chx - cht + q(x)) \;\; K(x,t) = 0, \; 0 < x < t, \tag{2.1}$$

$$K(x,x) = \frac{1}{2}\int_x^{\infty} q\,(t)\,dt, \tag{2.2}$$

$$\lim_{x+t\to\infty} K(x,t) = 0. \tag{2.3}$$

Setting $U(\xi,\eta) = U\left(\frac{t+x}{2},\frac{t-x}{2}\right) = K(x,t) = K(\xi-\eta,\xi+\eta)$, we reduce the problem (2.1) -(2.3) to the problem

$$L[U] \equiv \frac{\partial^2 U(\xi,\eta)}{\partial\xi\partial\eta} - 2sh\xi \cdot sh\eta U(\xi,\eta) = -U(\xi,\eta)\,q(\xi-\eta) \tag{2.4}$$

$$U(\xi,0) = \frac{1}{2}\int_{\xi}^{\infty} q(\alpha)\,d\alpha, \tag{2.5}$$

$$\lim_{\xi\to\infty} U(\xi,\eta) = 0, \; \eta > 0. \tag{2.6}$$

Let $\psi(\xi,\eta) = -U(\xi,\eta)q(\xi-\eta)$. Denote by $R(\xi,\eta;\xi_0,\eta_0)$ the Riemann function of the equation $L[U] = \psi(\xi,\eta)$, i.e., a function satisfying the equation

$$L^*(R) \equiv \frac{\partial^2 R}{\partial\xi\,\partial\eta} - 2sh\xi \cdot sh\eta \cdot R = 0, 0 < \eta < \eta_0 \le \xi_0 < \xi < \infty \tag{2.7}$$

and the following conditions on the characteristics:

$$R(\xi,\eta;\xi_0,\eta_0)\,|_{\xi=\xi_0} = 1, 0 \le \eta \le \eta_0, \tag{2.8}$$

$$R(\xi,\eta;\xi_0,\eta_0)\,|_{\eta=\eta_0} = 1, \; \xi_0 \le \xi < \infty. \tag{2.9}$$

It is easy to check that the function $R(\xi,\eta,\xi_0,\eta_0)$, defined by the formula

$$R(\xi,\eta,\xi_0,\eta_0) = J_0(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n!)^2}\left(\frac{z}{2}\right)^{2n}, \tag{2.10}$$

where $J_n(z)$ is the Bessel function of the first kind and

$$z = \sqrt{8\,(ch\xi - ch\xi_0)\,(ch\eta_0 - ch\eta)}, \; \xi_0 < \xi < \infty, \; 0 < \eta < \eta_0, \tag{2.11}$$

satisfies relations (2.8)-(2.9). On the other hand, from (2.10), (2.11) it follows that

$$\frac{\partial R}{\partial\xi} = 4sh\xi\,(ch\eta_0 - ch\eta)\,J_0'(z)\,z^{-1},$$
$$\frac{\partial^2 R}{\partial\xi\partial\eta} = -2sh\xi \cdot sh\eta J_0''(z) - 2sh\xi \cdot sh\eta J_0'(z)\,z^{-1},$$

whence we have

$$\frac{\partial^2 R}{\partial\xi\partial\eta} - 2sh\xi \cdot sh\eta\,R = -2sh\xi \cdot sh\eta\left(J_0''(z) + J_0'(z)\,z^{-1} + J_0(z)\right) = 0,$$

i.e., the function (2.10) is the Riemann function of Eq. (2.4). Further, since z takes real values, then, by virtue of the well-known inequality$|J_n(z)| \le 1$ (see [1]) we conclude that for all $\xi_0 \le \xi < \infty, \; 0 \le \eta \le \eta_0$ the following inequality holds:

$$|R| \le 1. \tag{2.12}$$

On other hand, using the known [1] relations $J_{n+1}(z)+J_{n-1}(z)=\frac{2n}{z}J_n(z)$, we obtain $\left|\frac{J_1(z)}{z}\right|\le 1$, $J_0''(z)-J_0'(z)\,z^{-1}=J_2(z)$. Differentiating (2.11) and taking into account the last relations, we can write

$$\frac{\partial R}{\partial \xi}=-4sh\xi\,(ch\eta_0-ch\eta)\,J_1(z)\,z^{-1},\ \frac{\partial R}{\partial \eta}=4\,(ch\xi-ch\xi_0)\ sh\eta J_1(z)\,z^{-1},$$

$$\frac{\partial^2 R}{\partial \xi^2}=-4ch\xi\,(ch\eta_0-ch\eta)\,J_1(z)\,z^{-1}+2sh^2\xi\frac{ch\eta_0-ch\eta}{ch\xi-ch\xi_0}J_2(z)\,,$$

$$\frac{\partial^2 R}{\partial \eta^2}=4ch\eta\,(ch\xi-ch\xi_0)\,J_1(z)\,z^{-1}+2sh^2\eta\frac{ch\xi-ch\xi_0}{ch\eta_0-ch\eta}J_2(z)\,,$$

$$\frac{\partial^2 R}{\partial \xi\partial\eta}=4sh\xi\cdot sh\eta\left[J_1(z)\,z^{-1}-\frac{1}{2}J_2(z)\right].$$

Hence it follows that for each $\eta\in(0,\,\eta_0)$ the following relations hold:

$$\begin{aligned}&\tfrac{\partial R}{\partial \xi}=O\left(e^{\xi}\right),\ \tfrac{\partial R}{\partial \eta}=O\left(e^{\xi}\right),\ \tfrac{\partial^2 R}{\partial \xi\partial\eta}=O\left(e^{\xi}\right),\xi\to\infty,\\ &\tfrac{\partial^2 R}{\partial \xi^2}=O\left(e^{\xi}\right),\ \tfrac{\partial^2 R}{\partial \eta^2}=O\left(e^{\xi}\right),\ \xi\to\infty.\end{aligned}\tag{2.13}$$

Applying the Riemann method (see, e.g., [3]) to Eq. (2.4), we obtain the following integral equation for $U(\xi_0,\eta_0)$:

$$\begin{aligned}U(\xi_0,\eta_0)=\frac{1}{2}\int_{\xi_0}^{\infty}R(\xi,0,\xi_0,\eta_0)\,q(\xi)\,d\xi-\\ -\int_{\xi_0}^{\infty}d\xi\int_0^{\eta_0}U(\xi,\eta)\,R(\xi,\eta,\xi_0,\eta_0)\,q(\xi-\eta)\,d\eta.\end{aligned}\tag{2.14}$$

Let us now deal with the question of the solvability of the integral equation (2.14). We use the method of successive approximations. Let us put

$$U_0(\xi_0,\eta_0)=\frac{1}{2}\int_{\xi_0}^{\infty}R(\xi,0;\xi_0,\eta_0)q(\xi)d\xi,$$

$$U_n(\xi_0,\eta_0)=-\int_{\xi_0}^{\infty}d\xi\int_0^{\eta_0}U_{n-1}(\xi,\eta)q(\xi-\eta)R(\xi,\eta;\xi_0,\eta_0)d\eta.$$

From (2.12) it follows that

$$|U_0(\xi_0,\eta_0)|\le\frac{1}{2}\int_{\xi_0}^{\infty}|\,R(\xi,0;\xi_0,\eta_0)\,|\ |q(\xi)|\,d\xi\le\frac{1}{2}\int_{\xi_0}^{\infty}|q(\xi)|\ d\xi,$$

since $\xi>\xi_0$, $\eta<\eta_0$. Then, we will have

$$|U_0(\xi_0,\eta_0)|\le\frac{1}{2}\sigma_0(\xi_0)\,.$$

Further, since the function $\sigma_0(\xi)$ is monotonically decreasing, taking into account (1.5), we find that

$$|U_1(\xi_0,\eta_0)|\le\int_{\xi_0}^{\infty}d\xi\int_0^{\eta_0}|U_0(\xi,\eta)|\cdot|q(\xi-\eta)|\cdot R(\xi,\eta;\,\xi_0,\eta_0)\,d\eta\le$$

$$\le\frac{1}{2}\int_{\xi_0}^{\infty}d\xi\int_0^{\eta_0}\sigma_0(\xi)\ \,|q(\xi-\eta)|\,d\eta\le\frac{1}{2}\int_{\xi_0}^{\infty}\sigma_0(\xi)\,d\xi\int_0^{\eta_0}|q(\xi-\eta)|\ d\eta\le$$

$$\leq \frac{\sigma_0(\xi_0)}{2}\int_{\xi_0}^{\infty}d\xi\int_0^{\eta_0}|q(\xi-\eta)|\,d\eta = \frac{\sigma_0(\xi_0)}{2}\int_{\xi_0}^{\infty}d\xi\int_{\xi-\eta_0}^{\xi}|q(\alpha)|\,d\alpha \leq$$

$$\leq \frac{\sigma_0(\xi_0)}{2}\int_{\xi_0}^{\infty}d\xi\int_{\xi-\eta_0}^{\infty}|q(\alpha)|\,d\alpha \leq$$

$$\leq \frac{\sigma_0(\xi_0)}{2}\int_{\xi_0}^{\infty}\sigma_0(\xi-\eta_0)\,d\xi = \frac{\sigma_0(\xi_0)}{2}\sigma_1(\xi_0-\eta_0).$$

Let now

$$|U_{n-1}(\xi_0,\eta_0)| \leq \frac{1}{2}\sigma(\xi_0)\frac{(\sigma_1(\xi_0-\eta_0))^{n-1}}{(n-1)!}.$$

In this case, we will have

$$|U_n(\xi_0,\eta_0)| \leq \int_{\xi_0}^{\infty}d\xi\int_0^{\eta_0}|q(\xi-\eta)R(\xi,\eta,\,\xi_0,\eta_0)U_{n-1}(\xi,\eta)|\,d\eta \leq$$

$$\frac{1}{2}\sigma_0(\xi_0)\int_{\xi_0}^{\infty}\frac{(\sigma_1(\xi-\eta_0))^{n-1}}{(n-1)!}\int_{\xi-\eta_0}^{\xi}|q(\alpha)|\,d\alpha d\xi \leq$$

$$\leq \frac{1}{2}\sigma_0(\xi_0)\int_{\xi_0}^{\infty}\frac{(\sigma_1(\xi-\eta_0))^{n-1}}{(n-1)!}\int_{\xi-\eta_0}^{\infty}|q(\alpha)|\,d\alpha d\xi \leq$$

$$= -\frac{1}{2}\sigma_0(\xi_0)\int_{\xi_0}^{\infty}\frac{(\sigma_1(\xi-\eta_0))^{n-1}}{(n-1)!}d\sigma_1(\xi-\eta_0) = \frac{1}{2}\sigma_0(\xi_0)\frac{(\sigma_1(\xi_0-\eta_0))^{n}}{n!}.$$

Hence, it obviously follows that the series $U(\xi_0,\eta_0) = \sum\limits_{n=0}^{\infty}U_n(\xi_0,\eta_0)$ is absolutely and uniformly convergent, its sum is a solution of (2.14) and $U(\xi_0,\eta_0)$ satisfies the inequality

$$|U(\xi_0,\eta_0)| \leq \frac{1}{2}\sigma_0(\xi_0)\,e^{\sigma_1(\xi_0-\eta_0)}. \tag{2.15}$$

Further, differentiating equations (2.14) and using (1.2), (2.13), and also taking into account the identity $R(\xi,\eta,\xi_0,\eta_0) = R(\xi_0,\eta_0,\xi,\eta)$, we find that the function $U(\xi_0,\eta_0)$ is twice continuously differentiable in the domain $0 \leq \eta_0 \leq \xi_0 < \infty$ and the relations

$$\begin{gathered}\tfrac{\partial U}{\partial \xi_0} = O\left(e^{\xi_0+\eta_0}\right),\ \tfrac{\partial U}{\partial \eta_0} = O\left(e^{\xi_0+\eta_0}\right),\ \tfrac{\partial^2 U}{\partial \xi_0\partial\eta_0} = O\left(e^{\xi_0+\eta_0}\right),\\ \tfrac{\partial^2 U}{\partial \xi_0^2} = O\left(e^{\xi_0+\eta_0}\right),\ \tfrac{\partial^2 U}{\partial \eta_0^2} = O\left(e^{\xi_0+\eta_0}\right),\ \xi_0+\eta_0\to\infty.\end{gathered} \tag{2.16}$$

From here and from (2.15) it follows that the function $K(x,t) = U\left(\frac{t+x}{2},\frac{t-x}{2}\right)$ is twice continuously differentiable in the domain $0 < x \leq t < \infty$ and relations (1.7), (1.8) are true. Moreover, due to (2.16), for each fixed x the relations

$$\frac{\partial K(x,t)}{\partial x} = O\left(e^t\right),\ \frac{\partial K(x,t)}{\partial t} = O\left(e^t\right),$$

$$\frac{\partial^2 K(x,t)}{\partial x^2} = O\left(e^t\right),\ \frac{\partial^2 K(x,t)}{\partial t^2} = O\left(e^t\right), t\to\infty.$$

are true. From here and from (2.15) it follows that the function $K(x,t) = U\left(\frac{t+x}{2},\frac{t-x}{2}\right)$ satisfies problem (2.1)-(2.3). This completes the proof of the theorem.

Acknowledgments. The authors express their deep gratitude to the referee for the useful comments that helped to improve this paper.

References

[1] M. Abramowitz and I.A. Stegun, *Handbook of Mathematical Funcions with Formulas, Graphs, and Mathematical Tables*, Natl. Bur. Stand. Appl. Math. Ser. 55, U.S. Government Printing Office, Washington, 1964.

[2] A.D. Alhaydari, Exponential confining potential well, *Theoret. and Math. Phys.,* **206** (2021), no.1, 84–96.

[3] R. Courant, D. Hilbert, *Methods of Mathematical Physics. Vol. II: Partial Differential Equations*, Interscience Publ., New York, 1962.

[4] M.G. Gasymov and B.A. Mustafaev, Inverse scattering problem for an anharmonic equation on the semiaxis. *Dokl. Akad. Nauk SSSR,* **228** (1976), no.1, 11–14.

[5] N.S. Grigoreva, Uniform asymptotic expansions of solutions of the Mathieu equation and the modified Mathieu equation *J. Soviet Math.*, **11** (1979), no.5, 700-721.

[6] C. Hunter and B. Guerrieri, The eigenvalues of ME and their branch points, *Studies in Appl. Math.*, **64** (1981), 113-141.

[7] A.P. Kachalov and Ya. V. Kurylev, Transformation operator method for the inverse scattering problem, "One-dimensional Stark effect", *J. Soviet Math.* **57** (1991), no. 3, 3111–3122.

[8] A.V. Kashevarov, Mathieu functions and coulomb spheroidal functions in the electrostatic probe theory, *Comput. Math. Math. Physics*, **51** (2011), 2137–2145.

[9] A.Kh. Khanmamedov, A.F. Mamedova, A Remark on the Inverse Scattering Problem for the Perturbed Hill Equation, *Math. Notes*, **112** (2022), no.2, 281–285

[10] B. M. Levitan, *Inverse Sturm–Liouville Problems. Translated from Russian by O. Efimov*, VSP, Zeist, 1987.

[11] V. A. Marchenko, *Sturm–Liouville Operators and Applications. Translated from Russian by A. Jacob*, Oper. Theory Adv. Appl. 22, Birkhauser, Basel, 1986.

[12] G. M. Masmaliev and A. K. Khanmamedov, Transformation operators for a perturbed harmonic oscillator, *Mat. Zametki*, **105** (2019), no.5, 740–746.

[13] N.W. McLachlan, *Theory and application of Mathieu functions*, Oxford., 1947.

[14] D.H. Orucov, On the transformation operator for the Schrodinger equation with an additional increasing potential, *Proc. Inst. Math. Mech. Natl. Acad. Sci. Azerb.* **48** (2022), no.2, 311–322.

[15] L. Ruby, Applications of the MF, *Am. J. Phys.,* **64** (1996), 39-44.

Agil Kh. Khanmamedov
Institute of Mathematics and Mechanics, B. Vahabzadeh 9, AZ1141, Baku, Azerbaijan
Baku State University, Z. Khalilov 23, AZ1148, Baku, Azerbaijan
Azerbaijan University, AZ1007, Baku, Azerbaijan
Baku Engineerig University, Khirdalan AZ 0101, Baku, Azerbaijan
E-mail address: agil_khanmamedov@yahoo.com

Gunay F. Rzayeva
Sumgayit State University, Baku Street 1, AZ5008, Sumgayit, Azerbaijan
E-mail address: haciyevag84@mail.ru

Received: May 16, 2023; Revised: September 6, 2023; Accepted: September 18, 2023