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GENERALIZED SEGAL-BARGMANN TRANSFORM AND ITS APPLICATIONS TO THE FIELDS OF UNCERTAINTY INEQUALITIES AND PDES

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Abstract. In this paper, we give some applications of the Dunkl-type Segal-Bargmann transform $\mathcal{B}_k$ to the field of uncertainty inequalities and to the field of partial differential equations. The resolution of the time-dependent Dunkl-Dirac Laplacian equation and the time-dependent generalized Dunkl-Schrödinger equation is based on the techniques of the intertwining operators on the Dunkl-type Fock space $\mathcal{F}_k(\mathbb{C}^d)$.

1. Introduction

The Fock space $\mathcal{F}(\mathbb{C}^d)$ is a Hilbert space consisting of entire functions on $\mathbb{C}^d$, square integrable with respect to the measure

$$\mathrm{d}m(z) := \frac{1}{\pi^d} e^{-|z|^2} \mathrm{d}x\mathrm{d}y, \quad z = x + iy,$$

where $|z|^2 = \sum_{i=1}^d |z_i|^2$ and $\mathrm{d}x\mathrm{d}y = \prod_{i=1}^d \mathrm{d}x_i \mathrm{d}y_i$. This space is equipped with the inner product

$$\langle f, g\rangle_{\mathcal{F}(\mathbb{C}^d)} := \int_{\mathbb{C}^d} f(z)\overline{g(z)}\mathrm{d}m(z).$$

The space $\mathcal{F}(\mathbb{C}^d)$ was introduced by Bargmann [3], is called also Segal-Bargmann space [6] and was used in many works [8, 30]. Precisely, Chen-Zhu [8] proved an uncertainty principle of Heisenberg type for the Fock space $\mathcal{F}(\mathbb{C}^d)$. Next, the Segal-Bargmann transform $\mathcal{B}$ was introduced for the first time by [3, 4] it has many applications in the quantized Yang-Mills theory on a space-time cylinder [11], in field of uncertainty inequalities [25] and in field of partial differential equations [7, 12, 15, 19].

In this paper we recall some properties for the Fock space $\mathcal{F}_k(\mathbb{C}^d)$ associated with the Dunkl operators $T_j(k)$, $j = 1, \dots, d$. The Dunkl-type Fock space $\mathcal{F}_k(\mathbb{C}^d)$ is firstly introduced by Soltani [20], next is studied by Ben Said-Orsted [5]. The space $\mathcal{F}_k(\mathbb{C}^d)$ is also used in many works [26, 27]. The Dunkl-type Segal-Bargmann transform $\mathcal{B}_k$ associated with a Coxeter group G is studied by

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Soltani et al. [20, 26] and Ben Said-Orsted [5]. This transform has a Plancherel and an inversion formulas [5, 20, 23]. In the one dimensional case this transform has some applications in field of uncertainty inequalities [24]. However, in d dimension, the application of this transform in field of uncertainty inequalities (local-type uncertainty principle, Heisenberg-type uncertainty principle,...) will be an open topic. This topic requires more details for the inner product of the Dunkl-type Fock space $\mathcal{F}_k(\mathbb{C}^d)$.

The purpose of this paper is to review the basic properties of the Dunkl-type Segal-Bargmann transform $\mathcal{B}_k$ in applications to the field of uncertainty inequalities and to the field of partial differential equations [7, 12, 15, 19]. More precisely, in Section 3, we give some intertwining relations on the Dunkl-type Fock space $\mathcal{F}_k(\mathbb{C}^d)$. In Section 4, we establish a Heisenberg type uncertainty principle related to the Dunkl-type Segal-Bargmann transform $\mathcal{B}_k$. In Sections 5 and 6, we study the time-dependent heat Cauchy problems associated with the complex operators

$$\mathcal{D}_k^+ := \frac{1}{2}\sum_{j=1}^{d}\left[T_j(k)+z_j\right]^2, \quad \mathcal{D}_k^- := \frac{1}{2}\sum_{j=1}^{d}\left[T_j(k)-z_j\right]^2.$$

The resolution of these problems are based on the techniques of intertwining relations

$$\mathcal{B}_k^{-1}\mathcal{D}_k^+ = |x|^2\mathcal{B}_k^{-1}, \quad \mathcal{B}_k^{-1}\mathcal{D}_k^- = \Delta_k\mathcal{B}_k^{-1},$$

where $\Delta_k := \sum_{j=1}^d T_j^2(k)$ is the Dunkl Laplacian. In the last section we describe the time-dependent generalized Schrödinger equation associated with the generalized Dunkl harmonic oscillator

$$L_k := \mathcal{B}_k^{-1}E\mathcal{B}_k,$$

where E is the complex Euler operator.

Throughout this paper we shall use on $\mathbb{C}^d$ the following notations. For all $z = (z_1,\dots,z_d)$, $w=(w_1,\dots,w_d)\in\mathbb{C}^d$, $w.z=\sum_{j=1}^d w_jz_j$, $|w|^2 = w.\overline{w} = \sum_{j=1}^d |w_j|^2$.

2. Dunk-type Segal-Bargmann space

In this section, we recall some properties for the Fock space $\mathcal{F}_k(\mathbb{C}^d)$ associated with the Dunkl operators.

For $\alpha\in\mathbb{R}^d\backslash\{0\}$, let σ_α be the reflection in the hyperplane $H_\alpha\subset\mathbb{R}^d$ orthogonal to α:

$$\sigma_\alpha y := y - 2\frac{\alpha.y}{|\alpha|^2}\alpha.$$

A finite set $\Re\subset\mathbb{R}^d\backslash\{0\}$ is called a root system, if $\Re\cap\mathbb{R}.\alpha=\{-\alpha,\alpha\}$ and $\sigma_\alpha\Re=\Re$ for all $\alpha\in\Re$. We assume that it is normalized by $|\alpha|^2=2$ for all $\alpha\in\Re$. For a root system $\Re$, the reflections σ_α, $\alpha\in\Re$, generate a finite group G. The Coxeter group G is a subgroup of the orthogonal group $O(d)$. All reflections in G correspond to suitable pairs of roots. For a given $\beta\in\mathbb{R}^d\backslash\bigcup_{\alpha\in\Re}H_\alpha$, we fix the positive subsystem $\Re_+ := \{\alpha\in\Re : \alpha.\beta>0\}$. Then for each $\alpha\in\Re$ either $\alpha\in\Re_+$ or $-\alpha\in\Re_+$.

Let $k : \Re \to \mathbb{C}$ be a multiplicity function on $\Re$ (a function which is constant on the orbits under the action of G). As an abbreviation, we introduce the index

$$\gamma = \gamma_k := \sum_{\alpha \in \Re_+} k(\alpha).$$

Throughout this paper, we will assume that the multiplicity is nonnegative, that is, $k(\alpha) \geq 0$ for all $\alpha \in \Re$.

The Dunkl operators $T_j(k)$, $j = 1, \dots, d$, on $\mathbb{R}^d$ associated with the finite reflection group G and multiplicity function k are given, for a function f of class C^1 on $\mathbb{R}^d$, by

$$T_j(k)f(x) := \frac{\partial}{\partial x_j} f(x) + \sum_{\alpha \in \Re_+} k(\alpha) \frac{\alpha_j}{\alpha . x} (f(x) - f(\sigma_\alpha x)).$$

For $y \in \mathbb{R}^d$, the initial value problem $T_j(k)u(., y)(x) = y_j u(x, y)$, $j = 1, \dots, d$, with $u(0, y) = 1$ admits a unique analytic solution on $\mathbb{R}^d$, which will be denoted by $E_k(x, y)$ and called Dunkl kernel [9, 10]. This kernel has a unique analytic extension to $\mathbb{C}^d \times \mathbb{C}^d$ (see [18]). We collect some further properties of the Dunkl kernel E_k. Let $w, z \in \mathbb{C}^d$ and $\lambda \in \mathbb{C}$, we have

$$E_k(w, z) = E_k(z, w), \quad E_k(\lambda w, z) = E_k(w, \lambda z), \quad \overline{E_k(w, z)} = E_k(\overline{w}, \overline{z}).$$

For $z, w \in \mathbb{C}^d$, we define

$$K(w, z) := E_k(z, \overline{w}).$$

From the properties of the Dunkl kernel [10, 18], the kernel K is continuous and the function $z \to K(w, z)$ is holomorphic for all $w \in \mathbb{C}^d$. Further, $K(w, z) = \overline{K(z, w)}$ and $K(w, z)$ is a positive definite kernel, i.e. for all $z^{(1)}, \dots, z^{(\ell)} \in \mathbb{C}^d$ and $a_1, \dots, a_\ell \in \mathbb{C}$:

$$\sum_{i,j=1}^{\ell} a_i \overline{a_j} K(z^{(i)}, z^{(j)}) \geq 0.$$

These properties of K lead to the following result.

Theorem 2.1. *(See* [5]*).*

(i) *There exists a Hilbert space $\mathcal{F}_k(\mathbb{C}^d)$ of holomorphic functions on $\mathbb{C}^d$, such that K is its reproducing kernel.*

(ii) *The Hilbert space $\mathcal{F}_k(\mathbb{C}^d)$ contains the $\mathbb{C}$-algebra $\mathcal{P}(\mathbb{C}^d)$ of polynomial functions on $\mathbb{C}^d$ as a dense subspace.*

In particular, if we denote by $\langle ., . \rangle_{\mathcal{F}_k(\mathbb{C}^d)}$ the inner product in $\mathcal{F}_k(\mathbb{C}^d)$, then

$$\langle p, q \rangle_{\mathcal{F}_k(\mathbb{C}^d)} = p(T(k))\overline{q(\overline{z})}\Big|_{z=0}, \quad p, q \in \mathcal{P}(\mathbb{C}^d),$$

where $p(T(k))$ is the operator formed by replacing z_j by $T_j(k)$ for $j = 1, \dots, d$. We shall call $\mathcal{F}_k(\mathbb{C}^d)$ the Fock space associated with the Coxeter group G or the Dunkl-type Fock space.

Remark 2.1. From the above theorem, $\mathcal{F}_k(\mathbb{C}^d)$ is defined by

$$\mathcal{F}_k(\mathbb{C}^d) := \overline{\langle K(w, .), \ w \in \mathbb{C}^d \rangle}.$$

Here the bar means the completion with respect to the norm $\|f\|_{\mathcal{F}_k(\mathbb{C}^d)} = \langle .,. \rangle^{1/2}_{\mathcal{F}_k(\mathbb{C}^d)}$. The Hilbert space $\mathcal{F}_k(\mathbb{C}^d)$ is uniquely determined by its reproducing kernel K. Notice that, for $k = 0$, $\mathcal{F}_0(\mathbb{C}^d)$ coincides with the classical Fock space $\mathcal{F}(\mathbb{C}^d)$, the space of holomorphic functions f on $\mathbb{C}^d$ such that

$$\|f\|^2_{\mathcal{F}(\mathbb{C}^d)} := \frac{1}{\pi^d} \int_{\mathbb{C}^d} |f(z)|^2 e^{-|z|^2} \mathrm{d}z < \infty.$$

Then, we have easily

Theorem 2.2. *For all $w \in \mathbb{C}^d$ and $f \in \mathcal{F}_k(\mathbb{C}^d)$, we have*

$$|f(w)| \leq [E_k(w, \overline{w})]^{1/2} \|f\|_{\mathcal{F}_k(\mathbb{C}^d)}.$$

3. Dunkl-type Segal-Bargmann transform

In this section, we recall some properties of the Dunkl-type Segal-Bargmann transform $\mathcal{B}_k$, and we give some intertwining relations on the Dunkl-type Fock space $\mathcal{F}_k(\mathbb{C}^d)$.

Let w_k denote the weight function

$$w_k(y) := \prod_{\alpha \in \Re_+} |\alpha.y|^{2k(\alpha)}, \quad y \in \mathbb{R}^d,$$

which is G-invariant and homogeneous of degree 2γ.

Let c_k be the Mehta-type constant given by

$$c_k := \left[\int_{\mathbb{R}^d} e^{-|y|^2/2} w_k(y) \mathrm{d}y \right]^{-1}.$$

We denote by μ_k the measure on $\mathbb{R}^d$ given by $\mathrm{d}\mu_k(y) := c_k w_k(y) \mathrm{d}y$; and by $L^2_k(\mathbb{R}^d)$, the space of measurable functions f on $\mathbb{R}^d$, such that

$$\|f\|_{L^2_k(\mathbb{R}^d)} := \left[\int_{\mathbb{R}^d} |f(y)|^2 \mathrm{d}\mu_k(y) \right]^{1/2} < \infty.$$

The following result is crucial in Dunkl's theory and its applications. Let $w, z \in \mathbb{C}^d$, we have

$$\int_{\mathbb{R}^d} e^{-|x|^2/2} E_k(w, x) E_k(z, x) \mathrm{d}\mu_k(x) = e^{(w.w + z.z)/2} E_k(w, z). \tag{3.1}$$

This interesting formula is used by many authors, see [5, 17, 20]. This is a formula that we also need in this work.

We denote by U_k the kernel given for $w \in \mathbb{C}^d$ and $x \in \mathbb{R}^d$, by

$$U_k(w, x) := 2^{(2\gamma+d)/4} e^{-(w.w + |x|^2)/2} E_k(w, \sqrt{2}x).$$

From (3.1), the kernel U_k satisfies the following properties [20].

(a) For all $w, z \in \mathbb{C}^d$, we have

$$E_k(w, z) = \int_{\mathbb{R}^d} U_k(w, x) U_k(z, x) \mathrm{d}\mu_k(x). \tag{3.2}$$

(b) For all $w \in \mathbb{C}^d$, the function $U_k(w,.)$ belongs to $L_k^2(\mathbb{R}^d)$, and

$$\|U_k(w,.)\|^2_{L_k^2(\mathbb{R}^d)} = E_k(w,\overline{w}).$$

The kernel U_k gives rise to an integral transform $\mathcal{B}_k$, which is called Dunkl-type Segal-Bargmann transform on $\mathbb{C}^d$, and defined for f in $L_k^2(\mathbb{R}^d)$, by

$$\mathcal{B}_k(f)(w) := \int_{\mathbb{R}^d} U_k(w,x)f(x)\mathrm{d}\mu_k(x), \quad w \in \mathbb{C}^d. \tag{3.3}$$

The following two theorems are proved in [5, 20].

Theorem 3.1. *The Dunkl-type Segal-Bargmann transform $\mathcal{B}_k$ is an isometric isomorphism of $L_k^2(\mathbb{R}^d)$ onto $\mathcal{F}_k(\mathbb{C}^d)$. In particular, we have*

$$\|\mathcal{B}_k(f)\|_{\mathcal{F}_k(\mathbb{C}^d)} = \|f\|_{L_k^2(\mathbb{R}^d)}, \quad f \in L_k^2(\mathbb{R}^d).$$

In the next part of this section we give some intertwining relations on the Dunkl-type Fock space $\mathcal{F}_k(\mathbb{C}^d)$.

We define the Dunkl Laplacian Δ_k for $f \in \mathcal{F}_k(\mathbb{C}^d)$ by

$$\Delta_k f(z) := \sum_{j=1}^d T_j^2(k)f(z).$$

The Dunkl Laplacian can be written as

$$\Delta_k f(z) = \Delta f(z) + 2\sum_{\alpha\in\Re_+} k(\alpha)\left[\frac{\nabla f(z).\alpha}{\alpha.z} - \frac{f(z)-f(\sigma_\alpha(z))}{(\alpha.z)^2}\right],$$

where Δ and ∇ denote the usual Laplacian and gradient operators, respectively. This operator is the goal of many works [5, 14, 17].

We define the complex Dunkl-Dirac Laplacian $\mathcal{D}_k^+ : \mathcal{F}_k(\mathbb{C}^d) \to \mathcal{F}_k(\mathbb{C}^d)$, by

$$\mathcal{D}_k^+ := \sum_{j=1}^d D_j^2(k), \tag{3.4}$$

where $D_j(k)$ are the complex Dunkl-Dirac operators given by

$$D_j(k) := \frac{1}{\sqrt{2}}\left[T_j(k) + z_j\right]. \tag{3.5}$$

The operators $D_j(k)$ are studied in [26, 28] and are involved in the demonstration of the Heisenberg's uncertainty principle for the Dunkl-type Fock space. The operators $D_j(k)$ are also used in the study of the Dunkl Riesz transforms [16]. In the classical case ($\alpha_j = -\frac{1}{2}$) these operators are studied in [8, 19] and are called complex Dirac operators.

As in the same way we define the complex operator $\mathcal{D}_k^- : \mathcal{F}_k(\mathbb{C}^d) \to \mathcal{F}_k(\mathbb{C}^d)$, by

$$\mathcal{D}_k^- := \sum_{j=1}^d Q_j^2(k), \tag{3.6}$$

where

$$Q_j(k) := \frac{1}{\sqrt{2}}\left[T_j(k) - z_j\right]. \tag{3.7}$$

Theorem 3.2. *For $f \in \mathcal{F}_k(\mathbb{C}^d)$, we have*

(i) $\mathcal{D}_k^+ f(z) = \mathcal{B}_k |x|^2 \mathcal{B}_k^{-1}(f)(z)$,

(ii) $\mathcal{D}_k^- f(z) = \mathcal{B}_k \Delta_k \mathcal{B}_k^{-1}(f)(z)$.

Proof. (i) For every $\varphi \in L_k^2(\mathbb{R}^d)$ and $x_j\varphi \in L_k^2(\mathbb{R}^d)$, with $j = 1, \dots, d$, we have

$$\begin{aligned} T_j(k)(\mathcal{B}_k(\varphi))(z) &= \int_{\mathbb{R}^d} T_j(k)(U_k(.,x))(z)\varphi(x)\mathrm{d}\mu_k(x) \\ &= 2^{(2\gamma+d)/4} \int_{\mathbb{R}^d} T_j(k)(e^{-z.z/2}E_k(z,\sqrt{2}x))e^{-|x|^2/2}\varphi(x)\mathrm{d}\mu_k(x) \\ &= \sqrt{2}\mathcal{B}_k(x_j\varphi)(z) - z_j\mathcal{B}_k(\varphi)(z). \end{aligned}$$

Hence

$$\mathcal{B}_k(x_j\varphi)(z) = D_j(k)\mathcal{B}_k(\varphi)(z), \quad j = 1, \dots, d. \tag{3.8}$$

By taking $\varphi = \mathcal{B}_k^{-1}(f)$, with $f \in \mathcal{F}_k(\mathbb{C}^d)$, we deduce that

$$\mathcal{B}_k |x|^2 \mathcal{B}_k^{-1}(f)(z) = \mathcal{D}_k^+ f(z).$$

The (i) is proved.

(ii) For every $\varphi \in L_k^2(\mathbb{R}^d)$ and $x_j\varphi \in L_k^2(\mathbb{R}^d)$, with $j = 1, \dots, d$. From ([26], page 145) we have

$$\mathcal{B}_k(T_j(k)\varphi)(z) = \mathcal{B}_k(x_j\varphi)(z) - \sqrt{2}z_j\mathcal{B}_k(\varphi)(z).$$

Combining this relation with (3.8) we get

$$\mathcal{B}_k(T_j(k)\varphi)(z) = Q_j(k)\mathcal{B}_k(\varphi)(z), \quad j = 1, \dots, d. \tag{3.9}$$

By taking $\varphi = \mathcal{B}_k^{-1}(f)$, with $f \in \mathcal{F}_k(\mathbb{C}^d)$, we deduce that

$$\mathcal{B}_k \Delta_k \mathcal{B}_k^{-1}(f)(z) = \mathcal{D}_k^- f(z).$$

The (ii) is proved. □

4. Uncertainty principles

In this section, we establish some uncertainty principles for the Dunk-type Segal-Bargmann space $\mathcal{F}_k(\mathbb{C}^d)$.

The Dunkl operators $T_j(k)$ and the multiplication operator by z_j satisfy [20]

$$[T_j(k), z_j] = I + P_j(k), \tag{4.1}$$

where I is the identity operator, and $P_j(k)$ are the operators given by

$$P_j(k)f(z) := \sum_{\alpha \in \Re_+} k(\alpha)(\alpha_j)^2 f(\sigma_\alpha z), \quad j = 1, \dots, d.$$

We define the domain of $P_j(k)$ denoted by $\mathrm{Dom}(P_j(k))$ as

$$\mathrm{Dom}(P_j(k)) := \left\{ f \in \mathcal{F}_k(\mathbb{C}^d) : P_j(k)f \in \mathcal{F}_k(\mathbb{C}^d) \right\}.$$

Lemma 4.1. *(See [5, 20]). The operators $T_j(k)$, z_j and $P_j(k)$ satisfy the following properties.*

(i) *$Dom(P_j(k)) = \mathcal{F}_k(\mathbb{C}^d)$,*

(ii) $T_j^*(k) = z_j$.

Lemma 4.2. *(See [13], Proposition 2.1). Let A and B be self-adjoint operators on a Hilbert space H, then*

$$\|(A-a)f\|_H \|(B-b)f\|_H \geq \frac{1}{2}\left|\langle [A,B]f, f\rangle_H\right|,$$

for all $f \in Dom([A,B])$ and all $a, b \in \mathbb{C}$.

Let $F_k^+(\mathbb{C}^d)$ be the space defined as

$$F_k^+(\mathbb{C}^d) := \left\{ f \in \mathcal{F}_k(\mathbb{C}^d) : \langle P_j(k)f, f\rangle_{\mathcal{F}_k(\mathbb{C}^d)} \geq 0, \quad j = 1, \dots, d \right\}.$$

Theorem 4.1. *Let $f \in F_k^+(\mathbb{C}^d)$. For all $a, b \in \mathbb{C}$, we have*

$$\|(D_j(k)-a)f\|_{\mathcal{F}_k(\mathbb{C}^d)} \|(Q_j(k)+ib)f\|_{\mathcal{F}_k(\mathbb{C}^d)} \geq \frac{1}{2}\|f\|^2_{\mathcal{F}_k(\mathbb{C}^d)}, \quad j = 1, \dots, d, \tag{4.2}$$

where $D_j(k)$ and $Q_j(k)$ are the operators given by (3.5) *and* (3.7)*, respectively.*

Proof. Let $f \in F_k^+(\mathbb{C}^d)$. Now, let A and B the operators defined by

$$Af(z) := D_j(k)f(z), \quad Bf(z) := iQ_j(k)f(z).$$

From (4.1) and Lemma 4.1, the operators A and B possess the following properties.

(i) $A^* = A$ and $B^* = B$,

(ii) $[A, B] = -i[T_j(k), z_j] = -i[I + P_j(k)]$,

(iii) $\mathrm{Dom}([A, B]) = \mathcal{F}_k(\mathbb{C}^d)$.

Thus, the inequality (4.2) follows from Lemma 4.2. □

This uncertainty principle give a generalization of the result of Chen-Zhu for the classical Fock space [8] and of the result of Soltani for the Bessel-type Fock space [29]. However we obtain the following result.

Theorem 4.2. *Let $\varphi \in L_k^2(\mathbb{R}^d)$ such that $\mathcal{B}_k(\varphi) \in F_k^+(\mathbb{C}^d)$. For all $a, b \in \mathbb{C}$, we have*

$$\|(x_j - a)\varphi\|_{L_k^2(\mathbb{R}^d)} \|(T_j(k)+ib)\varphi\|_{L_k^2(\mathbb{R}^d)} \geq \frac{1}{2}\|\varphi\|^2_{L_k^2(\mathbb{R}^d)}, \quad j = 1, \dots, d.$$

Proof. The result of this theorem follows from Theorem 4.1 by using relations (3.8) and (3.9) with Theorem 3.1. □

Theorem 4.3. *Let $f \in \mathcal{F}_k(\mathbb{C}^d)$. Then*

$$\|\mathcal{D}_k^+ f\|_{\mathcal{F}_k(\mathbb{C}^d)} \|\mathcal{D}_k^- f\|_{\mathcal{F}_k(\mathbb{C}^d)} \geq (\gamma + d/2)^2 \|f\|^2_{\mathcal{F}_k(\mathbb{C}^d)}.$$

Proof. Let $\varphi \in L_k^2(\mathbb{R}^d)$. From ([22], Theorem 1) we have

$$\||x|^2\varphi\|_{L_k^2(\mathbb{R}^d)} \|\Delta_k\varphi\|_{L_k^2(\mathbb{R}^d)} \geq (\gamma + d/2)^2 \|\varphi\|^2_{L_k^2(\mathbb{R}^d)}. \tag{4.3}$$

We take $f = \mathcal{B}_k(\varphi)$, $\varphi \in L_k^2(\mathbb{R}^d)$. Then by Theorem 3.1, we have

$$\|\varphi\|_{L_k^2(\mathbb{R}^d)} = \|\mathcal{B}_k(\varphi)\|_{\mathcal{F}_k(\mathbb{C}^d)} = \|f\|_{\mathcal{F}_k(\mathbb{C}^d)}.$$

However, by Theorem 3.2 we have

$$\||x|^2\varphi\|_{L_k^2(\mathbb{R}^d)} = \|\mathcal{B}_k|x|^2\mathcal{B}_k^{-1}(f)\|_{\mathcal{F}_k(\mathbb{C}^d)} = \|\mathcal{D}_k^+ f\|_{\mathcal{F}_k(\mathbb{C}^d)},$$

$$\|\Delta_k\varphi\|_{L_k^2(\mathbb{R}^d)} = \|\mathcal{B}_k\Delta_k\mathcal{B}_k^{-1}(f)\|_{\mathcal{F}_k(\mathbb{C}^d)} = \|\mathcal{D}_k^- f\|_{\mathcal{F}_k(\mathbb{C}^d)}.$$

We obtain the result of the theorem from relation (4.3). □

5. Dunkl-Dirac Laplacian equation

In this section we give application of the Dunkl-type Segal-Bargmann transform $\mathcal{B}_k$ to the time-dependent Dunkl-Dirac Laplacian equation associated with the complex operator $\mathcal{D}_k^+$.

For $w, z \in \mathbb{C}^d$ and $t \geq 0$, we denote by $h_k(w,z,t)$ the kernel

$$h_k(w,z,t) := \int_{\mathbb{R}^d} e^{-t|x|^2} U_k(w,x)U_k(z,x)\mathrm{d}\mu_k(x).$$

From (3.2), for $t = 0$ we have

$$h_k(w,z,0) = E_k(w,z).$$

Lemma 5.1. *For $w, z \in \mathbb{C}^d$, we have*

$$h_k(w,z,t) = \frac{e^{-t(w.w+z.z)/2(1+t)}}{(1+t)^{\gamma+d/2}} E_k\left(\frac{w}{\sqrt{1+t}}, \frac{z}{\sqrt{1+t}}\right). \tag{5.1}$$

Proof. For $w, z \in \mathbb{C}^d$, we have

$$\begin{aligned} h_k(w,z,t) &= \int_{\mathbb{R}^d} e^{-t|x|^2} U_k(w,x)U_k(z,x)\mathrm{d}\mu_k(x) \\ &= 2^{\gamma+d/2} e^{-(w.w+z.z)/2} \int_{\mathbb{R}^d} e^{-(1+t)|x|^2} E_k(w,\sqrt{2}x)E_k(z,\sqrt{2}x)\mathrm{d}\mu_k(x) \\ &= \frac{e^{-(w.w+z.z)/2}}{(1+t)^{\gamma+d/2}} \int_{\mathbb{R}^d} e^{-|x|^2/2} E_k\left(\frac{w}{\sqrt{1+t}}, x\right) E_k\left(\frac{z}{\sqrt{1+t}}, x\right) \mathrm{d}\mu_k(x). \end{aligned}$$

From relation (3.1), we have

$$\begin{aligned} \int_{\mathbb{R}^d} e^{-|x|^2/2} E_k\left(\frac{w}{\sqrt{1+t}}, x\right) E_k\left(\frac{z}{\sqrt{1+t}}, x\right) \mathrm{d}\mu_k(x) \\ = e^{(w.w+z.z)/2(1+t)} E_k\left(\frac{w}{\sqrt{1+t}}, \frac{z}{\sqrt{1+t}}\right). \end{aligned}$$

Then, we obtain

$$h_k(w,z,t) = \frac{e^{-t(w.w+z.z)/2(1+t)}}{(1+t)^{\gamma+d/2}} E_k\left(\frac{w}{\sqrt{1+t}}, \frac{z}{\sqrt{1+t}}\right).$$

We get the desired result. □

Let us consider the heat Cauchy problem

$$\begin{cases} \dfrac{\partial}{\partial t} F(t,z) = -\mathcal{D}_k^+ F(t,z), \quad t > 0,\ z \in \mathbb{C}^d, \\ F(0,z) = f(z) \in \mathcal{F}_k(\mathbb{C}^d), \end{cases} \tag{5.2}$$

where $\mathcal{D}_k^+$ is the complex Dunkl-Dirac Laplacian given by (3.4).

Theorem 5.1. *The solution of the heat Cauchy problem* (5.2) *is given by the formula*

$$F(t,z) = \langle f, h_k(.,\overline{z},t)\rangle_{\mathcal{F}_k(\mathbb{C}^d)},$$

where $h_k(w,z,t)$ *is the kernel given by* (5.1).

Proof. By applying the inverse of the Bargmann transform $\mathcal{B}_k^{-1}$, with Theorem 3.2 (i) to the Cauchy problem (5.2) we obtain

$$\begin{cases} \dfrac{\partial}{\partial t}\Phi(t,x) = -|x|^2\Phi(t,x), \quad t>0,\ x\in\mathbb{R}^d, \\ \Phi(0,x) = \varphi(x) \in L_k^2(\mathbb{R}^d), \end{cases}$$

where $\varphi = \mathcal{B}_k^{-1}(f)$ and $\Phi(t,.) = \mathcal{B}_k^{-1}(F(t,.))$. Thus $\Phi(t,x)$ satisfies

$$\Phi(t,x) = e^{-t|x|^2}\varphi(x).$$

This implies that

$$F(t,z) = \mathcal{B}_k(e^{-t|x|^2}\varphi(x))(z).$$

Therefore and by (3.3), we obtain

$$\begin{aligned} F(t,z) &= \int_{\mathbb{R}^d} U_k(z,x)e^{-t|x|^2}\varphi(x)\mathrm{d}\mu_k(x) \\ &= \int_{\mathbb{R}^d} \mathcal{B}_k^{-1}(f)(x)\overline{U_k(\overline{z},x)}e^{-t|x|^2}\mathrm{d}\mu_k(x). \end{aligned}$$

According to Theorem 3.1 and Lemma 5.1, we deduce that

$$F(t,z) = \langle f, \mathcal{B}_k(U_k(\overline{z},x)e^{-t|x|^2})\rangle_{\mathcal{F}_k(\mathbb{C}^d)} = \langle f, h_k(.,\overline{z},t)\rangle_{\mathcal{F}_k(\mathbb{C}^d)}.$$

The theorem is proved. □

6. Additional result

In this section we solve the time-dependent Cauchy problem associated with the complex operator $\mathcal{D}_k^-$.

The Dunkl-Weierstrass transform [17, 21] is defined for $f\in L_k^2(\mathbb{R}^d)$ and $t>0$, by

$$W_{k,t}(f)(x) := \int_{\mathbb{R}^d} \Gamma_k(x,y,t)f(y)\mathrm{d}\mu_k(y), \quad x\in\mathbb{R}^d, \tag{6.1}$$

where $\Gamma_k(x,y,t)$ is the kernel given by

$$\Gamma_k(x,y,t) := \frac{e^{-\left(|x|^2+|y|^2\right)/4t}}{(2t)^{\gamma+d/2}} E_k\left(\frac{x}{\sqrt{2t}}, \frac{y}{\sqrt{2t}}\right).$$

For $w,z\in\mathbb{C}^d$ and $t>0$, we denote by $q_k(w,z,t)$ the kernel

$$q_k(w,z,t) := \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} U_k(w,y)U_k(z,x)\Gamma_k(x,y,t)\mathrm{d}\mu_k(x)\mathrm{d}\mu_k(y).$$

Lemma 6.1. *For* $w,z\in\mathbb{C}^d$, *we have*

$$q_k(w,z,t) = \frac{e^{t(w.w+z.z)/2(1+t)}}{(1+t)^{\gamma+d/2}} E_k\left(\frac{w}{\sqrt{1+t}}, \frac{z}{\sqrt{1+t}}\right). \tag{6.2}$$

Proof. For $w, z \in \mathbb{C}^d$, we have

$$\begin{aligned} q_k(w,z,t) &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} U_k(w,y)U_k(z,x)\Gamma_k(x,y,t)\mathrm{d}\mu_k(x)\mathrm{d}\mu_k(y) \\ &= \frac{e^{-(w.w+z.z)/2}}{t^{\gamma+d/2}} \int_{\mathbb{R}^d} e^{-\frac{1+2t}{4t}|y|^2} E_k(w,\sqrt{2}y)J_k(y,z,t)\mathrm{d}\mu_k(y), \end{aligned}$$

where

$$J_k(y,z,t) = \int_{\mathbb{R}^d} e^{-\frac{1+2t}{4t}|x|^2} E_k(z,\sqrt{2}x)E_k\left(\frac{x}{\sqrt{2t}}, \frac{y}{\sqrt{2t}}\right)\mathrm{d}\mu_k(x).$$

According to the proof of Lemma 5.1 we get

$$J_k(y,z,t) = \left(\frac{2t}{1+2t}\right)^{\gamma+d/2} e^{\frac{2t}{1+2t}z.z+\frac{|y|^2}{4t(1+2t)}} E_k\left(\frac{\sqrt{2}z}{\sqrt{1+2t}}, \frac{y}{\sqrt{1+2t}}\right).$$

Therefore

$$\begin{aligned} q_k(w,z,t) &= \left(\frac{2}{1+2t}\right)^{\gamma+d/2} e^{-\frac{w.w}{2}+\frac{2t-1}{2(1+2t)}z.z} \\ &\times \int_{\mathbb{R}^d} e^{-\frac{1+t}{1+2t}|y|^2} E_k(w,\sqrt{2}y)E_k\left(\frac{\sqrt{2}z}{\sqrt{1+2t}}, \frac{y}{\sqrt{1+2t}}\right)\mathrm{d}\mu_k(y) \\ &= \frac{e^{t(w.w+z.z)/2(1+t)}}{(1+t)^{\gamma+d/2}} E_k\left(\frac{w}{\sqrt{1+t}}, \frac{z}{\sqrt{1+t}}\right). \end{aligned}$$

The lemma is proved. □

Let us consider the heat Cauchy problem

$$\begin{cases} \dfrac{\partial}{\partial t}F(t,z) = \mathcal{D}_k^- F(t,z), \quad t>0,\ z\in\mathbb{C}^d, \\ F(0,z) = f(z) \in \mathcal{F}_k(\mathbb{C}^d), \end{cases} \tag{6.3}$$

where $\mathcal{D}_k^-$ is the complex operator given by (3.6).

Theorem 6.1. *The solution of the heat Cauchy problem* (6.3) *is given by the formula*

$$F(t,z) = \langle f, q_k(.,\overline{z},t)\rangle_{\mathcal{F}_k(\mathbb{C}^d)},$$

where $q_k(w,z,t)$ is the kernel given by (6.2).

Proof. By applying the inverse of the Segal-Bargmann transform $\mathcal{B}_k^{-1}$, with Theorem 3.2 (ii), to the Cauchy problem (6.3) we obtain

$$\begin{cases} \dfrac{\partial}{\partial t}\Phi(t,x) = \Delta_k\Phi(t,x), \quad t>0,\ x\in\mathbb{R}^d, \\ \Phi(0,x) = \varphi(x) \in L_k^2(\mathbb{R}^d), \end{cases}$$

where $\varphi = \mathcal{B}_k^{-1}(f)$ and $\Phi(t,.) = \mathcal{B}_k^{-1}(F(t,.))$. Thus $\Phi(t,x)$ satisfies

$$\Phi(t,x) = W_{k,t}(\varphi)(x),$$

where $W_{k,t}(\varphi)$ is the Dunkl-Weierstrass transform given by (6.1) .
This implies that

$$F(t,z) = \mathcal{B}_k(W_{k,t}(\varphi))(z).$$

Then by (3.3) and (6.1) we have

$$F(t,z) = \int_{\mathbb{R}^d} U_k(z,x) W_{k,t}(\varphi)(x) \mathrm{d}\mu_k(x)$$
$$= \int_{\mathbb{R}^d} \mathcal{B}_k^{-1}(f)(y) \overline{V_k(y,\overline{z},t)} \mathrm{d}\mu_k(y),$$

where

$$V_k(y,z,t) := \int_{\mathbb{R}^d} U_k(z,x) \Gamma_k(x,y,t) \mathrm{d}\mu_k(x).$$

According to Theorem 3.1 and Lemma 6.1, we deduce that

$$F(t,z) = \langle f, \mathcal{B}_k(V_k(.,\overline{z},t))\rangle_{\mathcal{F}_k(\mathbb{C}^d)} = \langle f, q_k(.,\overline{z},t)\rangle_{\mathcal{F}_k(\mathbb{C}^d)}.$$

The theorem is proved. □

7. Generalized Dunkl harmonic oscillator

In this section we give application of the Dunkl-type Segal-Bargmann transform $\mathcal{B}_k$ to time-dependent generalized Dunkl-Schrödinger equation.

We define the Dunkl harmonic oscillator $H_k : L_k^2(\mathbb{R}^d) \to L_k^2(\mathbb{R}^d)$ by

$$H_k := -\frac{1}{2}\Delta_k + \frac{1}{2}|x|^2 - \frac{1}{2}(2\gamma + d).$$

Let $\{\xi_1, \ldots, \xi_d\}$ be any orthonormal basis of $\mathbb{C}^d$. On $\mathcal{F}_k(\mathbb{C}^d)$, Ben Said-Orsted ([5], Theorem 4.12) proved the following intertwining relation

$$\mathcal{B}_k H_k \mathcal{B}_k^{-1} = \sum_{j=1}^{d} \xi_j \frac{\partial}{\partial \xi_j}.$$

The Dunkl harmonic oscillator H_k (called also the Dunkl-Schrödinger operator) is studied by many authors [1, 2, 5, 16].

We define the generalized Dunkl harmonic oscillator $L_k : L_k^2(\mathbb{R}^d) \to L_k^2(\mathbb{R}^d)$ by

$$L_k\varphi(x) := \mathcal{B}_k^{-1} E \mathcal{B}_k \varphi(x), \tag{7.1}$$

where E is the complex Euler operator given by

$$E := \sum_{j=1}^{d} z_j \frac{\partial}{\partial z_j}, \quad z = (z_1, \ldots, z_d) \in \mathbb{C}^d.$$

Let us consider the time-dependent generalized Dunkl-Schrödinger equation

$$\begin{cases} i\dfrac{\partial}{\partial t}\Phi(t,x) = L_k\Phi(t,x), & t > 0,\ x \in \mathbb{R}^d, \\ \Phi(0,x) = \varphi(x) \in L_k^2(\mathbb{R}^d). \end{cases} \tag{7.2}$$

By applying the Segal-Bargmann transform $\mathcal{B}_k$, with (7.1) to the Cauchy problem (7.2) we obtain

$$\begin{cases} i\dfrac{\partial}{\partial t}F(t,z) = EF(t,z), & t > 0,\ z \in \mathbb{C}^d, \\ F(0,z) = f(z) \in \mathcal{F}_k(\mathbb{C}^d), \end{cases} \tag{7.3}$$

where $f = \mathcal{B}_k(\varphi)$ and $F(t,.) = \mathcal{B}_k(\Phi(t,.))$.

Lemma 7.1. *For every $f \in \mathcal{F}_k(\mathbb{C}^d)$, there exists the unique solution to the Cauchy problem* (7.3) *given by*

$$F(t,z) = f(ze^{-it}).$$

Proof. We want to solve the partial differential equation

$$\frac{\partial}{\partial t}F(t,z) + i\sum_{j=1}^{d} z_j \frac{\partial}{\partial z_j}F(t,z) = 0,$$

with the condition

$$F(0,z) = f(z) \in \mathcal{F}_k(\mathbb{C}^d).$$

By the transport equations $w_j = \log z_j$ and $\tau = it$, we have

$$\frac{\partial}{\partial \tau}G(\tau,w) + \sum_{j=1}^{d} \frac{\partial}{\partial w_j}G(\tau,w) = 0.$$

Then

$$G(\tau,w) = g(w_1 - \tau, \dots, w_d - \tau),$$

where g is found from the initial condition $g(\log z_1, \dots, \log z_d) = f(z)$, which implies that $g(z) = f(e^{z_1}, \dots, e^{z_d})$. Therefore,

$$F(t,z) = G(\tau,w) = g(\log z_1 - it, \dots, \log z_d - it) = f(ze^{-it}).$$

The lemma is proved. □

Theorem 7.1. *The solution of the time-dependent Dunkl-Schrödinger equation* (7.2) *is given by*

$$\Phi(t,x) = \mathcal{B}_k^{-1}(F(t,.))(x),$$

where

$$F(t,z) = \mathcal{B}_k(\varphi)(ze^{-it}).$$

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References

[1] B. Amri and A. Hammi, Dunkl-Schrödinger operators, *Complex Anal. Oper. Theory* **13** (2019), no. 3, 1033–1058.

[2] B. Amri and A. Hammi, Semigroup and Riesz transform for the Dunkl-Schrödinger operators, *Semigroup Forum* **101** (2020), no. 3, 507–533.

[3] V. Bargmann, On a Hilbert space of analytic functions and an associated integral transform part I, *Comm. Pure Appl. Math.* **14** (1961), no. 3, 187–214.

[4] V. Bargmann, On a Hilbert space of analytic functions and an associated integral transform part II. A family of related function spaces application to distribution theory, *Comm. Pure Appl. Math.* **XX** (1967), no. 1, 1–101.

[5] S. Ben Said and B. Orsted, Segal-Bargmann transforms associated with finite Coxeter groups, *Math. Ann.* **334** (2006), no. 2, 281–323.

[6] C.A. Berger and L.A. Coburn, Toeplitz operators on the Segal-Bargmann space, *Trans. Amer. Math. Soc.* **301** (1987), no. 2, 813–829.
[7] N. Berline, E. Getzler and M. Vergne, *Heat kernels and Dirac operator*, Grundlehren, Text Editions, Springer Verlag, 2004.
[8] Y. Chen and K. Zhu, Uncertainty principles for the Fock space, *Sci. Sinica Math.* **45** (2015), no. 11, 1847–1854.
[9] C.F. Dunkl, Integral kernels with reflection group invariance, *Can. J. Math.* **43** (1991), no. 6, 1213–1227.
[10] C.F. Dunkl, Hankel transforms associated to finite reflection groups. Proc. of the special session on hypergeometric functions on domains of positivity, Jack polynomials and applications. Proceedings, Tampa 1991, *Contemp. Math.* **138** (1992), 123–138.
[11] B.K. Driver and B.C. Hall, Yang-Mills theory and the Segal-Bargmann transform, *Commun. Math. Phys.* **201** (1999), no. 2, 249–290.
[12] L.C. Evans, *Partial differential equation*, Graduate Studies in Mathematics, Vol. 19, American Mathematical Society, 1998.
[13] G.B. Folland and A. Sitaram, The uncertainty principle: A mathematical survey, *J. Fourier Anal. Appl.* **3** (1997), no. 3, 207–238.
[14] M.F. de Jeu, The Dunkl transform, *Inv. Math.* **113** (1993), no. 1, 147–162.
[15] A.Y.O. Mohameden and M.V.O. Moustapha, Wave kernels for the Dirac, Euler operators and the harmonic oscillator, *J. Math. Phys.* **55** (2014), no. 3, 031503.
[16] A. Nowak and K. Stempak, Riesz transforms for the Dunkl harmonic oscillator, *Math. Z.* **262** (2009) 539–556.
[17] M. Rösler, Generalized Hermite polynomials and the heat equation for Dunkl operators, *Comm. Math. Phys.* **192** (1998), no. 3, 519–542.
[18] E.M. Opdam, Dunkl operators. Bessel functions and the discriminant of a finite Coxeter group, *Compos. Math.* **85** (1993), no. 3, 333–373.
[19] A. Saidi, A.Y. Mahmoud and M.V.O. Moustapha, Bargmann transform with application to time-dependent Schrödinger equation, *Int. J. Sci. Tech. Research* **9** (2020), no. 2, 760–763.
[20] F. Soltani, Generalized Fock spaces and Weyl commutation relations for the Dunkl kernel, *Pacific J. Math.* **214** (2004), no. 2, 379–397.
[21] F. Soltani, Inversion formulas in the Dunkl-type heat conduction on $\mathbb{R}^d$, *Appl. Anal.* **84** (2005), no. 6, 541–553.
[22] F. Soltani, A general form of Heisenberg-Pauli-Weyl uncertainty inequality for the Dunkl transform, *Integral Transform. Spec. Funct.* **24** (2013), no. 5, 401–409.
[23] F. Soltani, Inversion formulas for the Dunkl-type Segal-Bargmann transform, *Integral Transform. Spec. Funct.* **26** (2015), no. 5, 325–339.
[24] F. Soltani, Uncertainty principles for the Dunkl-type Segal-Bargmann transform, *Complex Anal. Oper. Theory* **11** (2017), no. 3, 475–490.
[25] F. Soltani, Uncertainty principles for the Segal-Bargmann transform, *J. Math. Res. Appl.* **37** (2017), no. 5, 563–576.
[26] F. Soltani and A. Nemri, Analytical and numerical approximation formulas on the Dunkl-type Fock spaces, *Acta Math. Vietnam.* **42** (2017), no. 1, 129–147.
[27] F. Soltani and M. Nenni, Difference and primitive operators on the Dunkl-type Fock space $\mathcal{F}_\alpha(\mathbb{C}^d)$, *J. Math. Sci.* **266** (2022), no. 6, 917–932.
[28] F. Soltani and M. Nenni, Heisenberg uncertainty principles for the Dunkl-type Fock space, *Complex Anal. Oper. Theory* **17** (2023), no. 6, Art. 105.
[29] F. Soltani, Reproducing kernel Hilbert spaces (RKHS) for the higher order Bessel operator, *Bol. Soc. Mat. Mex.* **29** (2023), no. 1, Art. 20.

[30] K. Zhu, *Analysis on Fock spaces*, Vol. 263, Graduate Texts in Mathematics, Springer, 2012.

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