

ON THE CLASS OF n -QUASI- m -SYMMETRIC OPERATORS

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Abstract. The aim of this paper is to extend some properties of m -symmetric operators to the class of n -quasi- m -symmetry. It is shown that if T is an n_1 -quasi- m -symmetry and S is an n_2 -quasi- l -symmetry such that T and S are double commuting, then TS is an n -quasi- $(m + l - 1)$ -symmetry where $n = \max\{n_1, n_2\}$. Also we study some spectral properties and \mathcal{C}_0 -semigroup of this class.

1. Introduction

Let \mathcal{H} be a complex Hilbert space, and $\mathcal{B}(\mathcal{H})$ denote to the algebra of all bounded linear operators on \mathcal{H} . For every $T \in \mathcal{B}(\mathcal{H})$, we denote T^* , $N(T)$ and $R(T)$ the adjoint, the null space and the range of T , respectively. As usual \overline{M} denotes the closure of $M \subset \mathcal{H}$, while $\sigma(T)$, $\sigma_p(T)$ and $\sigma_{ap}(T)$ stand for the spectrum, the point spectrum and the approximate point spectrum of T , respectively.

An operator $T \in \mathcal{B}(\mathcal{H})$ is said to be m -isometry for some positive integer m if

$$\sum_{k=0}^m (-1)^{m-k} \binom{m}{k} T^{*k} T^k = 0,$$

where $\binom{m}{k}$ is the binomial coefficient. This class has been generalized to the class of n -quasi- m -isometry i.e., T is an n -quasi- m -isometry if

$$T^{*n} \left(\sum_{k=0}^m (-1)^{m-k} \binom{m}{k} T^{*k} T^k \right) T^n = 0,$$

for some positive integers m and n . This class has been studied in [2, 4, 8, 10, 13].

Let m be a positive integer. $T \in \mathcal{B}(\mathcal{H})$ is said to be m -symmetry if it satisfies

$$\alpha_m(T) = \sum_{k=0}^m (-1)^{m-k} \binom{m}{k} T^{*(m-k)} T^k = 0.$$

For $m = 1$, we obtain that T is selfadjoint. Moreover, if T is selfadjoint, then T is m -symmetric for every positive integer m ; hence the class of m -symmetry is a generalization of selfadjoint operators. This class has been studied by many authors see [1, 3, 6, 7, 11, 12, 14].

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In [16] the authors generalize the class of m -symmetry to the class of n -quasi- m -symmetry where m and n are positive integers, i.e., T is an n -quasi- m -symmetry if

$$T^{*n} \left(\sum_{k=0}^m (-1)^{m-k} \binom{m}{k} T^{*(m-k)} T^k \right) T^n = 0.$$

Recall that T is an n -quasi strict m -symmetry if T is an n -quasi- m -symmetry, and T is not an n -quasi- $(m-1)$ -symmetry. It is shown that if T is an n -quasi- m -symmetry and $R(T^n)$ is dense, then T is an m -symmetric operator [16]. It has been proved in [16] that T is an n -quasi- m -symmetric operator if and only if

$$T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}$$

on $\mathcal{H} = \overline{R(T^n)} \oplus N(T^{*n})$, where T_1 is an m -symmetric operator and $T_3^n = 0$.

A well-known property of n -quasi- m -symmetries is the power of any n -quasi- m -symmetry is also an n -quasi- m -symmetry. Moreover, if T is an n -quasi- m -symmetry, then T is an k -quasi- l -symmetry for $k \geq n$ and $l \geq m$.

Our purpose in this paper is to extend the properties of m -symmetry to the class of n -quasi- m -symmetry and we discuss which properties remain valid and which properties are not valid for the class of n -quasi- m -symmetry. The paper is organized as follows. Section 2 begins with some lemmas that are needed throughout this work. Then we study some properties of m -symmetric operators. Next we devote our interest to the study of properties of n -quasi- m -symmetries. Section 3 is dedicated to discuss \mathcal{C}_0 -semigroup of n -quasi- m -symmetry. More precisely, we prove that if $\{T(t)\}_{t \geq 0} \subset \mathcal{B}(\mathcal{H})$ is a \mathcal{C}_0 -semigroup on \mathcal{H} , then $T(t)$ is an n -quasi- m -symmetry for all $t \geq 0$ if and only if $T(t)$ is an m -symmetry for all $t \geq 0$.

2. Properties of n -quasi- m -symmetry

First, we present some lemmas which are needed throughout this work.

Let $T, S \in \mathcal{B}(\mathcal{H})$. T and S are said to be double commuting if T commutes with S and S^* .

Lemma 2.1. [5] *Let $T, S, Q \in \mathcal{B}(\mathcal{H})$. Then*

(i) *If T and Q commutes, then*

$$\alpha_m(T + Q) = \sum_{k=0}^m \sum_{j=0}^{m-k} (-1)^k \binom{m}{k} \binom{m-k}{j} Q^{*j} \alpha_{m-k-j}(T) Q^k.$$

(ii) *If T and S are double commuting, then*

$$\alpha_m(TS) = \sum_{k=0}^m \binom{m}{k} T^{*k} \alpha_k(S) \alpha_{m-k}(T) S^{m-k}.$$

Lemma 2.2. [5] *Let $T, S \in \mathcal{B}(\mathcal{H})$. Assume that T and S are double commuting. If T is m -symmetric and S is ℓ -symmetric, then operator TS is $(m + \ell - 1)$ -symmetric.*

Lemma 2.3. [9] *Let $T \in \mathcal{B}(\mathcal{H})$ be an $2m$ -symmetry. Then T is an $(2m - 1)$ -symmetry.*

Proposition 2.1. *Let $T \in \mathcal{B}(\mathcal{H})$ be invertible. If T^r and T^{r+1} are selfadjoint for some $r \in \mathbb{N}$, then T is selfadjoint.*

Proof. Since T is invertible with T^r and T^{r+1} are selfadjoint, then

$$\begin{aligned} T^{r+1} &= (T^{r+1})^* = (T^r)^* T^* = T^r T^* \\ \implies T^{r+1} &= T^r T^* \\ \implies T^* &= T. \end{aligned}$$

Therefore, T is selfadjoint. \square

The following example shows that Proposition 2.1 is not necessarily true if T is not invertible.

Example 2.1. Let $\mathcal{H} = \mathbb{C}^3$, and $T = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, it is clear that T is not selfadjoint even though T^2 and T^3 are selfadjoint, since $T^2 = T^3 = 0$.

Theorem 2.1. *Let $T \in \mathcal{B}(\mathcal{H})$ be an invertible operator. If T^r is an m -symmetry and T^s is an l -symmetry with $T^{*r}T^s = T^sT^{*r}$ for some positive integers r and s , then T^h is an $(m + l - 1)$ -symmetry where $h = \max\{r, s\} - \min\{r, s\}$.*

Proof. Assume that $r \leq s$, since $T^r T^s = T^s T^r = T^{r+s}$, then $T^{-r} T^s = T^s T^{-r}$. Similarly, since $T^{*r} T^s = T^s T^{*r}$, we deduce that $(T^*)^{-r} T^s = T^s (T^*)^{-r}$. That means T^{-r} and T^s are double commuting, and since T^r is an m -symmetry, thus so is T^{-r} . Applying Lemma 2.2 on T^s and T^{-r} , we get $T^h = T^s T^{-r}$ is an $(m + l - 1)$ -symmetry where $h = s - r$. \square

Corollary 2.1. *Let $T \in \mathcal{B}(\mathcal{H})$ be an invertible operator. Then we have*

- (1) *If T^r is an m -symmetry and T^{r+1} is an l -symmetry for some $r \in \mathbb{N}$, such that $T^{*r} T^{r+1} = T^{r+1} T^{*r}$, then T is an $(m + l - 1)$ -symmetry.*
- (2) *If T^r is selfadjoint and T^{r+1} is an m -symmetry, then T is an m -symmetry.*

Theorem 2.2. *Let $T, S \in \mathcal{B}(\mathcal{H})$. If T is an n_1 -quasi- m -symmetry and S is an n_2 -quasi- l -symmetry such that T and S are double commuting, then TS is an n -quasi- $(m + l - 1)$ -symmetry, where $n = \max\{n_1, n_2\}$. Moreover, TS is an n -quasi strict $(m + l - 1)$ -symmetry if and only if $T^{*(n+l-1)} \alpha_{m-1}(T) T^n S^{*n} \alpha_{l-1}(S) S^{n+l-1}$ is not the zero operator.*

Proof. Since $TS = ST$ and $T^* S = S T^*$, and by Lemma 2.1, we obtain

$$\begin{aligned} &(T^* S^*)^n \alpha_{m+l-1}(TS)(TS)^n \\ &= T^{*n} S^{*n} \sum_{k=0}^{m+l-1} \binom{m+l-1}{k} T^{*k} \alpha_k(S) \alpha_{m+l-1-k}(T) S^{m+l-1-k} T^n S^n. \end{aligned}$$

For $k \leq l - 1$, then $m + l - 1 - k \geq m$, thus for all $k \leq l - 1$, we have

$$T^{*n} \alpha_{m+l-1-k}(T) T^n = 0.$$

For $k \geq l$, then $S^{*n}\alpha_k(S)S^n = 0$ for all $k \geq 0$, hence

$$(T^*S^*)^n\alpha_{m+l-1}(TS)(TS)^n = 0.$$

Similarly, we get

$$(T^*S^*)^n\alpha_{m+l-2}(TS)(TS)^n = T^{*(n+l-1)}\alpha_{m-1}(T)T^nS^{*n}\alpha_{l-1}(S)S^{n+l-1}.$$

Therefore, TS is an n -quasi strict $(m+l-1)$ -symmetry if and only if $T^{*(n+l-1)}\alpha_{m-1}(T)T^nS^{*n}\alpha_{l-1}(S)S^{n+l-1}$ is not the zero operator. \square

The following example shows that Theorem 2.2 is not necessarily true if T and S are not double commuting.

Example 2.2. Let $\mathcal{H} = \mathbb{C}^3$, and $T = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix}$ and $S = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$, by a simple calculation we obtain that T is an 1-quasi-3-symmetry, and S is clearly selfadjoint. We have that T and S are not commuting, then by straightforward calculation we get that TS is not a 1-quasi-3-symmetry.

Theorem 2.3. Let $T, Q \in \mathcal{B}(\mathcal{H})$. If T is an n -quasi- m -symmetry and Q is a nilpotent operator of order p such that $TQ = QT$, then $T + Q$ is an $(n+p-1)$ -quasi- $(m+2p-2)$ -symmetry. Furthermore, $T + Q$ is an $(n+p-1)$ -quasi strict $-(m+2p-2)$ -symmetry if and only if $(T^* + Q^*)^{n+p-1}Q^{*(p-1)}\alpha_{m-1}(T)Q^{p-1}(T+Q)^{n+p-1}$ is not the zero operator.

Proof. Set $r = n+p-1$, $l = m+2p-2$ and $R = T+Q$, since $TQ = QT$ and applying Lemma 2.1, we obtain

$$\begin{aligned} & R^{*r}\alpha_l(R)R^r \\ &= (T^* + Q^*)^r \sum_{k=0}^l \sum_{j=0}^{l-k} (-1)^k \binom{l}{k} \binom{l-k}{j} Q^{*j}\alpha_{l-k-j}(T)Q^k(T+Q)^r \\ &= \sum_{i=0}^r a_i T^{*r-i} Q^{*i} \sum_{k=0}^l \sum_{j=0}^{l-k} (-1)^k b_k c_{k,j} Q^{*j}\alpha_{l-k-j}(T)Q^k \sum_{i=0}^r a_i T^{r-i} Q^i, \end{aligned}$$

where $a_i = \binom{r}{i}$, $b_k = \binom{l}{k}$ and $c_{k,j} = \binom{l-k}{j}$.

Note that if $k \geq p$ or $j \geq p$, then $Q^k = 0$ or $Q^{*j} = 0$, thus

$$Q^{*j}\alpha_{l-k-j}(T)Q^k = 0.$$

If $k \leq p-1$ and $j \leq p-1$, we obtain

$$l-k-j = m+2p-2-k-j \geq m+2p-2-(p-1)-(p-1) = m.$$

Since T is an n -quasi- m -symmetry and Q is a nilpotent operator of order p , we get

$$T^{*n+p-1-i}\alpha_{l-k-j}(T)T^{n+p-1-i} = 0 \quad \text{for } i \leq p-1.$$

For $p \leq i \leq n+p-1$, we obtain

$$T^{*n+p-1-i}Q^{*i}\alpha_{l-k-j}(T)T^{n+p-1-i}Q^i = 0.$$

Then, $R^{*r}\alpha_l(R)R^r = 0$ as desired.

Similarly, we obtain

$$\begin{aligned} & R^{*r}\alpha_{m+2p-3}(R)R^r \\ &= (-1)^{m+p-2} \binom{m+2p-3}{p-1} \binom{m+p-2}{p-1} R^{*r}Q^{*(p-1)}\alpha_{m-1}(T)Q^{p-1}R^r. \end{aligned}$$

Hence $T + Q$ is an $(n + p - 1)$ -quasi strict $(m + 2p - 2)$ -symmetry if and only if $(T^* + Q^*)^{n+p-1}Q^{*(p-1)}\alpha_{m-1}(T)Q^{p-1}(T + Q)^{n+p-1}$ is not the zero operator. \square

Let $\mathcal{H} \otimes \mathcal{H}$ denote the completion, endowed with a reasonable uniform cross-norm of the algebraic tensor product $\mathcal{H} \otimes \mathcal{H}$ of \mathcal{H} and \mathcal{H} . For $T, S \in \mathcal{B}(\mathcal{H})$, $T \otimes S \in \mathcal{B}(\mathcal{H} \otimes \mathcal{H})$ denotes to the tensor product operator defined by T and S .

Lemma 2.4. *Let $T \in \mathcal{B}(\mathcal{H})$. Then T is an n -quasi- m -symmetry if and only if $T \otimes I$ and $I \otimes T$ are n -quasi- m -symmetry.*

Proof. We have

$$\alpha_m(T \otimes I) = \alpha_m(T) \otimes I.$$

Then,

$$(T \otimes I)^{*n}\alpha_m(T \otimes I)(T \otimes I)^n = (T^{*n} \otimes I)(\alpha_m(T) \otimes I)(T^n \otimes I) = T^{*n}\alpha_m(T)T^n \otimes I.$$

Hence T is an n -quasi- m -symmetry if and only if $T \otimes I$, and the same with $I \otimes T$. \square

Proposition 2.2. *Let $T, S \in \mathcal{B}(\mathcal{H})$. If T is an n_1 -quasi- m -symmetry and S is an n_2 -quasi- l -symmetry, then $T \otimes S$ is an n -quasi- $(m + l - 1)$ -symmetry where $n = \max\{n_1, n_2\}$.*

Proof. Since T is an n_1 -quasi- m -symmetry, and S is an n_2 -quasi- l -symmetry, by Lemma 2.4 we obtain that $T \otimes I$ is an n_1 -quasi- m -symmetry, and $I \otimes S$ is an n_2 -quasi- l -symmetry. We observe that

$$T \otimes S = (T \otimes I)(I \otimes S) = (I \otimes S)(T \otimes I).$$

Similarly, we obtain

$$T \otimes S^* = (T \otimes I)(I \otimes S^*) = (T \otimes I)(I \otimes S)^* = (I \otimes S)^*(T \otimes I).$$

Hence $T \otimes I$ and $I \otimes S$ are double commuting, by applying Theorem 2.2 on $T \otimes I$ and $I \otimes S$, we obtain that $T \otimes S$ is an n -quasi- $(m + l - 1)$ -symmetry, where $n = \max\{n_1, n_2\}$. \square

Proposition 2.3. *Let $T, Q \in \mathcal{B}(\mathcal{H})$. If T is an n -quasi- m -symmetry and Q is a nilpotent of order p , then $T \otimes I + I \otimes Q$ is an $(n + p - 1)$ -quasi- $(m + 2p - 2)$ -symmetry.*

Proof. Since T is an n -quasi- m -symmetry, then so is $T \otimes I$, as Q is a nilpotent of order p , thus $I \otimes Q$ is a nilpotent of order p as well, we have

$$T \otimes Q = (T \otimes I)(I \otimes Q) = (I \otimes Q)(T \otimes I).$$

Applying Theorem 2.3 on $T \otimes I$ and $I \otimes Q$, we obtain that $T \otimes I + I \otimes Q$ is an $(n + p - 1)$ -quasi- $(m + 2p - 2)$ -symmetry. \square

The following example shows that if T is an n_1 -quasi- m -symmetry and S is an n_2 -quasi- l -symmetry with T and S are double commuting, then $T + S$ is not necessarily n -quasi- $(m + l - 1)$ -symmetry.

Example 2.3. Let $\mathcal{H} = \mathbb{C}^3$, $T = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix}$ and $S = I$, T is an 1-quasi-3-symmetry and S is selfadjoint with T and S are double commuting. By a simple calculation we obtain that $T + I$ is not n -quasi-3-symmetry for any $n \in \mathbb{N}$.

Theorem 2.4. Let $T, S \in \mathcal{B}(\mathcal{H})$. If T is an n_1 -quasi- m -symmetry and S is an n_2 -quasi- l -symmetry such that $TS = ST = T^*S = ST^* = 0$, then $T + S$ is an n -quasi- q -symmetry, where $n = \max\{n_1, n_2\}$ and $q = \max\{m, l\}$.

Proof. Since $TS = ST = 0$, then for any strictly positive integer r , we have

$$(T + S)^r = \sum_{i=0}^r \binom{r}{i} T^{r-i} S^i = T^r + S^r,$$

$$(T^* + S^*)^r = \sum_{i=0}^r \binom{r}{i} T^{*(r-i)} S^{*i} = T^{*r} + S^{*r}.$$

Therefore,

$$\begin{aligned} & \alpha_q(T + S) \\ &= \sum_{k=0}^q (-1)^{q-k} \binom{q}{k} (T^{*(q-k)} + S^{*(q-k)})(T^k + S^k) \\ &= (-1)^q T^{*q} + (-1)^q S^{*q} + \sum_{k=1}^{q-1} (-1)^{q-k} \binom{q}{k} (T^{*(q-k)} + S^{*(q-k)})(T^k + S^k) \\ & \quad + T^q + S^q. \end{aligned}$$

Since $T^*S = ST^* = 0$, we obtain

$$\begin{aligned} \alpha_q(T + S) &= (-1)^q T^{*q} + (-1)^q S^{*q} + \sum_{k=1}^{q-1} (-1)^{q-k} \binom{q}{k} (T^{*(q-k)} T^k + S^{*(q-k)} S^k) \\ & \quad + T^q + S^q \\ &= \alpha_q(T) + \alpha_q(S). \end{aligned}$$

We observe that

$$(T + S)^{*n} \alpha_q(T + S) (T + S)^n = (T^{*n} + S^{*n})(\alpha_q(T) + \alpha_q(S))(T^n + S^n).$$

We have $TS = ST = T^*S = ST^* = 0$, then

$$(T + S)^{*n} \alpha_q(T + S) (T + S)^n = T^{*n} \alpha_q(T) T^n + S^{*n} \alpha_q(S) S^n.$$

Since $n = \max\{n_1, n_2\}$ and $q = \max\{m, l\}$, using the assumption that T is an n_1 -quasi- m -symmetry and S be an n_2 -quasi- l -symmetry, we get

$$T^{*n} \alpha_q(T) T^n = S^{*n} \alpha_q(S) S^n = 0.$$

Hence $(T + S)^{*n} \alpha_q(T + S) (T + S)^n = 0$ as required. \square

Proposition 2.4. *Let $T \in \mathcal{B}(\mathcal{H})$ be an n -quasi- $2m$ -symmetry. Then T is an n -quasi- $(2m - 1)$ -symmetry.*

Proof. • If $R(T^n)$ is dense, then T is an $2m$ -symmetry, by Lemma 2.3, we get that T is an $(2m - 1)$ -symmetry.

• If $R(T^n)$ is not dense, then $T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}$ on $\mathcal{H} = \overline{R(T^n)} \oplus N(T^{*n})$, where T_1 is an $2m$ -symmetric operator and $T_3^n = 0$.

By Lemma 2.3, we obtain that T_1 is an $(2m - 1)$ -symmetry. Hence we get that T is an n -quasi- $(2m - 1)$ -symmetry. \square

Theorem 2.5. *Let $T \in \mathcal{B}(\mathcal{H})$ be an n -quasi- m -symmetry. The following statements hold:*

- (1) $\sigma(T) \subset \mathbb{R}$.
- (2) *If $\lambda \in \sigma_p(T) \setminus \{0\}$ i.e., there exists $x \in \mathcal{H}$ such that $Tx = \lambda x$, if $x \notin N(T^{*n})$, then $\lambda \in \sigma_p(T^*)$.*
- (3) *If $\lambda \in \sigma_{ap}(T) \setminus \{0\}$ i.e., there exists a sequence $(x_j) \subset \mathcal{H}$ of unit vectors such that $\lim_{j \rightarrow \infty} (T - \lambda)x_j = 0$, if $\lim_{j \rightarrow \infty} T^{*n}x_j \neq 0$, then $\lambda \in \sigma_{ap}(T^*)$.*

Proof. (1) Let $\lambda \in \mathbb{C} \setminus \mathbb{R}$. If λ is an approximate point spectrum of T , then there exists a sequence of unit vectors $(x_j) \subset \mathcal{H}$ such that

$$\lim_{j \rightarrow \infty} (T - \lambda)x_j = 0.$$

Then we can easily prove that for any strictly positive integer r

$$\lim_{j \rightarrow \infty} (T^r - \lambda^r)x_j = 0.$$

Now, we prove that for any strictly positive integers r and s

$$\lim_{j \rightarrow \infty} \langle T^r x_j, T^s x_j \rangle = \lambda^r \bar{\lambda}^s.$$

Observe that

$$\begin{aligned} |\langle T^r x_j, T^s x_j \rangle - \lambda^r \bar{\lambda}^s| &= |\langle T^r x_j, T^s x_j \rangle - \langle \lambda^r x_j, \lambda^s x_j \rangle| \\ &= |\langle T^r x_j, T^s x_j \rangle - \langle \lambda^r x_j, T^s x_j \rangle + \langle \lambda^r x_j, T^s x_j \rangle \\ &\quad - \langle \lambda^r x_j, \lambda^s x_j \rangle| \\ &= |\langle (T^r - \lambda^r)x_j, T^s x_j \rangle + \langle \lambda^r x_j, (T^s - \lambda^s)x_j \rangle| \\ &\leq |\langle (T^r - \lambda^r)x_j, T^s x_j \rangle| + |\langle \lambda^r x_j, (T^s - \lambda^s)x_j \rangle| \\ &\leq \|(T^r - \lambda^r)x_j\| \|T^s x_j\| + \|\lambda^r x_j\| \|(T^s - \lambda^s)x_j\| \\ &\leq \|T^s\| \|(T^r - \lambda^r)x_j\| + |\lambda|^r \|(T^s - \lambda^s)x_j\|. \end{aligned}$$

We have

$$\lim_{j \rightarrow \infty} (T^r - \lambda^r)x_j = \lim_{j \rightarrow \infty} (T^s - \lambda^s)x_j = 0,$$

we deduce that

$$\lim_{j \rightarrow \infty} |\langle T^r x_j, T^s x_j \rangle - \lambda^r \bar{\lambda}^s| = 0.$$

Since T is an n -quasi- m -symmetry, we obtain

$$\begin{aligned} 0 &= \sum_{k=0}^m (-1)^{m-k} \binom{m}{k} \langle T^{*(n+m-k)} T^{k+n} x_j, x_j \rangle \\ &= \sum_{k=0}^m (-1)^{m-k} \binom{m}{k} \langle T^{k+n} x_j, T^{n+m-k} x_j \rangle. \end{aligned}$$

Since

$$\lim_{j \rightarrow \infty} \langle T^r x_j, T^s x_j \rangle = \lambda^r \bar{\lambda}^s,$$

we obtain

$$\begin{aligned} 0 &= \sum_{k=0}^m (-1)^{m-k} \binom{m}{k} \lambda^{n+k} \bar{\lambda}^{n+m-k} \\ &= |\lambda|^{2n} \sum_{k=0}^m (-1)^{m-k} \binom{m}{k} \lambda^k \bar{\lambda}^{m-k} \\ &= |\lambda|^{2n} (\lambda - \bar{\lambda})^m. \end{aligned}$$

Thus,

$$2^m |\lambda|^{2n} \operatorname{Im}(\lambda)^m = 0.$$

We have $\lambda \in \mathbb{C} \setminus \mathbb{R}$, then $\lambda \neq 0$, this implies that $\operatorname{Im}(\lambda) = 0$.

This contradicts the hypothesis of $\lambda \in \mathbb{C} \setminus \mathbb{R}$. Hence $\lambda \notin \sigma_{ap}(T)$, then $\sigma_{ap}(T) \subset \mathbb{R}$. It follows that $\partial\sigma(T) \subset \sigma_{ap}(T) \subset \mathbb{R}$. Therefore, $\sigma(T) \subset \mathbb{R}$.

- (2) Let $\lambda \in \sigma_p(T) \setminus \{0\}$ i.e. there exists $x \in \mathcal{H}$ such that $Tx = \lambda x$, and assume that $x \notin N(T^{*n})$. Since T is an n -quasi- m -symmetry, we get

$$\begin{aligned} 0 &= \sum_{k=0}^m (-1)^{m-k} \binom{m}{k} T^{*(n+m-k)} T^{k+n} x \\ &= \sum_{k=0}^m (-1)^{m-k} \binom{m}{k} T^{*(n+m-k)} \lambda^{k+n} x \\ &= \lambda^n T^{*n} \sum_{k=0}^m (-1)^{m-k} \binom{m}{k} \lambda^k T^{*(m-k)} x \\ &= (-1)^m \lambda^n T^{*n} (T^* - \lambda)^m x. \end{aligned}$$

Since $\lambda \neq 0$, we obtain

$$T^{*n} (T^* - \lambda)^m x = 0.$$

Let $S \in \mathcal{B}(\mathcal{H})$ such that $N(S) = \{0\}$. Then $N(S^r) = \{0\}$ for any positive integer r . By induction, for $r = 1$, we have $N(S) = \{0\}$.

Assume that $N(S^r) = \{0\}$, we prove that $N(S^{r+1}) = \{0\}$.

Suppose that there exists a nonzero $x \in N(S^{r+1})$. Using $N(S) = N(S^r) = \{0\}$, we get

$$S^{r+1}x = 0 \implies S(S^r x) = 0 \implies S^r x = 0 \implies x = 0.$$

Therefore, $N(S^{r+1}) = \{0\}$ as desired.

If $\lambda \notin \sigma_p(T^*)$, then $N(T^* - \lambda) = \{0\}$, thus $N((T^* - \lambda)^m) = \{0\}$.

Observe that

$$T^{*n}(T^* - \lambda)^m x = (T^* - \lambda)^m T^{*n} x = 0 \implies T^{*n} x = 0.$$

This contradicts the assumption of $x \notin N(T^{*n})$. Hence $\lambda \in \sigma_p(T^*)$.

- (3) Let $\lambda \in \sigma_{ap}(T) \setminus \{0\}$ i.e., there exists a sequence $(x_j) \subset \mathcal{H}$ of unit vectors such that $\lim_{j \rightarrow \infty} (T - \lambda)x_j = 0$. Assume that $\lim_{j \rightarrow \infty} T^{*n}x_j \neq 0$.

Since T is an n -quasi- m -symmetry,

$$\left\| \sum_{k=0}^m (-1)^{m-k} \binom{m}{k} T^{*(n+m-k)} T^{n+k} x_j \right\| = 0.$$

Hence

$$\begin{aligned} & \left\| \lambda^n T^{*n} \sum_{k=0}^m (-1)^{m-k} \binom{m}{k} \lambda^k T^{*(m-k)} x_j \right\| \\ &= \left\| \lambda^n T^{*n} \sum_{k=0}^m (-1)^{m-k} \binom{m}{k} \lambda^k T^{*(m-k)} x_j \right\| \\ & - \left\| \sum_{k=0}^m (-1)^{m-k} \binom{m}{k} T^{*(n+m-k)} T^{n+k} x_j \right\| \\ & \leq \left\| T^{*n} \sum_{k=0}^m (-1)^{m-k} \binom{m}{k} T^{*(m-k)} (T^{k+n} - \lambda^{k+n}) x_j \right\| \\ & \leq \sum_{k=0}^m \binom{m}{k} \|T^{*(n+m-k)}\| \| (T^{k+n} - \lambda^{k+n}) x_j \|. \end{aligned}$$

Note that for every $0 \leq k \leq m$,

$$\lim_{j \rightarrow \infty} \| (T^{k+n} - \lambda^{k+n}) x_j \| = 0.$$

Therefore,

$$\lim_{j \rightarrow \infty} \left\| \lambda^n T^{*n} \sum_{k=0}^m (-1)^{m-k} \binom{m}{k} \lambda^k T^{*(m-k)} x_j \right\| = 0.$$

We have

$$\lambda^n T^{*n} \sum_{k=0}^m (-1)^{m-k} \binom{m}{k} \lambda^k T^{*(m-k)} x_j = (-1)^m \lambda^n T^{*n} (T^* - \lambda)^m x_j.$$

Since $\lambda \neq 0$, we get

$$\lim_{j \rightarrow \infty} \|T^{*n} (T^* - \lambda)^m x_j\| = 0.$$

If $\lambda \notin \sigma_{ap}(T^*)$, then $T^* - \lambda$ is bounded from below

$$\exists c > 0, \forall x \in \mathcal{H} : \|(T^* - \lambda)x\| \geq c\|x\|.$$

We observe that for all $x \in \mathcal{H}$

$$\|(T^* - \lambda)^m x\| \geq c^m \|x\|.$$

Thus,

$$\|T^{*n}(T^* - \lambda)^m x_j\| = \|(T^* - \lambda)^m T^{*n} x_j\| \geq c^m \|T^{*n} x_j\|.$$

Since $\lim_{j \rightarrow \infty} \|T^{*n}(T^* - \lambda)^m x_j\| = 0$, we deduce that

$$\lim_{j \rightarrow \infty} \|T^{*n} x_j\| = 0.$$

This contradicts the assumption of $\lim_{j \rightarrow \infty} \|T^{*n} x_j\| \neq 0$.

Hence $\lambda \in \sigma_{ap}(T^*)$. □

Proposition 2.5. *Let $T \in \mathcal{B}(\mathcal{H})$ be a hyponormal operator (i.e., $T^*T \geq TT^*$). If T is an n -quasi- m -symmetry, then T is selfadjoint.*

Proof. By Theorem 2.5, we have $\sigma(T) \subset \mathbb{R}$. From [15, Corollary 3], it follows that T is selfadjoint. □

Corollary 2.2. *Let $T \in \mathcal{B}(\mathcal{H})$. If T is a strict n -quasi- m -symmetry, then T is not hyponormal.*

Proposition 2.6. *Let $T \in \mathcal{B}(\mathcal{H})$ be an n -quasi- m -symmetry. If there exists a strictly positive integer $r \leq n - 1$ such that $N(T^{*r}) = N(T^{*(r+1)})$, then T is an r -quasi- m -symmetry.*

Proof. Since $N(T^{*r}) = N(T^{*(r+1)})$, we can prove that $N(T^{*r}) = N(T^{*n})$.

We have T is an n -quasi- m -symmetry, thus

$$\begin{aligned} T^{*n} \sum_{k=0}^m (-1)^{m-k} \binom{m}{k} T^{*(m-k)} T^k T^n &= 0 \\ \implies T^{*r} \sum_{k=0}^m (-1)^{m-k} \binom{m}{k} T^{*(m-k)} T^k T^n &= 0 \\ \implies (T^{*r} \sum_{k=0}^m (-1)^{m-k} \binom{m}{k} T^{*(m-k)} T^k T^n)^* &= 0 \\ \implies T^{*n} \sum_{k=0}^m (-1)^{m-k} \binom{m}{k} T^{*k} T^{m-k} T^r &= 0 \\ \implies T^{*r} \sum_{k=0}^m (-1)^{m-k} \binom{m}{k} T^{*k} T^{m-k} T^r &= 0. \end{aligned}$$

Hence T is an r -quasi- m -symmetry. □

3. \mathcal{C}_0 -semigroup of n -quasi- m -symmetry

Let $\{T(t)\}_{t \geq 0} \subset \mathcal{B}(\mathcal{H})$. $\{T(t)\}_{t \geq 0}$ is said to be \mathcal{C}_0 -semigroup if it satisfies the following:

- (1) $T(0) = I$,
- (2) $T(t+s) = T(t)T(s)$, for all $t, s \geq 0$,
- (3) $\lim_{t \rightarrow 0^+} T(t)x = x$, for all $x \in \mathcal{H}$ in the strong operator topology.

Lemma 3.1. *Let $\{T(t)\}_{t \geq 0} \subset \mathcal{B}(\mathcal{H})$ be a \mathcal{C}_0 -semigroup on \mathcal{H} . The following statements are equivalent:*

- (i) $T(t)$ is an n -quasi- m -symmetry for all $t \geq 0$.
- (ii) $T(t)$ is an n -quasi- m -symmetry for all $t \in]0, t']$ with $t' > 0$.

Proof. (i) \implies (ii) is obvious.

(ii) \implies (i) For any $t \geq t'$, there exists $r = [\frac{t}{t'}] + 1$ and $s \in]0, t']$ such that $t = rs$, where $[\frac{t}{t'}]$ denotes the greatest integer $\leq \frac{t}{t'}$. Since $s \in]0, t']$, $T(s)$ is an n -quasi- m -symmetry, thus $T^r(s)$ is an n -quasi- m -symmetry as well. Observe that $T^r(s) = T(rs) = T(t)$. Hence $T(t)$ is an n -quasi- m -symmetry. \square

Theorem 3.1. *Let $\{T(t)\}_{t \geq 0} \subset \mathcal{B}(\mathcal{H})$ be a \mathcal{C}_0 -semigroup on \mathcal{H} . Then $T(t)$ is an n -quasi- m -symmetry for all $t \geq 0$ if and only if $T(t)$ is an m -symmetry for all $t \geq 0$.*

Proof. • If $T(t)$ is an m -symmetry for all $t \geq 0$, then $T(t)$ is clearly an n -quasi- m -symmetry for all $t \geq 0$.

• Suppose that $T(t)$ is an n -quasi- m -symmetry for all $t \geq 0$, then by Lemma 3.1 for all $t' > 0$, we have $T(t)$ is an n -quasi- m -symmetry for all $t \in]0, t']$. Observe that for all $x \in \mathcal{H}$

$$\sum_{k=0}^m (-1)^{m-k} \binom{m}{k} \langle T^{*(n-k)}(t) T^k(t) T^n(t)x, T^n(t)x \rangle = 0.$$

Hence for all $x \in \overline{R(T^n(t))}$, we have

$$\sum_{k=0}^m (-1)^{m-k} \binom{m}{k} \langle T^{*(n-k)}(t) T^k(t)x, x \rangle = 0.$$

Therefore, for all $x \in \bigcap_{t \in]0, t']} \overline{R(T^n(t))}$, we have

$$\sum_{k=0}^m (-1)^{m-k} \binom{m}{k} \langle T^{*(n-k)}(t) T^k(t)x, x \rangle = 0.$$

However, we can easily prove that $\bigcap_{t \in]0, t']} \overline{R(T^n(t))} = \overline{R(T^n(t'))}$. Then for all

$t \in]0, t']$ and for all $x \in \overline{R(T^n(t'))}$, we have

$$\sum_{k=0}^m (-1)^{m-k} \binom{m}{k} \langle T^{*(n-k)}(t) T^k(t)x, x \rangle = 0.$$

That means for all $t \in]0, t']$, $T(t)$ is an m -symmetry on $\overline{R(T^n(t'))}$. Using Lemma 3.1 again, we obtain that $T(t)$ is an m -symmetry on $\overline{R(T^n(t'))}$ for all $t > 0$ i.e.

$$\sum_{k=0}^m (-1)^{m-k} \binom{m}{k} \langle T^{*(n-k)}(t) T^k(t)x, x \rangle = 0, \forall x \in \overline{R(T^n(t'))}, \forall t > 0.$$

We observe for all $t > 0$ and $x \in \mathcal{H}$

$$\sum_{k=0}^m (-1)^{m-k} \binom{m}{k} \langle T^{*(n-k)}(t)T^k(t)T^n(t')x, T^n(t')x \rangle = 0.$$

Now, for all $x \in \mathcal{H}$ we have

$$\begin{aligned} & |\langle T^{*(n-k)}(t)T^k(t)T^n(t')x, T^n(t')x \rangle - \langle T^{*(n-k)}(t)T^k(t)x, x \rangle| \\ &= |\langle T^{*(n-k)}(t)T^k(t)T^n(t')x, T^n(t')x \rangle - \langle T^{*(n-k)}(t)T^k(t)x, T^n(t')x \rangle \\ &+ \langle T^{*(n-k)}(t)T^k(t)x, T^n(t')x \rangle - \langle T^{*(n-k)}(t)T^k(t)x, x \rangle| \\ &= |\langle T^{*(n-k)}(t)T^k(t)T^n(t')x - T^{*(n-k)}(t)T^k(t)x, T^n(t')x \rangle \\ &+ \langle T^{*(n-k)}(t)T^k(t)x, T^n(t')x - x \rangle| \\ &\leq \|T^{*(n-k)}(t)T^k(t)T^n(t')x - T^{*(n-k)}(t)T^k(t)x\| \|T^n(t')x\| \\ &+ \|T^{*(n-k)}(t)T^k(t)x\| \|T^n(t')x - x\| \\ &\leq (\|T^{*(n-k)}(t)T^k(t)\| \|T^n(t')x\| + \|T^{*(n-k)}(t)T^k(t)x\|) \|T^n(t')x - x\|. \end{aligned}$$

Since $\lim_{s \rightarrow 0^+} T(s)x = x$, we have

$$\lim_{t' \rightarrow 0^+} \|T^n(t')x - x\| = \lim_{t' \rightarrow 0^+} \|T(nt')x - x\| = 0.$$

Therefore,

$$\lim_{t' \rightarrow 0^+} |\langle T^{*(n-k)}(t)T^k(t)T^n(t')x, T^n(t')x \rangle - \langle T^{*(n-k)}(t)T^k(t)x, x \rangle| = 0.$$

Thus, for all $t > 0$ and $x \in \mathcal{H}$

$$\begin{aligned} & \lim_{t' \rightarrow 0^+} \sum_{k=0}^m (-1)^{m-k} \binom{m}{k} \langle T^{*(n-k)}(t)T^k(t)T^n(t')x, T^n(t')x \rangle = 0, \\ & \implies \sum_{k=0}^m (-1)^{m-k} \binom{m}{k} \langle T^{*(n-k)}(t)T^k(t)x, x \rangle = 0, \\ & \implies \sum_{k=0}^m (-1)^{m-k} \binom{m}{k} T^{*(n-k)}(t)T^k(t)x = 0. \end{aligned}$$

Hence $T(t)$ is an m -symmetry for all $t \geq 0$. \square

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