Proceedings of the Institute of Mathematics and Mechanics,
National Academy of Sciences of Azerbaijan
Volume 00, Number 0, XXXX, Pages 000–000

BALAYAGE THEOREMS FOR CONNECTEDNESS PROBLEMS IN UNIFORMLY CONVEX SPACES

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Abstract. For sets with connected values of the operator of best approximation (in particular, for Chebyshev sets), we study connectedness of their intersections with closed balls. For a given set M, we introduce M-acting points and M-uniformly convex spaces. Given a set M, a point s of the unit sphere S is called an M-acting point if $s \in (P_M x - x)/\rho(x, P_M x)$ for some $x \notin M$, where $\rho(x, M)$ is the distance from x to M, and $P_M x$ is the set of all best approximants from M to x. We show that, in many problems of geometric approximation theory, it suffices to consider not the entire unit sphere, but only its M-acting points. In particular, if the metric projection P_M onto M has connected values (for example, M is a Chebyshev set), the space is complete and M-uniformly convex, then the set M has connected intersections with closed balls.

0. Introduction

Below, X is a real normed linear space. We will follow the definitions from [6], [8]. The main definitions are given below. Next:

$B(x, r)$ is the closed ball with centre x and radius r;

$\mathring{B}(x, r)$ is the open ball with centre x and radius r;

$S(x, r)$ is the sphere with centre x and radius r.

For brevity, $B := B(0, 1)$ is the unit ball, $S = S(0, 1)$ is the unit sphere.

The best approximation (the distance) from a given element x in a normed linear space X to a given nonempty set $M \subset X = (X, \|\cdot\|)$ is defined by

$$\rho(x, M) := \inf_{y \in M} \|x - y\|.$$

A point $y \in M$ is a nearest point for a point $x \in X$ from N if $\|x - y\| = \rho(x, M)$. The set of all nearest points (elements of best approximation) from M for a given x is denoted by $P_M x$, i.e.,

$$P_M x := \big\{ y \in M \mid \|x - y\| = \rho(x, M) \big\}.$$

(the operator P_M is known as the metric projection operator to the set M).

2010 *Mathematics Subject Classification.* 41A65.

Key words and phrases. M-acting point, M-uniformly convex space, B-connectedness set, Chebyshev set.

If Q denotes some property (for example, "connectedness"), we say that a closed set M has the property

P-Q if for all $x \in X$ the set $P_M x$ is nonempty and has the property Q;

P_0-Q if $P_M x$ has the property Q for all $x \in X$;

B-Q if $M \cap B(x, r)$ has the property Q for all $x \in X$, $r > 0$;

$\mathring{B}$-Q if $M \cap \mathring{B}(x, r)$ has the property Q for all $x \in X$, $r > 0$.

For example, a closed subset of a finite-dimensional space is P-nonempty, i.e., is an existence set; M is P_0-connected if, for each x, the set $P_M x$ of its nearest points is connected (or empty); M is $\mathring{B}$-connected if its intersection with any open ball is connected.

It is well known (see, for example, [15]) that $\mathring{B}$-connectedness of a set implies its connectedness and local connectedness. It is easily verified (see, for example, [20]) that a B-connected set is $\mathring{B}$-connected. The converse implication may fail to hold for closed sets. For further details, see, for example, [15], [9], [2], and [19].

Vlasov [20] proved that, in a complete uniformly convex space, any P-connected set is B-connected (in particular, any Chebyshev set[1] is B-connected). Ch. Dunham constructed a disconnected Chebyshev set in $C[0, 1]$ (see [6, § 7.3]). Tsar'kov [15] showed that, in a Efimov–Stechkin space, any closed P_0-connected set is $\mathring{B}$-connected. For further advances, see also § § 2 and 3 below.

The purpose of the paper is to show that, in many problems of geometric approximation theory, it suffices to consider not the entire unit sphere, but rather its subset consisting of the M-acting points (see Definition 1.1). Such results can be looked upon as balayage theorems of geometric approximation theory, whose remote classical ancestors are the Fermat's Rule from calculus and the Chebyshev equioscillation theorem from analytic approximation theory (see, for example, [13]). In results of this kind, one gets rid of the unnecessary points in the domain of a given functional without chancing its optimal value. For recent results on balayage type theorems in geometric approximation theory, see [3] and [4].

It is well-known that in a reflexive (LUR)-space any P_0-connected set is $\mathring{B}$-connected (see, for example, [15, § 3]). Below, we obtain a balayage type theorem for this result. Namely, given a set M, we show that in the problem of B-connectedness of M it suffices to test for uniform convexity only M-acting points of the unit sphere, rather than the entire sphere (Theorem 2.1). We also give examples which illustrate how this theorem works. Density properties of points of uniqueness are studied in § 3.

1. M-acting points and M-spaces

Let us recall the definition of an M-acting point.

Definition 1.1 (see [1]). Given a nonempty set $M \subset X$, a point $s \in S$ is called an *M-acting point* (here, "M" means the set M under consideration) if

$$s \in (P_M x - x)/\rho(x, P_M x) \quad \text{for some} \quad x \notin M.$$

This means that if s is an M-acting point for a set M, then some ball $B(x, r)$ "touches" the set M by an analogue $y \in S(x, r)$ of the point s, i.e., $s = (y - x)/r$.

[1] A set $M \subset X$ is a Chebyshev set if it is a set of existence and uniqueness, i.e., for each $x \in X$, $P_M x$ is a singleton.

Definition 1.2. A space X is called *locally uniformly convex* ($X \in$ (LUR)) if any point s of the unit sphere S is an LUR-*point*, which means by definition that $\|s_n\| = 1$, $\|s_n + s\|/2 \to 1$ implies that $s_n \to s$.

The concept of an LUR-point was introduced by various authors.

Definition 1.3 (see [3]). Let $\varnothing \neq M \subset X$. A space X is called *M-locally uniformly convex* ($X \in$ (M-LUR)) if any M-acting point s of the unit sphere S is an LUR-*point* (see Definition 1.2).

For some results of geometric approximation theory for M-locally uniformly convex spaces, see [3].

A space X is called *uniformly convex* if, for each $\varepsilon > 0$, there exists $\delta > 0$ such that if $x, y \in X$, $\|x\| = \|y\| = 1$ and $\|x + y\|/2 > 1 - \delta/2$, then $\|x - y\| < \varepsilon$.

Definition 1.4 (see [3]). Let $\varnothing \neq M \subset X$. A space X is called *M-uniformly convex* ($X \in$ (M-UR)) if any M-acting point s of the unit sphere S is an LUR-point (see Definition 1.3) and, for each $\varepsilon > 0$, there exists $\delta > 0$ such that $\|x - y\| < \varepsilon$ provided that x, y are M-acting points of the unit sphere S and $\|x + y\|/2 > 1 - \delta/2$.

2. The main theorem

The following result is well known (Vlasov, [20]): *any P-connected subset of a complete uniformly convex space is B-connected.* Tsar'kov [15, Theorem 3] strengthened this result as follows: in any reflexive (CLUR)-space[2] (and, in particular, in any complete uniformly convex space), each closed P_0-connected set is B-connected. On the other hand, in any nonreflexive space, an example of a disconnected P_0-connected set is given by two parallel hyperplanes generated by a non-norm-attaining functional.) For some new results on connectedness of intersections with balls of P_0- and P-connected sets in symmetric and asymmetric spaces, see [7], [19], [3]. Some balayage-type theorems in the problem of connectedness of sets can be found in [3]. In this regard, we note that according to [3] if $M \subset X$ is a sun and any M-acting luminosity point is an LUR-point, then M is B-connected. A partially converse result was also proved in [3]: if $\varnothing \neq M \subset X$ is $\mathring{B}$-connected and $X \in$ (M-LUR), then M is unimodal (an LG-set, or, equivalently, a strict protosun). Among the recent results in this direction, we mention one result of Tsar'kov [16], who constructed an example of a four-dimensional polyhedral space and non-B-connected sun in it. In the same paper, I. G. Tsar'kov also constructed a disconnected sun in a three-dimensional asymmetric normed space.

The following main result can be looked upon as a balayage theorem of the result of Vlasov from [20].

Theorem 2.1. *Let $\varnothing \neq M \subset X$ be P-connected, where X is a Banach M-uniformly convex space ($X \in$ (M-UR)). Then M is $\mathring{B}$-connected.*

In particular, if $M \subset X$ is a Chebyshev subset of a Banach space $X \in$ (M-UR), then M has connected intersections with open balls.

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[2](CLUR) is the class of spaces such that the condition $x \in S$, $y_n \in S$, $\|x + y_n\|/2 \to 1$ implies that (y_n) has a convergent subsequence. Note that any reflexive (CLUR)-space is a Efimov–Stechkin space (see, for example, [15]).

Remark 2.1. In connection to Theorem 2.1 we mention the following result (see [3, Theorem 5]): Let $M \subset X$ be a sun[3] and let that any M-acting luminosity point be an LUR-point. Then M is B-connected. A kind of converse result also holds: if $\varnothing \neq M \subset X$ is $\mathring{B}$-connected and $X \in (M\text{-LUR})$, then M is unimodal (see [3, Theorem 6]).

Proof of Theorem 2.1. We follow some ideas of [15], [7]. We need the following auxiliary result [15, Lemma 4]. Let $x \in X$, $r > 0$ and $M \cap B(x,r) = A_1 \sqcup C_1$, where $A_1, C_1 \in \mathscr{F}(B(x,r))$, $\max\{\rho(x,A_1), \rho(x,C_1)\} < r$. Then there exist $x_1 \in X$ and $r_1 > 0$ such that

$$B(x_1,r_1) \subset B(x,r) \quad \text{and} \quad \rho(x_1,M) = \rho(x_1,A_1) = \rho(x_1,C_1) < r_1 \tag{2.1}$$

Here and in what follows, $\mathscr{F}(N)$ is the class of closed subsets of a set N.

Next, let $x \in X$, $r > 0$ and $M \cap B(x,r) = A_1 \sqcup C_1$, where $A_1, C_1 \in \mathscr{F}(B(x,r))$ and $d := \rho(x,A_1) = \rho(x,C_1) < r$. Let us show that, for each $\varepsilon > 0$, there exist $x_0 \in X$ and $r_0 > 0$ such that

$$\begin{gathered} B(x_0,r_0) \subset \mathring{B}(x,r), \quad \rho(x_0,A_1) = \rho(x_0,C_1) < r_0 \ \text{ and} \\ A_1 \cap B(x_0,r_0) \subset \mathring{B}(y_0,\varepsilon) \ \text{ for some } \ y_0 \in A_1. \end{gathered} \tag{2.2}$$

Let $z \in C_1$ be such that $\|z - x\| < d + (r-d)/10$. Let $\delta := \min\{(r-d)/10, \rho(x,A_1)\}/3$. The distance function is continuous, and hence then there exists $z_1 \in (x,z)$ such that

$$\rho(z_1,C_1) = \rho(z_1,A_1) - 2\delta. \tag{2.3}$$

Recall that $a \in X$ is a point of approximative uniqueness for N (written $a \in \mathrm{AU}(N)$) if the set of nearest points from M for a consists of a single point $y \in N$ and $\|y - y_n\| \to 0$ as $n \to \infty$ for any sequence $(y_n) \subset N$ such that $\|y_n - a\| \to \rho(a,N)$. It is clear that if $y \in \mathrm{AU}(N)$, then $y \in P_N a$. Approximative uniqueness of sets is actively studied at present (see, for example, [5], [6], [18], and [17]).

In [18, Theorem 3] and [7, Theorem 3.8] it was shown (in a more general setting of complete asymmetric locally uniformly convex spaces) that, for a nonempty closed set $M \subset X$, the set of points approximative uniqueness is dense in X, i.e., in a complete $X \in (M\text{-UR})$

$$\mathrm{AU}(M) \text{ is a set of second category.} \tag{2.4}$$

This result carried out verbatim to M-uniformly convex spaces. As a corollary (see [7, Lemma 3.6]), in a complete M-uniformly convex space X (M is a fixed closed set), the set of points of existence for M is dense in X.

By (2.4), there exists a point $z_2 \in \mathrm{AU}(A_1)$ such that $\|z_2 - z_1\| \le \delta$. Let us show that

$$B(z_2, \rho(z_2,A_1) + \delta) \subset \mathring{B}(x,r). \tag{2.5}$$

[3]A set M is a sun if, for any $x \notin M$, there is a best approximant $y' in M$ for x such that $y \in P_M z$ for any z on the ray emanating from y and passing through x. Suns sometimes called generalized Kolmogorov sets, because they can be characterized in terms of the generalized Kolmogorov criterion for best approximation.

By the triangle inequality, $\rho(u,N) \leqslant \|v-u\| + \rho(v,N)$, for all $u,v \in X$, for a given $N \subset X$. Consequently,

$$\begin{aligned}\rho(z_2,A_1)+\delta &\le \rho(z_1,A_1)+2\delta = \rho(z_1,C_1)+4\delta\\ &\le \|z-z_1\|+4\delta = \|z-x\|-\|z_1-x\|+4\delta\\ &\le \|z-x\|-(\|z_2-x\|-\|z_2-z_1\|)+4\delta\\ &\le \|z-x\|-\|z_2-x\|+5\delta\\ &\le d+(r-d)/10-\|z_2-x\|+(r-d)/2\\ &= (3r+2d)/5-\|z_2-x\| < r-\|z_2-x\|.\end{aligned}$$

Now using the well-known equivalence

$$B(x,r)\subset \mathring{B}(x',r') \iff \|x-x'\|<r'-r \tag{2.6}$$

(see, for example, [6, Proposition 1.2]) we arrive at (2.5). Given $\delta>0$, we set

$$P_M^\delta x := \big\{y\in M \mid \|y-x\|\le \rho(x,M)+\delta\big\}.$$

By the assumption, X is M-uniformly convex, and hence, by the definition of a point of approximative uniqueness, for all $x\in \mathrm{AU}(M)$ and $\varepsilon>0$, there exists $\delta>0$ such that

$$P_M^\delta x \subset \mathring{B}(y,\varepsilon), \qquad \text{where } \ P_M x=\{y\}. \tag{2.7}$$

As a result, we have $z_2\in \mathrm{AU}(A_1)$. Now employing (2.7), we have, for some $\delta=\delta(\varepsilon)>0$,

$$P_{A_1}^\delta z_2 \subset \mathring{B}(y_1,\varepsilon), \qquad \text{where } \ P_{A_1} z_2=\{y_1\}. \tag{2.8}$$

Let $\alpha_1=\alpha_1(\varepsilon)$ be such that $\rho(z_2,A_1)<\alpha_1<\rho(z_2,A_1)+\delta$. For each $z_3\in \mathring{B}(z_2,\alpha_1)\cap A_1$, we have

$$\begin{aligned}&\rho(z_2,C_1)-\rho(z_2,A_1) < (\rho(z_1,C_1)+\delta)-(\rho(z_1,A_1)-\delta) \overset{(2.3)}{=} 0,\\ &\rho(z_3,C_1)-\rho(z_3,A_1)=\rho(z_3,C_1)>0.\end{aligned}$$

Next, since the distance function is continuous, there exists $x_0\in[z_2,z_3]$ such that $\rho(x_0,A_1)=\rho(x_0,C_1)$. We set $r_0:=\alpha_1-\|x_0-z_2\|$ (here, r_0 is positive, because $\rho(z_2,A_1)<\alpha_1$ and $x_0\in[z_2,z_3]$). We have

$$B(x_0,r_0)\subset B(z_2,\alpha_1)\subset B(z_2,\rho(z_2,A_1)+\delta)\subset \mathring{B}(x,r). \tag{2.9}$$

As a corollary, $(B(x_0,r_0)\cap A_1) \overset{(2.9)}{\subset} (B(z_2,\alpha_1)\cap A_1) \overset{(2.8)}{\subset} \mathring{B}(y_1,\varepsilon)$, and

$$\rho(x_0,A_1)=\rho(x_0,C_1)\le \|z_3-x_0\| = \|z_3-z_2\|-\|x_0-z_2\| < \alpha_1-\|x_0-z_2\| = r_0,$$

which verifies (2.2) with $y_0=y_1$.

Now let us proceed with the proof of Theorem 2.1. By (2.1), there exist $x_1\in X$ and $r_1>0$ such that

$$B(x_1,r_1)\subset B(x,r) \quad \text{and} \quad \rho(x_1,M)=\rho(x_1,A_0)=\rho(x_1,C_0).$$

We set

$$A_1=B(x_1,r_1)\cap A_0, \quad C_1=B(x_1,r_1)\cap C_0.$$

We induct on i. Assume that there exist $x_i\in X$ and $r_i>0$ such that $B(x_i,r_i)\cap M=A_i\sqcup C_i$, where $A_i,C_i\in \mathscr{F}(X)$ and $\rho(x_i,A_i)=\rho(x_i,C_i)<r_i$.

Using (2.2), one can find $x_{i+1} \in X$ and $r_{i+1} > 0$ such that

$$\begin{gathered} B(x_{i+1}, r_{i+1}) \subset \mathring{B}(x_i, r_i), \qquad \rho(x_{i+1}, A_i) = \rho(x_{i+1}, C_i) < r_{i+1}, \\ C_{i+1} := (B(x_{i+1}, r_{i+1}) \cap A_i) \subset \mathring{B}(y_i, r/2^{(i+1)}), \quad \text{where} \ \ y_i \in A_i. \end{gathered} \tag{2.10}$$

We set $A_{i+1} = B(x_{i+1}, r_{i+1}) \cap C_i$. For $m > n$, we have

$$\|x_m - x_n\| \le \sum_{k=n}^{m-1} \|x_{k+1} - x_k\| \overset{(2.6),(2.10)}{\le} \sum_{k=n}^{m-1} (r_{k+1} - r_k) = r_m - r_n.$$

Hence (x_i) is a Cauchy sequence. Let x_0 be the limit of this sequence. The bounded sequence (r_i) $(r_i > r_{i+1} \ge 0)$ converges to some number $r_0 > 0$ $(r_0 \ne 0$, because $A_0 \cap C_0 = \varnothing)$. It is clear that

$$B(x_0, r_0) = \bigcap_{i=1}^{\infty} \mathring{B}(x_i, r_i). \tag{2.11}$$

Each of the sequences $(A_{2k})_{k=1}^{\infty}$, $(A_{2k-1})_{k=1}^{\infty}$ consists of nested closed sets. In addition, from (2.10) for each $k \in \mathbb{N}$ we have $y_{2k-1} \in A_{2k-1} \subset \mathring{B}(y_{2k-1}, r/2^{2k})$. The Cauchy sequence (y_{2k-1}) converges to some point $u' \in M$. A similar argument shows that, $y_{2k} \in A_{2k} \subset \mathring{B}(y_{2k}, r/2^{2k-1})$ for each $k \in \mathbb{N}$. The space is complete, and hence (y_{2k}) converges to some $u'' \in M$. It is clear that $\widetilde{A} := \bigcap_k A_{2k-1} = \{u'\}$, $\widetilde{C} := \bigcap_k A_{2k} = \{u''\}$. By (2.11), $\widetilde{A} = B(x_0, r_0) \cap A_0$, $\widetilde{C} = B(x_0, r_0) \cap C_0$, and, therefore,

$$\rho(x_0, \widetilde{A}) = \rho(x_0, A_0) = \rho(x_0, \widetilde{C}) = \rho(x_0, C_0) = \rho(x_0, M).$$

So, x_0 has precisely two (distinct) nearest points from M, where $u' \in A_0$, $u'' \in C_0$, which contradicts the assumption that M is P-connected. This contradiction verifies that M is $\mathring{B}$-connected. Theorem 2.1 is proved. $\square$

Example 2.1. Let $X = \mathbb{R}^2 \oplus_1 \mathbb{R}$. The unit ball B of X is the convex hull of the Euclidean 2-dimensional unit ball and the interval $[-1, 1]$, i.e., $B = \{(\xi_1, \xi_2, c) \mid c \in [-1, 1],\ \xi_1^2 + \xi_2^2 \le 1\}$. Recall that if $x, y \in Y$, $x \ne y$, then the set

$$\begin{aligned} \mathring{K}(y, x) :&= \bigcup_{r>0} \mathring{B}\big(-ry + (r+1)x, (r+1)\|y - x\|\big) \\ &= \big\{z \in Y \mid [z, y] \cap \mathring{B}(x, \|y - x\|) \ne \varnothing\big\} \end{aligned}$$

is the (open) support cone (to the ball $(x, \|y - x\|)$ at its boundary point y). It is well known that the set $Y \setminus \mathring{K}(y, x)$ is a sun (see, for example, Theorem 3 in [10]). Let $s \in S$ be an arbitrary point of the unit sphere. By the above, the set $M := X \setminus \mathring{B}(s, 0)$ is a sun (which is not a Chebyshev set). Now an application of Theorem 2.1 and Lemma 15 in [15] shows that M is B-connected. In addition, the metric projection operator onto M has connected values. Moreover, the intersection of M with any ball is contractible. However, it is unknown at present whether *any* sun in the space $X = \mathbb{R}^2 \oplus_1 \mathbb{R}$ has connected intersections with closed balls.

3. Density properties of points of uniqueness

We need the following result on the structure of the unit sphere of a locally uniformly convex space (see [11], [12], [14]). In [12], this result is called the Stechkin lens lemma, because the set M_ε in this lemma resembles a lens. We require the following local analogue of this result (see [3, Lemma 1]).

Lemma 3.A. *Let $x_0 \in S$ be an* LUR*-point of the unit sphere S and let $0 < \alpha < 1$. Consider the set*

$$M_\varepsilon := \{X \setminus \mathring{B}(0,1)\} \cap B(\alpha x_0, 1-\alpha+\varepsilon).$$

Then

$$\operatorname{diam} M_\varepsilon \to 0 \quad \textit{as} \quad \varepsilon \to +0.$$

Recall that a set $M \subset X$ is called nowhere dense if its closure does not contain interior points in X. A set of first category (or a meager set) is a union of countably many nowhere dense sets. The class of subsets of X of first category is denoted by (I), and the class of complements of sets of first category in X is denoted by (II).

We set

$$\mathrm{U}(M) := \{x \in X \mid P_M x \text{ is either empty or a singleton.}\}$$

Any point $x \in \mathrm{U}(M)$ is called a point of uniqueness for M.

The following result in a particular case of locally uniformly convex spaces was established by S. B. Stechkin (see [14, Theorem 4], and also [11, Theorem 2.7]). We extend this result to the case of M-locally uniformly convex spaces (see Definition 1.3). For boundedly precompact sets M of an M-strictly convex space, a similar result was established in [3, Theorem 4].

Theorem 3.1. *Let $M \subset X$ be a closed subset of a complete M-locally uniformly convex space X ($X \in (M\text{-LUR})$). Then*

$$\mathrm{U}(M) \in \mathrm{II} \quad (\textit{i.e., } \mathrm{U}(M) \textit{ is a set of second category}).$$

We set

$$d_0(x) = \lim_{\delta \to 0+} \operatorname{diam}\big(M \cap B(x, \rho(x,M)+\delta) \cap \mathscr{O}_\delta(P_M x)\big);$$

here and in what follows, $\mathscr{O}_\delta(A)$ is the δ-neighborhood of the set A.

Proof of Theorem 3.1. Following [14] and [11], we set

$$\Phi_\alpha : = \{x \in X \mid \operatorname{diam} P_M(x) \geq 1/\alpha\},$$
$$F_\alpha : = \{x \in X \mid \operatorname{diam} d_0(x) \geq 1/\alpha\}.$$

It is clear that $F_\alpha \supset \Phi_\alpha$ ($\alpha > 0$). Let us show that each set Φ_n, $n \in \mathbb{N}$, is nowhere dense, i.e., any open ball $\mathring{B}_1$ contains an open ball $\mathring{B}_2$ disjoint from Φ_n. If $\mathring{B}_1 \cap \Phi_n = \varnothing$, then there is nothing to prove. So, we now assume that $x \in \mathring{B}_1 \cap \Phi_n$. In this case, $x \notin M$. Let $y_0 \in P_M x$ and $z \in (x, y_0)$. By Lemma 3.A, $\operatorname{diam} P_M^\varepsilon(z) \to 0$ as $\varepsilon \to 0$, where, as above, $P_M^\varepsilon(z) := M \cap B(z, \rho(z,M)+\varepsilon)$, i.e., $z \notin F_n$. By Lemma 3 in [14] (see also [11, Proposition 2.13]), the set F_α is closed for each α (in this lemma, X is an arbitrary normed space). Therefore, there exists an open ball $\mathring{B}_2 = \mathring{B}(z, \gamma)$ lying in $X \setminus F_n$. For each $z' \in B_2$, we have $z' \notin F_n$. Hence $z' \notin \Phi_n$, i.e., $\mathring{B}_2 = \cap \Phi_n = \varnothing$. So, each Φ_n is nowhere dense in X. Now the required result follows: $\mathrm{U}(M) = X \setminus \bigcup_n^\infty \Phi_n$ is a set of second category. □

Acknowledgements. This research was carried out at Lomonosov Moscow State University with the financial support of the Russian Science Foundation (grant no. 22-11-00129).

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Received: February 1, 2024; Accepted: March 15, 2024